Title: PSI 2019/2020 - Gravitational Physics - Lecture 7

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LECTURE 7 Black Holes & Beyond

Recall the geodesic eqn:

$$\dot{r}^2 + V_{eff}(r) = 0$$

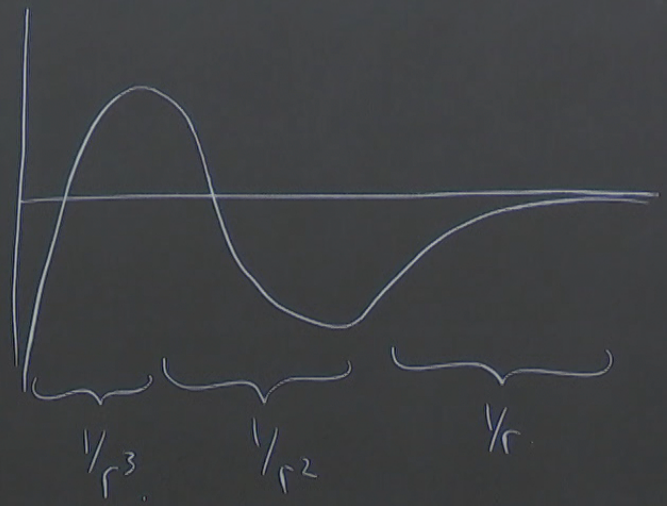
where $V_{eff} = \frac{\eta - p^2}{-g_{rr}}$

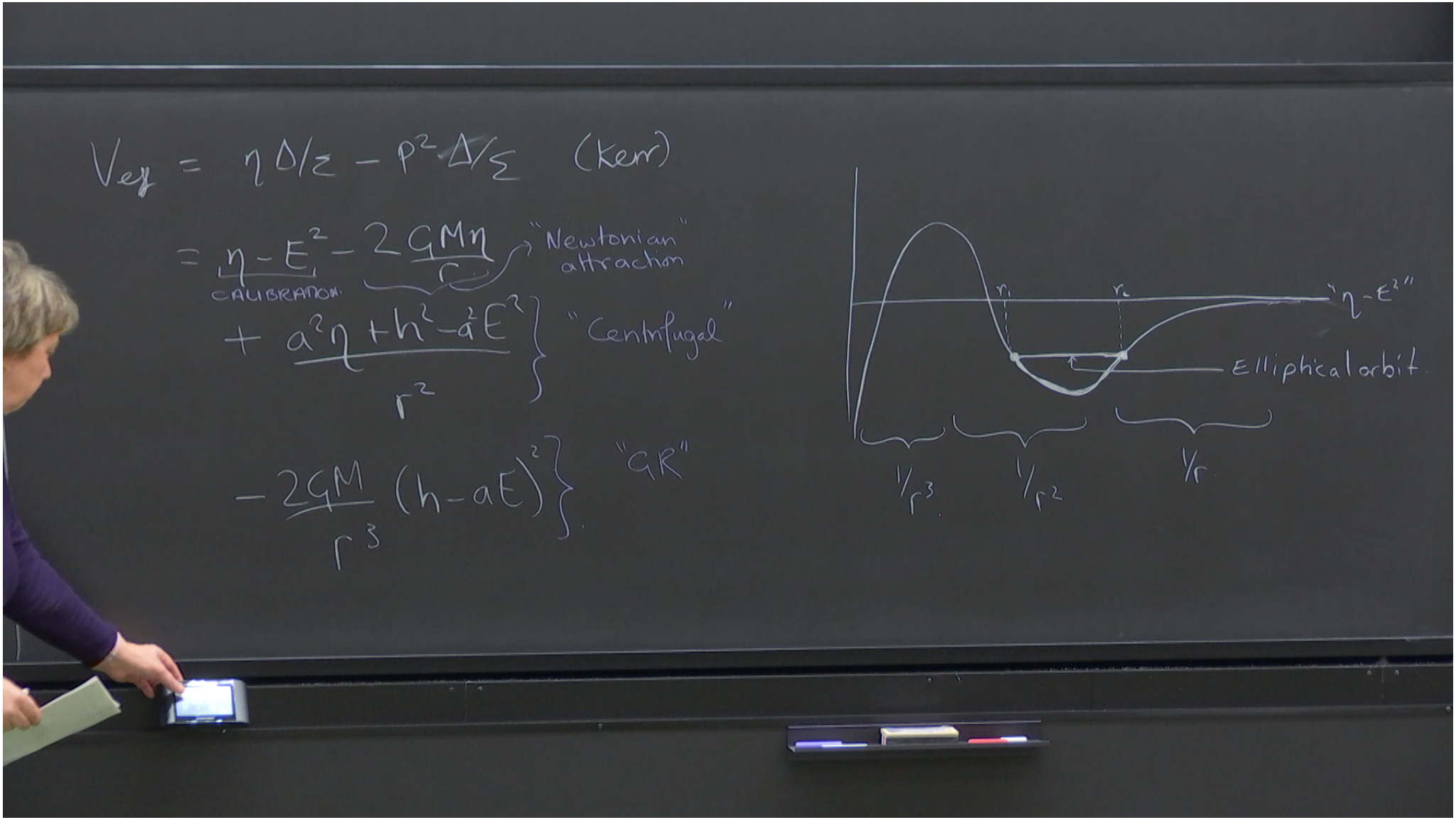
$$V_{eff} = \eta \Delta/\Sigma - p^2 \cdot \Delta/\Sigma \quad \text{(Kerr)}$$

