Title: PSI 2019/2020 - Gravitational Physics - Lecture 6

Speakers: Ruth Gregory

Collection: PSI 2019/2020 - Gravitational Physics

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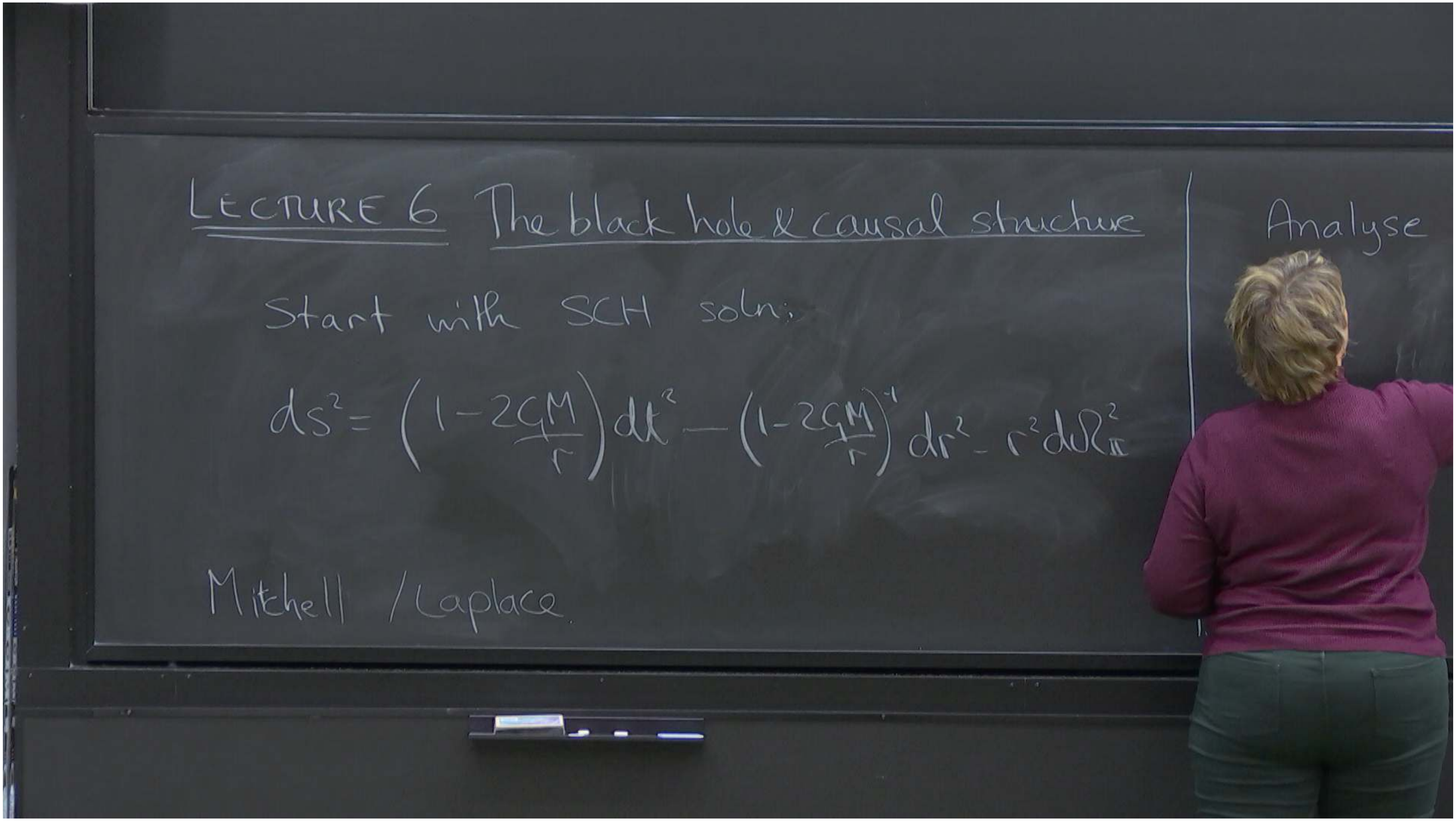
LECTURE 6 The black hole & causal structure

Start with SCH soln:

$$ds^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1}dr^2 - r^2\, d\Omega_{II}^2$$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Mitchell / Laplace
Analyse

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$$(dt + dr^*)(dt - dr^*)$$

Mitchell / Laplace.