$$= \underbrace{\eta - E^2}_{\text{CALIBRATION}} - \underbrace{\frac{2GM\eta}{r}}_{\text{"Newtonian attraction"}}$$

$$+ \left\{ \frac{a^2\eta + h^2 - a^2E^2}{r^2} \right\} \text{"Centrifugal"}$$

$$\left. - \frac{2GM}{r^3}(h - aE)^2 \right\} \text{"GR"}$$



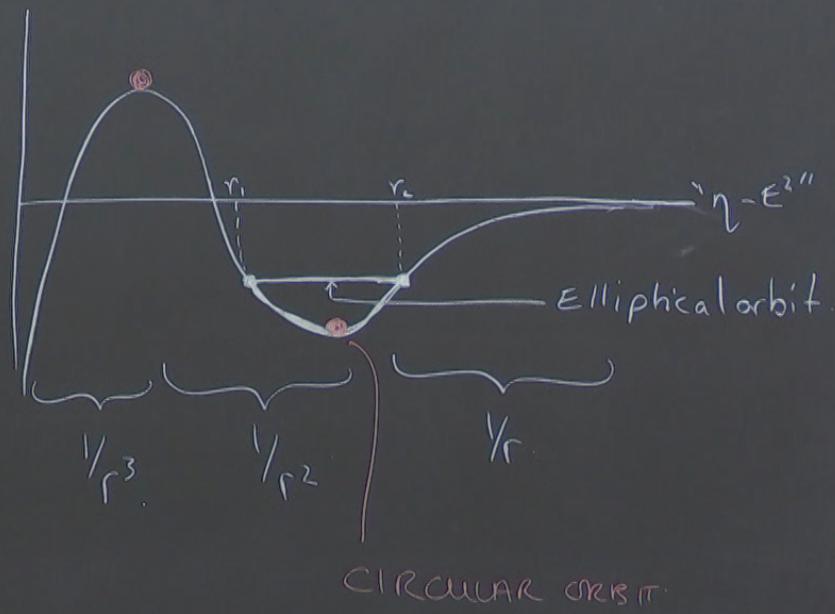

$V_{eff} = \eta \Delta/\Sigma - p^2 \Delta/\Sigma$ (Kerr)
$= \underbrace{\eta - E^2}_{\text{CALIBRATION}} - \underbrace{\frac{2GM\eta}{r}}_{\text{"Newtonian attraction"}}$
$+ \frac{a^2\eta + h^2 - a^2E^2}{r^2}$ } "Centrifugal"
$- \frac{2GM}{r^3}(h - aE)^2$ } "GR"
r_1
r_2
"$\eta - E^2$"
Elliptical orbit
$1/r^3$
$1/r^2$
$1/r$

$$V_{eff} = \eta \Delta/\Sigma - p^2 \Delta/\Sigma \quad \text{(Kerr)}$$

$$= \underbrace{\eta - E^2}_{\text{CALIBRATION}} - \underbrace{\frac{2GM\eta}{r}}_{\text{"Newtonian attraction"}}$$

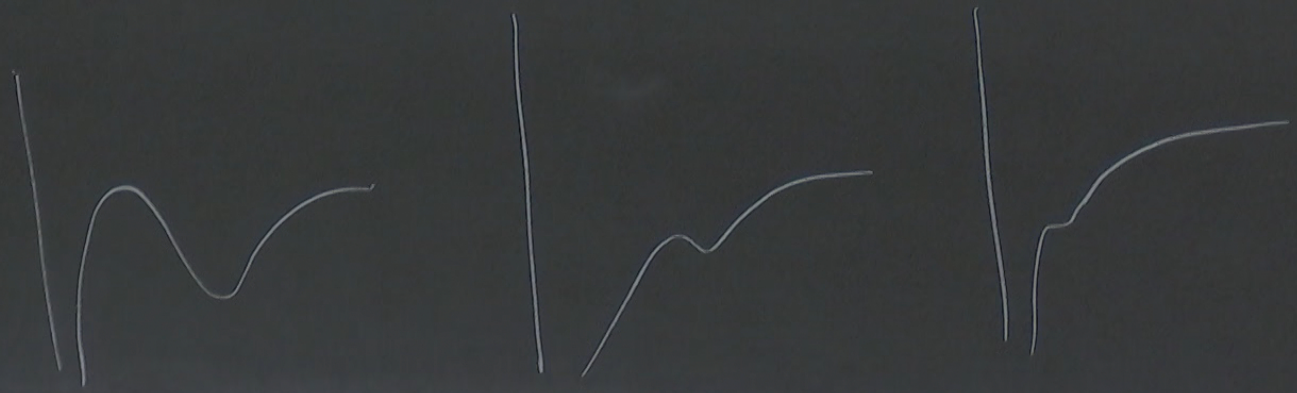
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Around a black hole, matter forms an accretion disc - modelled by concentric circular orbits. The innermost edge of the accretion disc is a marginally stable orbit

Around a black hole, matter forms an accretion disc - modelled by concentric circular orbits. The innermost edge of the accretion disc is a marginally stable orbit



This is the Innermost Stable Circular Orbit or ISCO

In general, a circular orbit is characterized by

$$V_{eff} = V'_{eff} = 0$$

$V''_{eff} > 0$ stable

$V''_{eff} < 0$ unstable

} $V''_{eff} = 0$ for ISCO.