Analyse

TORTOIS

Analyse $r = 2GM$, think about light rays

TORTOISE : $$r^* = \int \frac{dr}{1 - \frac{2GM}{r}} = r + 2GM \ln\left(\frac{r - 2GM}{2GM}\right)$$

or $$r + r_+ \ln\left(\frac{r - r_+}{r_+}\right)$$

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Next, take null coords.

KRUSKAL : $U = -r_+ \,\mathrm{Exp}\left[\frac{-(t-r^*)}{2r_+}\right]$

$V = r_+ \,\mathrm{Exp}\left[\frac{(t+r^*)}{2r_+}\right]$

$(dt+dr^*)(dt-dr^*)$

$$\Rightarrow dU\,dV = -\left(dt^2 - dr^{*2}\right)\frac{UV}{4r_+^2}$$

$$= -\left(dt^2 - dr^{*2}\right)\frac{e^{r^*/r_+}}{4}$$

$$(dt + dr^*)(dt - dr^*)$$

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$$= -(dt^2 - dr^{*2})\,\frac{e^{r^*/r_+}}{4}$$

$$= (dt^2 - dr^{*2})\,\frac{e^{r/r_+}}{4}\left(\frac{r - r_+}{r_+}\right)$$

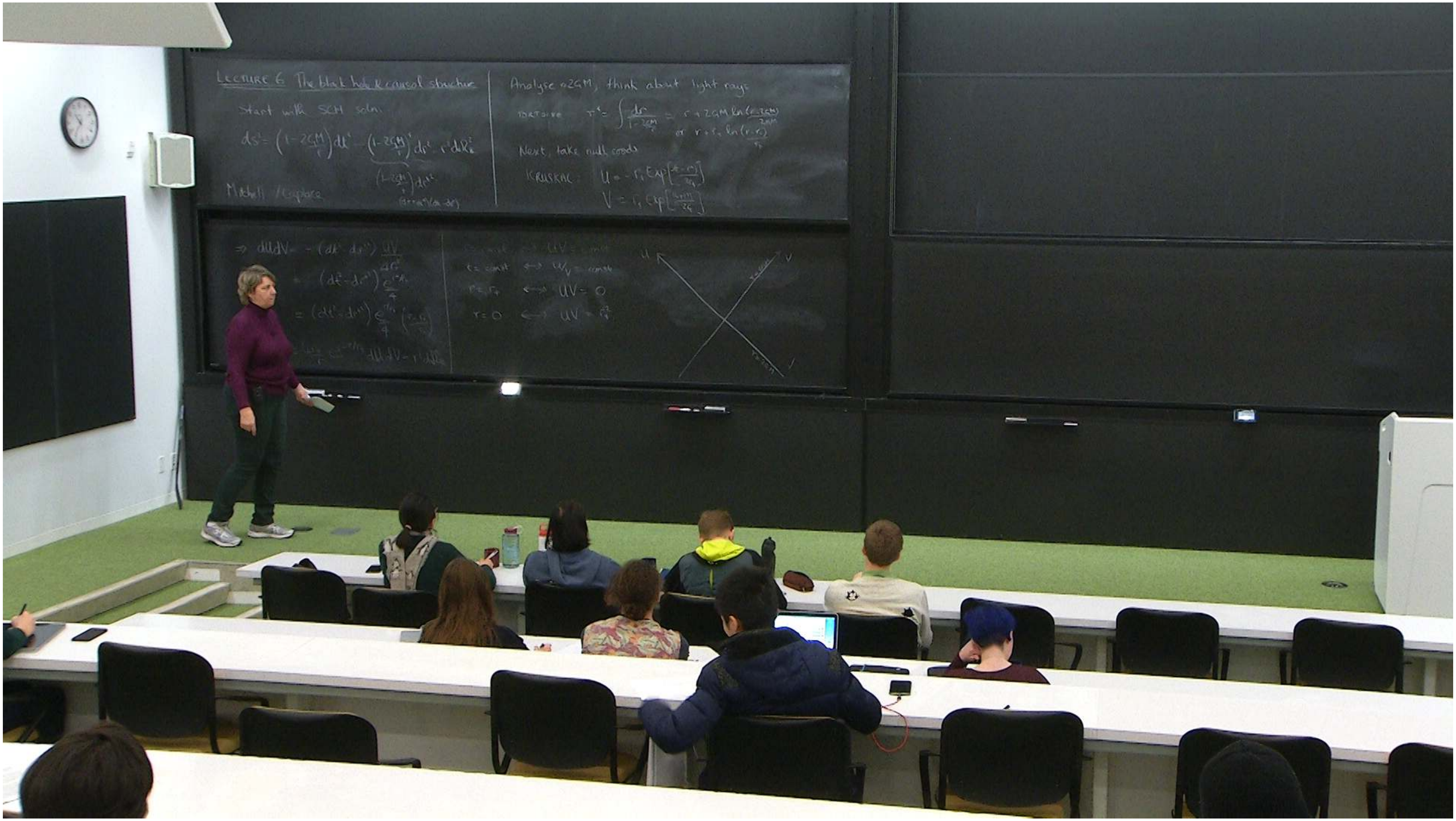
$$\Rightarrow ds^2 = \frac{4r_+}{r}\,e^{-r/r_+}\,dU\,dV - r^2\,d\Omega_{II}^2$$