Circular Orbit

bit is

$V''_{eff} = 0$
for ISCO.

e.g. SCH

$$V = 1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3} = 0$$

$$3r^4 V' + r^5 V'' = 2GMr^2 - 6GMh^2 = 0$$

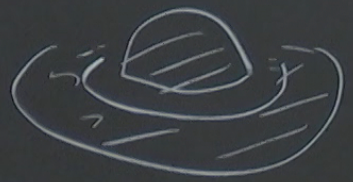
$$4r^3 V' + r^4 V'' = 4GMr - 2h^2 = 0$$

$$\Rightarrow r^2 = 3h^2 \;\&\; r = h^2/2GM$$

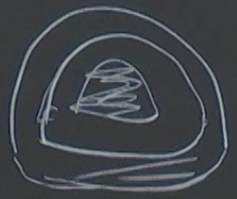
$$\Rightarrow h^2 = 12G^2M^2 \;\&\; r = 6GM$$

r_{isco} drops as a increases,

& $r_{isco} = a = r_+$ for extremal Kerr.



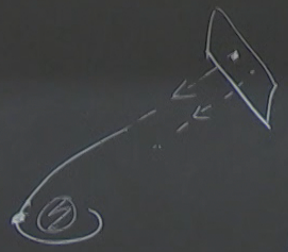
TRADITIONAL
ARTISTS IMPRESSION.



"INTERSTELLAR"
PICTURE

$V''_{eff} > 0$ stable
$V''_{eff} < 0$ unstable
$\left.\begin{matrix} V'_{eff} = 0 \\ \text{for ISCO} \end{matrix}\right\}$

$\Rightarrow h^2 = 12 G^2 M^2$ & $r = 6GM$



Beyond? What if there are > 4 dims?

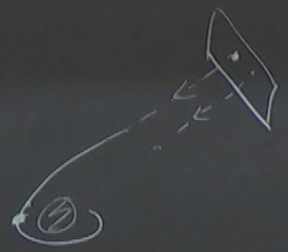
$D = p + n + 2$

$$ds^2 = A^2(r)\, g_{\mu\nu} dx^\mu dx^\nu - dr^2 - C^2(r)\, \gamma_{\alpha\beta} dy^\alpha dy^\beta$$

$V''_{eff} > 0$ stable $\left\{ V''_{eff} = 0 \text{ for ISCO} \right.$

$V''_{eff} < 0$ unstable

$\Rightarrow h^2 = 12 G^2 M^2$ & $r = 6GM$



Beyond? What if there are > 4 dims?

$D = p + n + 2$

$\alpha, \beta = 1 \ldots n$

$ds^2 = A^2(r)\, g_{\mu\nu} dx^\mu dx^\nu - dr^2 - C^2(r)\, \gamma_{\alpha\beta} dy^\alpha dy^\beta$

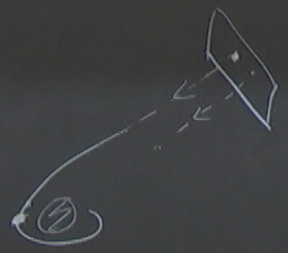
$\eta_{ab}\, \omega^a_{\cdot\mu}\, \omega^b_{\cdot\nu}\, dx^\mu dx^\nu$

$\delta_{ij}\, \omega^i_{\cdot\alpha}\, \omega^j_{\cdot\beta}\, dy^\alpha dy^\beta$

$V''_{eff} > 0$ stable $\left\{ \begin{array}{l} V''_{eff} = 0 \\ \text{for ISCO} \end{array} \right.$

$V''_{eff} < 0$ unstable

$\Rightarrow h^2 = 12 G^2 M^2$ & $r = 6GM$



Beyond? What if there are > 4 dims?

$D = p + n + 2$

$\alpha, \beta = 1 \ldots n$

$ds^2 = A^2(r) \underbrace{g_{\mu\nu} dx^\mu dx^\nu}_{p+1 \text{ dim}} - dr^2 - C^2(r) \underbrace{\gamma_{\alpha\beta} dy^\alpha dy^\beta}_{i,j}$

$\eta_{ab} \omega^a_{\cdot\mu} \omega^b_{\cdot\nu} dx^\mu dx^\nu$

$\delta_{ij} \omega^i_{\cdot\alpha} \omega^j_{\cdot\beta} dy^\alpha dy^\beta$

Use Cartan:

$$\underline{\omega}^{\hat{a}} = A\,\underline{\omega}_0^{\hat{a}} \; ; \quad \underline{\omega}^{\hat{r}} = dr \; ; \quad \underline{\omega}^{\hat{i}} = C\,\underline{\omega}_0^{\hat{i}}$$

$$\underline{d}\,\underline{\omega}^{\hat{a}} = A'\underline{dr} \wedge \underline{\omega}_0^{\hat{a}} + A\,\underline{d}\,\underline{\omega}_0^{\hat{a}}$$

$$= \frac{A'}{A}\,\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{a}} - A\,\underline{\Theta}_0^{\hat{a}}{}_{\hat{b}} \wedge \underline{\omega}_0^{\hat{b}}$$

$$= -\frac{A'}{A}\,\underline{\omega}^{\hat{a}} \wedge \underline{\omega}^{\hat{r}} - \underline{\Theta}_0^{\hat{a}}{}_{\hat{b}} \wedge \underline{\omega}^{\hat{b}}$$