$$r = \text{const} \longleftrightarrow UV = \text{const}.$$

$$t = \text{const} \longleftrightarrow U/V = \text{const}$$

$$r = r_+ \longleftrightarrow UV = 0.$$

$$r = 0 \longleftrightarrow UV = r_+^2.$$



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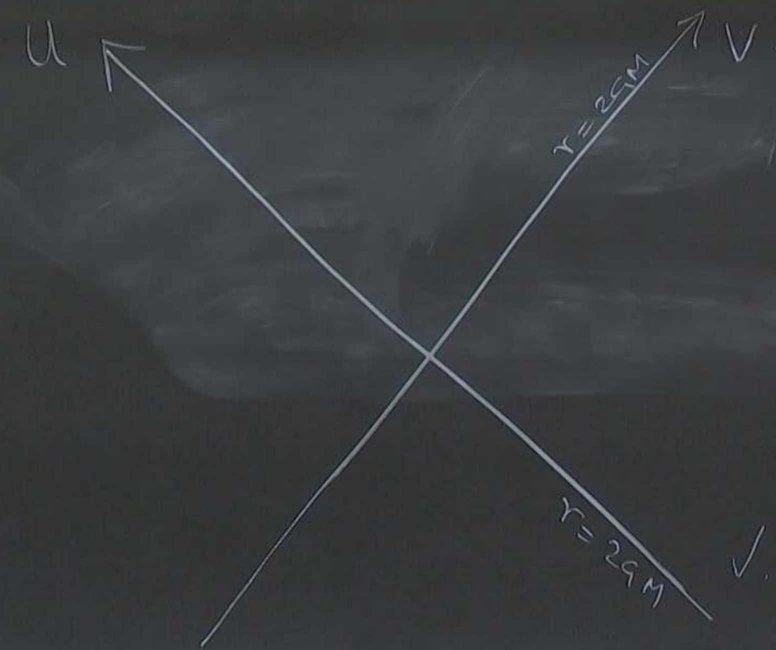
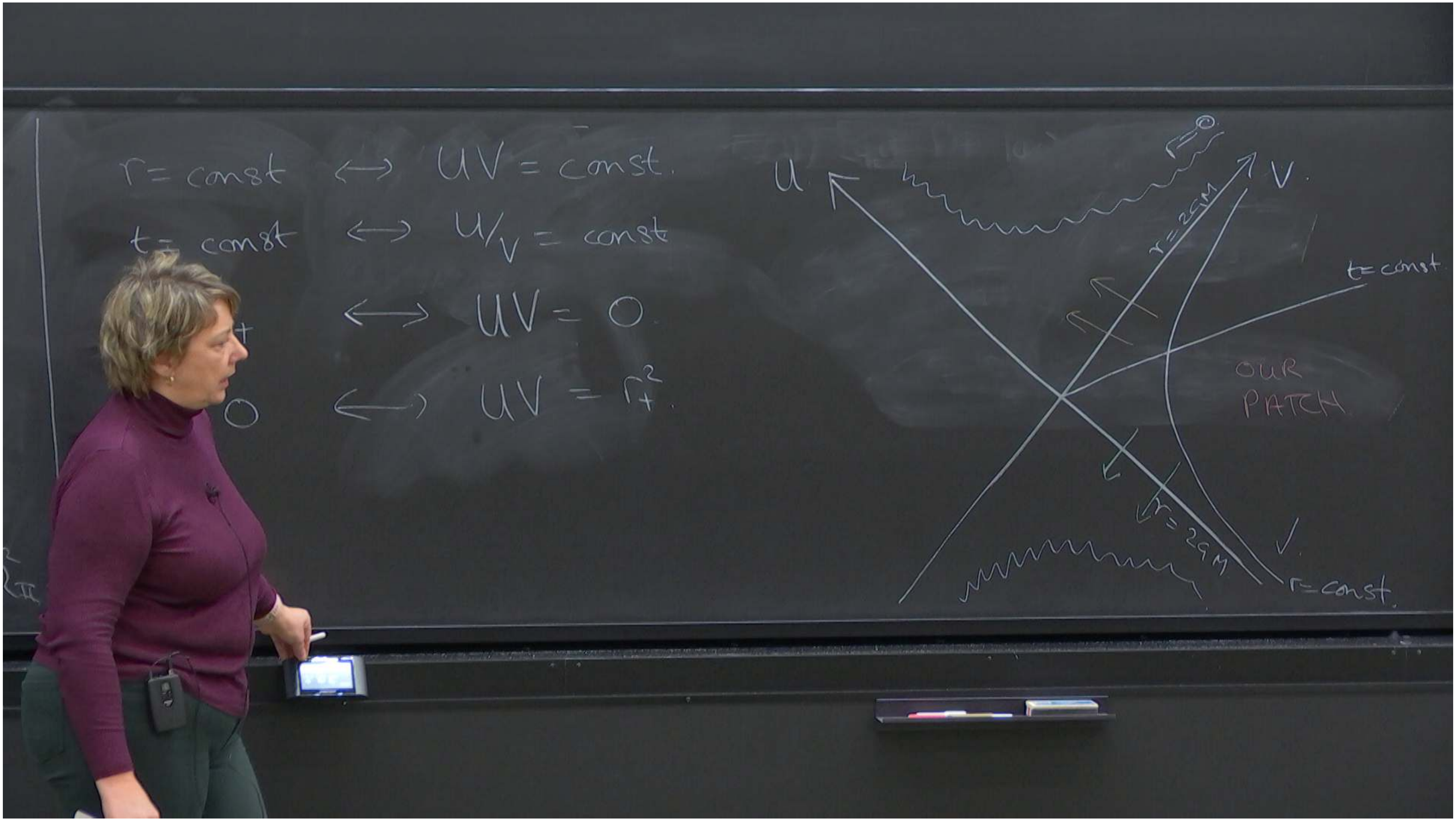
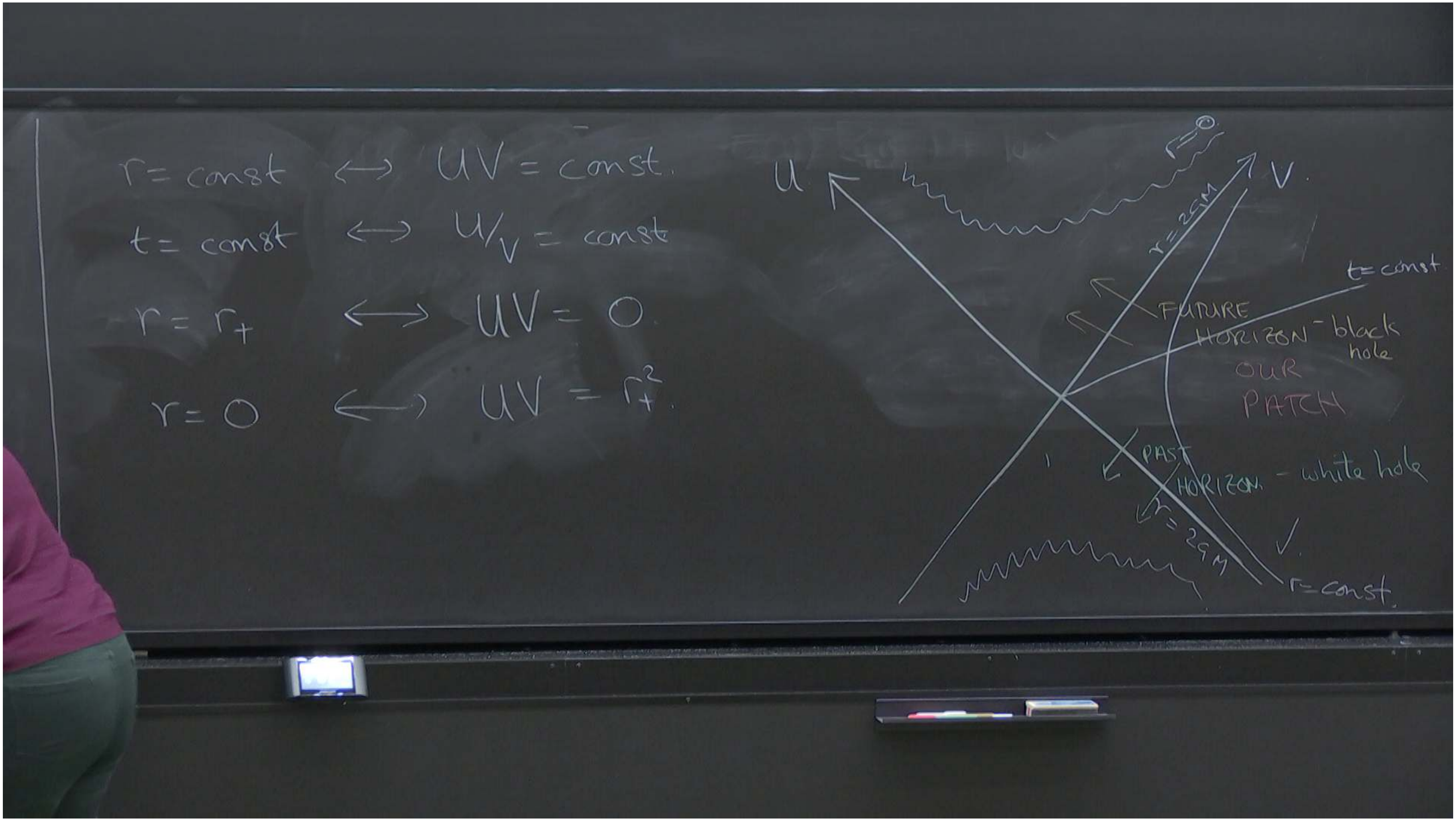