$\underset{\sim}{\omega}^{\hat{r}} = dr \; ; \quad \underset{\sim}{\omega}^{\hat{i}} = C \, \underset{\sim}{\omega}_0^{\hat{i}}$

$+ A \, d\underset{\sim}{\omega}_0^{\hat{a}}$

$- A \, \underset{\sim}{\Theta}_0^{\hat{a}}{}_{\hat{b}} \wedge \underset{\sim}{\omega}_0^{\hat{b}}$

$\underset{\sim}{\omega}^{\hat{r}} - \underset{\sim}{\Theta}_0^{\hat{a}}{}_{\hat{b}} \wedge \underset{\sim}{\omega}^{\hat{b}}$

$\underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{b}} = \underset{\sim}{\Theta}_0^{\hat{a}}{}_{\hat{b}}$

$\underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{r}} = \frac{A'}{A} \underset{\sim}{\omega}^{\hat{a}}$

Similarly

$\underset{\sim}{\Theta}^{\hat{i}}{}_{\hat{j}} = \underset{\sim}{\Theta}_0^{\hat{i}}{}_{\hat{j}}$

$\underset{\sim}{\Theta}^{\hat{i}}{}_{\hat{r}} = \frac{C'}{C} \underset{\sim}{\omega}^{\hat{i}}$

$\underset{\sim}{R}^{\hat{a}}$

$$\underset{\sim}{R}^{\hat{a}}{}_{\hat{b}} = \underset{\sim}{d}\,\underset{\sim}{\Theta}_0{}^{\hat{a}}{}_{\hat{b}} + \underset{\sim}{\Theta}_0{}^{\hat{a}}{}_{\hat{c}} \wedge \underset{\sim}{\Theta}_0{}^{\hat{c}}{}_{\hat{b}} + \underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{r}} \wedge \underset{\sim}{\Theta}^{\hat{r}}{}_{\hat{b}}$$

$$= \underset{\sim}{R}_0{}^{\hat{a}}{}_{\hat{b}} + \left(\frac{A'}{A}\right)^2 \underset{\sim}{\omega}^{\hat{a}} \wedge \underset{\sim}{\omega}_{\hat{b}} \qquad \leftarrow \eta_{\hat{b}\hat{c}}\,\underset{\sim}{\omega}^{\hat{c}}$$

$$\underset{\sim}{R}^{\hat{a}}{}_{\hat{r}} = \underset{\sim}{d}\,\underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{r}} + \underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{b}} \wedge \underset{\sim}{\Theta}^{\hat{b}}{}_{\hat{r}}$$

$$= A'' \underset{\sim}{dr} \wedge \underset{\sim}{\omega}_0{}^{\hat{a}} + A' \underset{\sim}{d\omega}_0{}^{\hat{a}} + A' \underset{\sim}{\Theta}_0{}^{\hat{a}}{}_{\hat{b}} \wedge \underset{\sim}{\omega}_0{}^{\hat{b}}$$

$$= \frac{A''}{A} \underset{\sim}{\omega}^{\hat{r}} \wedge \underset{\sim}{\omega}^{\hat{a}}$$

$$\underset{\sim}{R}^{\hat{a}}{}_{\hat{i}} = \underset{\sim}{\Theta}^{\hat{a}}{}_{\hat{r}} \wedge \underset{\sim}{\Theta}^{\hat{r}}{}_{\hat{i}} = \frac{A'C'}{AC}\,\underset{\sim}{\omega}^{\hat{a}} \wedge \underset{\sim}{\omega}_{\hat{i}}$$

By analogy

$$\underline{R}^{\hat{\imath}}{}_{\hat{\jmath}} = \underline{R}_0{}^{\hat{\imath}}{}_{\hat{\jmath}} + \frac{C^R}{C^2}\,\underline{\omega}^{\hat{\imath}} \wedge \underline{\omega}_{\hat{\jmath}}$$

$$\underline{R}^{\hat{\imath}}{}_{\hat{r}} = \frac{C''}{C}\,\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\imath}}$$

Note $\underline{\omega}_{\hat{\imath}} = -\delta_{\hat{\imath}\hat{\jmath}}\,\underline{\omega}^{\hat{\jmath}}$