r = const ⟷ UV = const.
t = const ⟷ U/V = const
⟷ UV = 0
⟷ UV = r₊²
U
V
r = 2GM
t = const
OUR PATCH
r = const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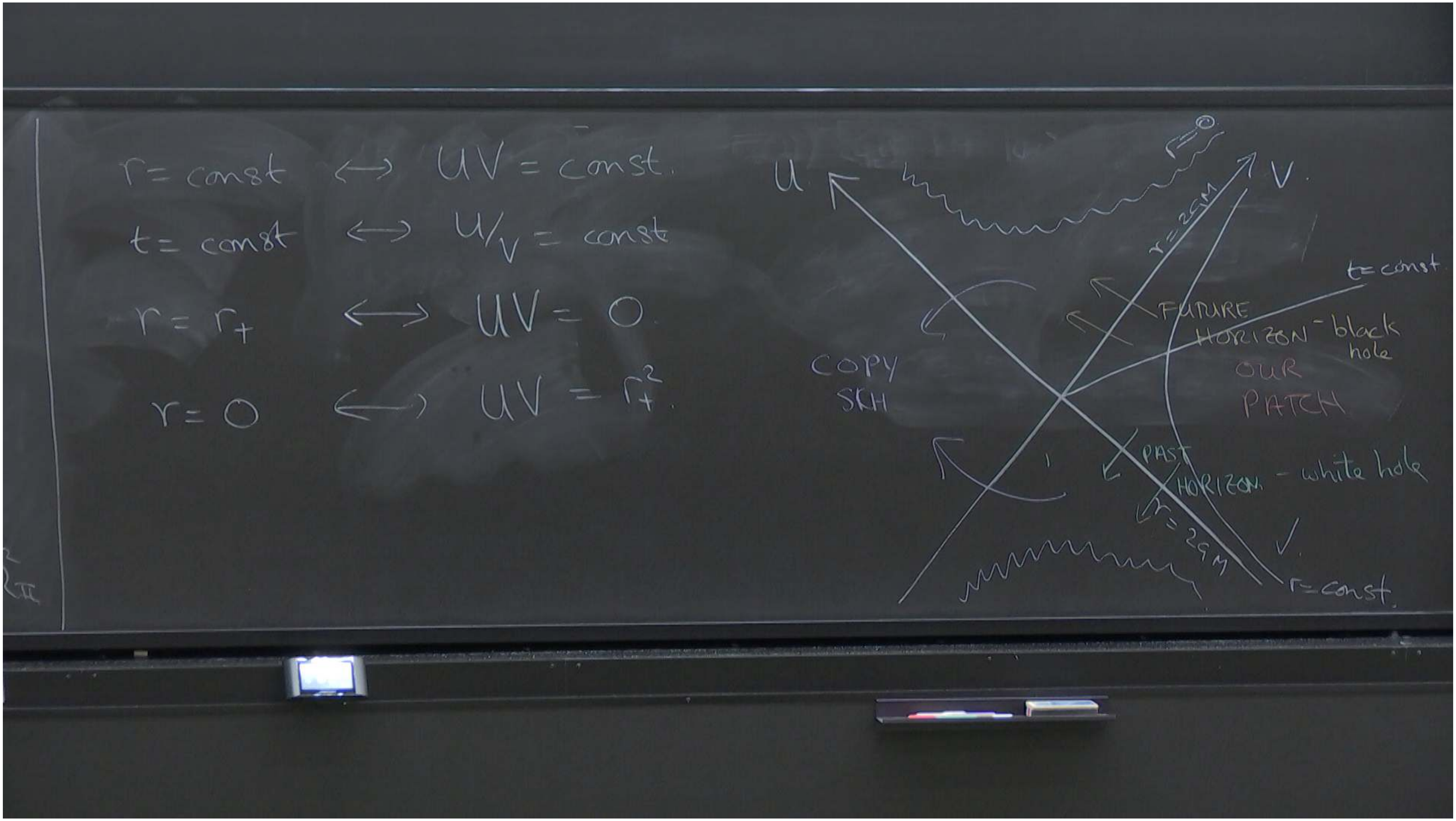
FUTURE HORIZON - black hole
OUR PATCH
PAST HORIZON - white hole
r = 2GM
r = const