By analogy

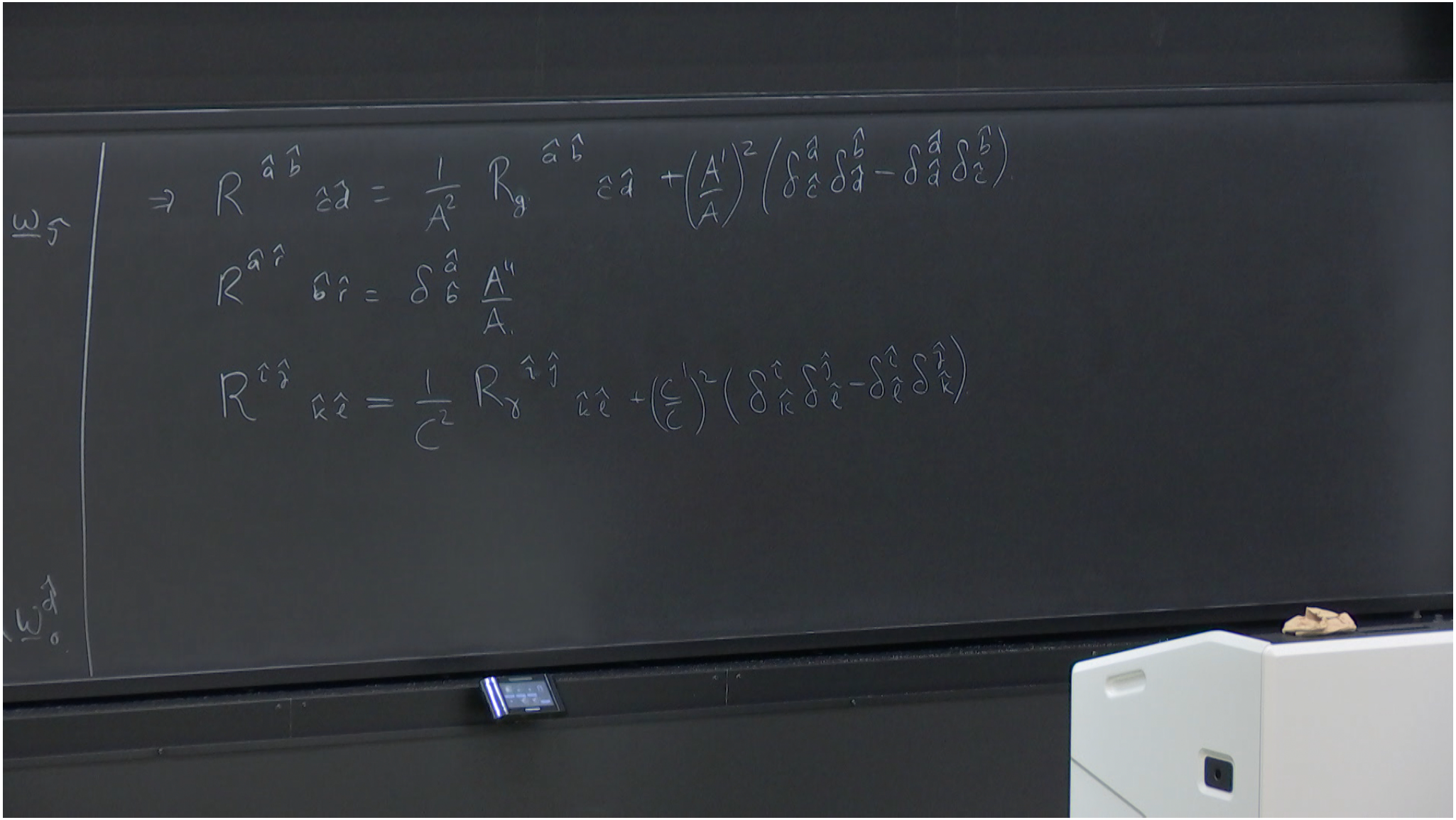
$$\underline{R}^{\hat{i}}{}_{\hat{j}} = \underline{R}_0{}^{\hat{i}}{}_{\hat{j}} + \frac{c'^2}{c^2}\,\underline{\omega}^{\hat{i}} \wedge \underline{\omega}_{\hat{j}}$$

$$\underline{R}^{\hat{i}}{}_{\hat{j}} = \frac{c''}{c}\,\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{i}}$$

Note $\underline{\omega}_{\hat{i}} = -\delta_{\hat{i}\hat{j}}\,\underline{\omega}^{\hat{j}}$

For Riemann - be careful!

$$\underline{R}_0{}^{\hat{a}}{}_{\hat{b}} = \frac{1}{2} R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\,\underline{\omega}^{\hat{c}}_0 \wedge \underline{\omega}^{\hat{d}}_0$$


$\Rightarrow R^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} = \frac{1}{A^2} R_g{}^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} + \left(\frac{A'}{A}\right)^2 \left(\delta^{\hat{a}}_{\hat{c}}\delta^{\hat{b}}_{\hat{d}} - \delta^{\hat{a}}_{\hat{d}}\delta^{\hat{b}}_{\hat{c}}\right)$
$R^{\hat{a}\hat{r}}{}_{\hat{b}\hat{r}} = \delta^{\hat{a}}_{\hat{b}} \frac{A''}{A}$
$R^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} = \frac{1}{C^2} R_\gamma{}^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} + \left(\frac{C'}{C}\right)^2 \left(\delta^{\hat{i}}_{\hat{k}}\delta^{\hat{j}}_{\hat{\ell}} - \delta^{\hat{i}}_{\hat{\ell}}\delta^{\hat{j}}_{\hat{k}}\right)$

$$\Rightarrow R^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} = \frac{1}{A^2} R_{\gamma}{}^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} + \left(\frac{A'}{A}\right)^2 \left(\delta^{\hat{a}}_{\hat{c}}\delta^{\hat{b}}_{\hat{d}} - \delta^{\hat{a}}_{\hat{d}}\delta^{\hat{b}}_{\hat{c}}\right)$$

$$R^{\hat{a}\hat{r}}{}_{\hat{b}\hat{r}} = \delta^{\hat{a}}_{\hat{b}} \frac{A''}{A} \quad \& \quad R^{\hat{i}\hat{r}}{}_{\hat{j}\hat{r}} = \frac{C''}{C}\delta^{\hat{i}}_{\hat{j}}$$

$$R^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} = \frac{1}{C^2} R_{\gamma}{}^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} + \left(\frac{C'}{C}\right)^2 \left(\delta^{\hat{i}}_{\hat{k}}\delta^{\hat{j}}_{\hat{\ell}} - \delta^{\hat{i}}_{\hat{\ell}}\delta^{\hat{j}}_{\hat{k}}\right)$$

$$R^{\hat{a}\hat{i}}{}_{\hat{b}\hat{j}} = \frac{A'C'}{AC}\delta^{\hat{a}}_{\hat{b}}\delta^{\hat{i}}_{\hat{j}}$$

$$\underline{\Theta}^{\hat{i}}{}_{\hat{r}} = \frac{C'}{C}\,\underline{\omega}^{\hat{i}}$$

$$\underline{R}^{\hat{a}}{}_{\hat{i}} = \underline{\Theta}^{\hat{a}}{}_{\hat{r}} \wedge \underline{\Theta}^{\hat{r}}{}_{\hat{i}} = \frac{A'C'}{AC}\,\underline{\omega}^{\hat{a}} \wedge \underline{\omega}_{\hat{i}}$$

$$\Rightarrow R^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} = \frac{1}{A^2} R_g{}^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} + \left(\frac{A'}{A}\right)^2 \left(\delta^{\hat{a}}_{\hat{c}}\delta^{\hat{b}}_{\hat{d}} - \delta^{\hat{a}}_{\hat{d}}\delta^{\hat{b}}_{\hat{c}}\right)$$

$$R^{\hat{a}\hat{r}}{}_{\hat{b}\hat{r}} = \delta^{\hat{a}}_{\hat{b}}\frac{A''}{A} \quad \& \quad R^{\hat{r}\hat{r}}{}_{\hat{j}\hat{r}} = \frac{C''}{C}\delta^{\hat{r}}_{\hat{j}}$$

$$R^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} = \frac{1}{C^2} R_\gamma{}^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} + \left(\frac{C'}{C}\right)^2 \left(\delta^{\hat{i}}_{\hat{k}}\delta^{\hat{j}}_{\hat{\ell}} - \delta^{\hat{i}}_{\hat{\ell}}\delta^{\hat{j}}_{\hat{k}}\right)$$

$$R^{\hat{a}\hat{i}}{}_{\hat{b}\hat{j}} = \frac{A'C'}{AC}\delta^{\hat{a}}_{\hat{b}}\delta^{\hat{i}}_{\hat{j}}$$

Easy to convert to coord basis

$$R^{M}{}_{\nu} = \frac{1}{A^2} R_g{}^{M}{}_{\nu} + p\frac{A'^2}{A^2}\delta^{M}_{\nu} + \frac{A''}{A}\delta^{M}_{\nu} + n\frac{A'C'}{AC}\delta^{M}_{\nu}$$

$$R^{\alpha}{}_{\beta} = \frac{1}{C^2} R_\gamma{}^{\alpha}{}_{\beta} + (n-1)\frac{C'^2}{C^2}\delta^{\alpha}_{\beta} + \frac{C''}{C}\delta^{\alpha}_{\beta} + (p+1)\frac{A'C'}{AC}\delta^{\alpha}_{\beta}$$

$$R^{r}{}_{r} = (p+1)\frac{A''}{A} + n\frac{C''}{C}$$

Usually g, γ are constant curvature

$$R^{\alpha\beta}{}_{\gamma\delta} = -\kappa_\gamma \left(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma \right)$$

$$R^\alpha{}_\beta = -\kappa_\gamma (n-1) \delta^\alpha_\beta$$

$$R^M_{\ N} = \delta^M_N \left[\frac{A''}{A} + p\frac{A'^2}{A^2} + n\frac{A'C'}{AC} - p\frac{\kappa_\rho}{A^2} \right]$$

$$R^\alpha_{\ \beta} = \delta^\alpha_\beta \left[\frac{C''}{C} + (n-1)\frac{C'^2}{C^2} + (p+1)\frac{A'C'}{AC} - (n-1)\frac{\kappa_\sigma}{C^2} \right]$$

Black hole in D dims

$p=0$, $n=D-2$, $\kappa_\gamma = 1$.

$$\frac{A''}{A} + p\frac{A'^2}{A^2} + n\frac{A'C'}{AC} - p\frac{\kappa_p}{A^2}\Big]$$

$$\frac{C''}{C} + (n-1)\frac{C'^2}{C^2} + (p+1)\frac{A'C'}{AC} - (n-1)\frac{\kappa_\gamma}{C^2}\Big]$$

e in D dims

$= 0-2$, $\kappa_\gamma = 1$.

Also, to take area gauge

$C \to \rho$ $\quad dr = B(\rho)\,d\rho$

$$R^t_t = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + (D-2)\frac{A'}{A\rho}\right]$$

$$R^\rho_\rho = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} - (D-2)\frac{B'}{B\rho}\right]$$

$$R^\alpha_\beta = \delta^\alpha_\beta\left[\frac{1}{B^2}\left(\frac{(D-3)}{\rho^2} - \frac{B'}{B\rho} + \frac{A'}{A\rho}\right) - \frac{(D-3)}{\rho^2}\right]$$

Notice $R^t_t - R^\rho_\rho = \frac{(D-2)}{B^2\rho}\left(\frac{A'}{A} + \frac{B'}{B}\right)$

$$G^t_t = (D-2)\frac{B'}{B^3\rho} - \frac{(D-2)(D-3)}{2B^2\rho^2} + \frac{(D-2)(D-3)}{2\rho^2}$$

$$= \frac{(D-2)}{2}\left[-\frac{(B^{-2}\rho^{D-3})'}{\rho^{D-2}} + \frac{(D-3)}{\rho^2}\right]$$

$$\frac{-2)}{s^2\rho}\left(\frac{A'}{A}+\frac{B'}{B}\right)$$

$$\frac{)(D-3)}{B^2\rho^2}+\frac{(D-2)(D-3)}{2\rho^2}$$

$$\left.\right)'+\frac{(D-3)}{\rho^2}\Big]$$

$$\left(B^{-2}\rho^{D-3}\right)'=(D-3)\rho^{D-4}-\frac{16\pi G}{D-2}\rho^{D-2}T^0_0$$

$$B^{-2}=1-\frac{16\pi G}{D-2}\underbrace{\int\rho^{D-2}T^0_0}_{\frac{1}{A_{D-2}}M(\rho)}$$

$$B^{-2}=A^{-2}=1-\left(\frac{\rho_+}{\rho}\right)^{D-3}$$

$$\left(\frac{1}{r} + \frac{B'}{B}\right)$$

$$+ \frac{(D-2)(D-3)}{2\rho^2}$$

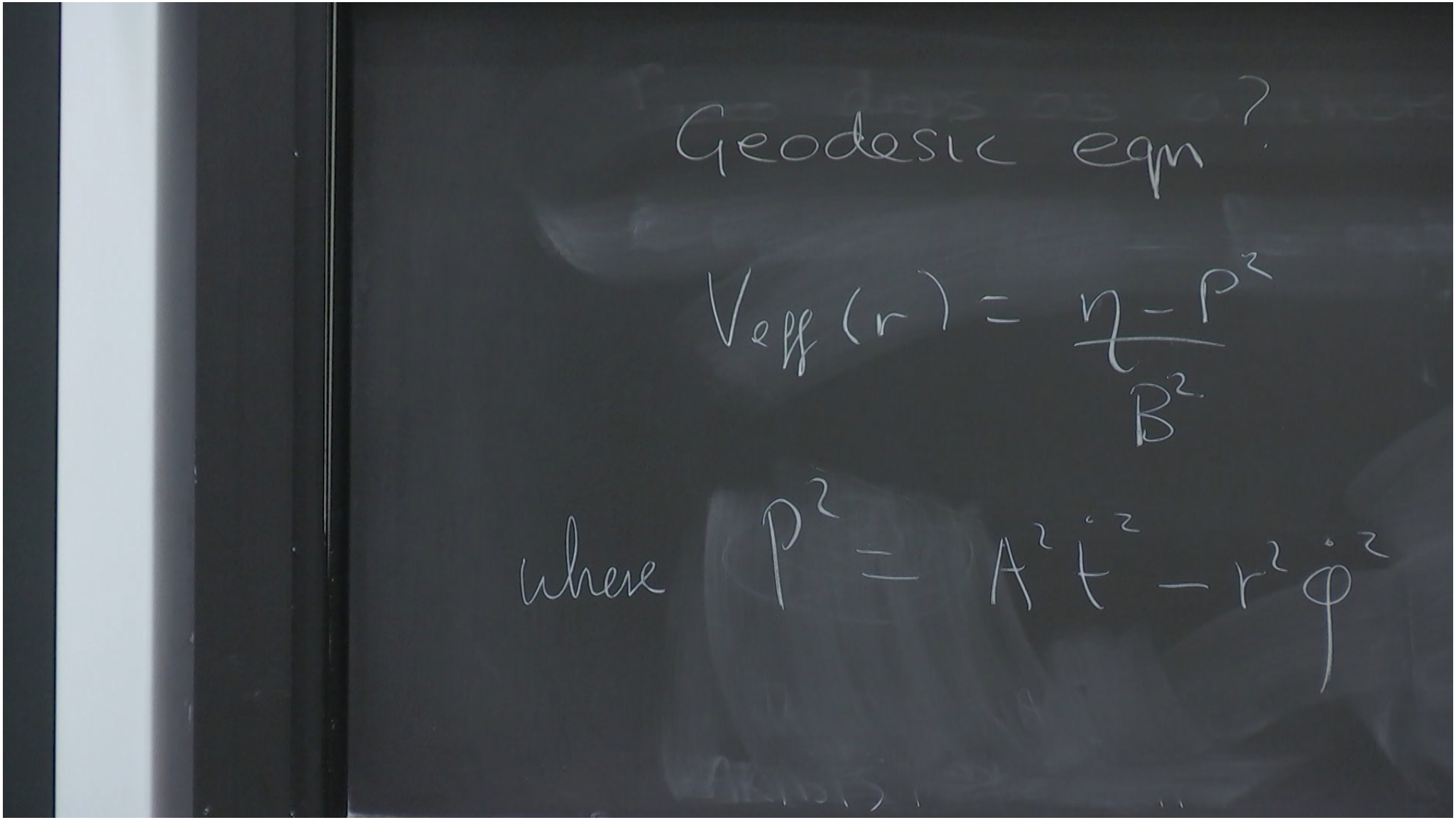
$$\frac{D-3}{\rho^2}\Big].$$

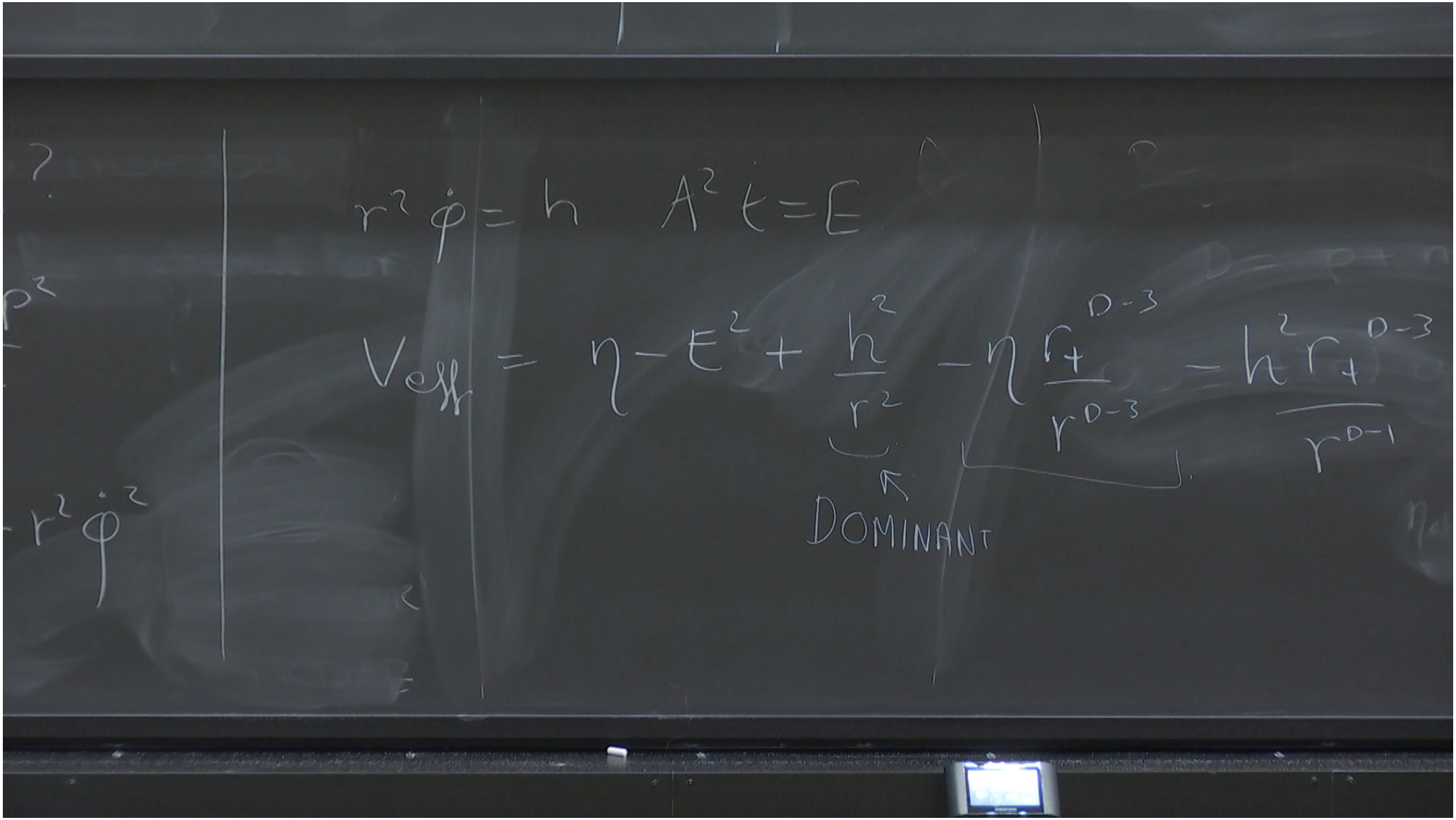
$$\left(B^{-2}\rho^{D-3}\right)' = (D-3)\rho^{D-4} - \frac{16\pi G}{D-2}\rho^{D-2}T^0_0.$$

$$B^{-2} = 1 - \frac{16\pi G}{D-2}\underbrace{\int \rho^{D-2} T^0_0}_{\frac{1}{A_{D-2}} M(\rho)}$$

$$B^{-2} = A^2 = 1 - \left(\frac{\rho_+}{\rho}\right)^{D-3}$$

$$M = \frac{(D-2)A_{D-2}\rho_+^{D-3}}{16\pi G}$$


Geodesic eqn?
$V_{eff}(r) = \frac{\eta - P^2}{B^2}$
where $P^2 = A^2\dot{t}^2 - r^2\dot{\varphi}^2$


$r^2\dot{\varphi} = h \quad A^2\dot{t} = E$
$V_{eff} = \eta - E^2 + \frac{h^2}{r^2} - \eta\frac{r_+^{D-3}}{r^{D-3}} - h^2\frac{r_+^{D-3}}{r^{D-1}}$
DOMINANT