r = const ⟷ UV = const.
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U
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FUTURE HORIZON - black hole
OUR PATCH
COPY
PAST HORIZON - white hole
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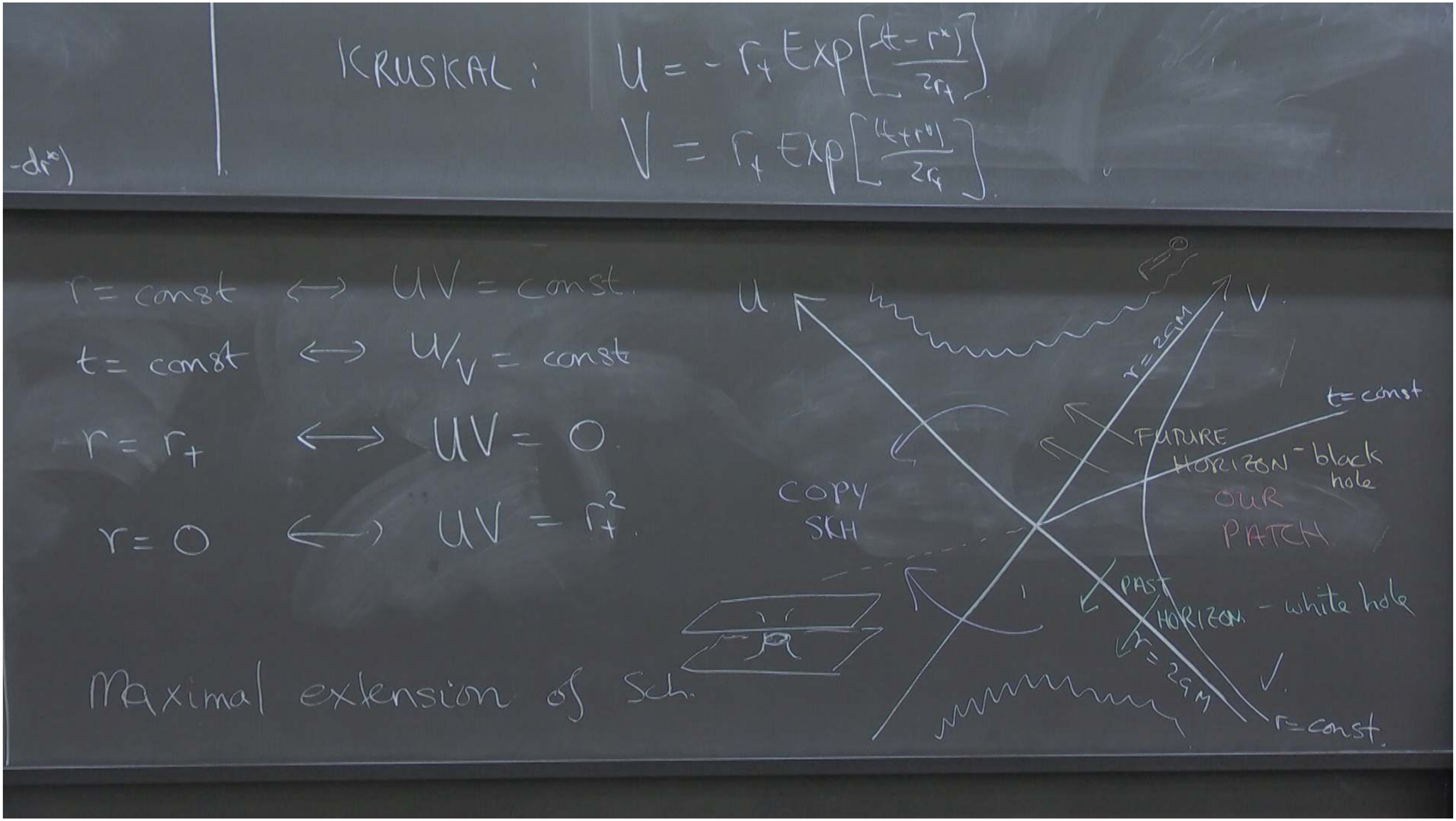
-dr*)
KRUSKAL: $U = -r_+ \operatorname{Exp}\left[\frac{-(t-r^*)}{2r_+}\right]$
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Maximal extension of Sch.
U
V
r = 2GM
t = const
FUTURE HORIZON - black hole
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COPY SCH
PAST HORIZON - white hole
r = 2GM
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$$\Rightarrow ds^2 = \frac{4r_+}{r} e^{-r/r_+} dU\,dV - r^2 d\Omega_{II}^2$$

Maximal exten

Convenient to have a compact picture.

define $p = \arctan V/r_+$

$q = \arctan U/r_+$

$V = 0 \Leftrightarrow p = 0$

$V = r_+ \Leftrightarrow p = \pi/4$

$V \to \infty \Leftrightarrow p = \pi/2$.

$$\Rightarrow ds^2 = \frac{4r_+}{r} e^{-r/r_+} dU\,dV - r^2 d\Omega_{II}^2$$

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$$\tan(p+q) = \frac{\tan p + \tan q}{1 - \tan p \tan q} = \frac{U+V}{r_+^2 - UV}.$$

so $UV = r_+^2 \Leftrightarrow p + q = \pi/4$

Maximal extension of Sch.

r = const.

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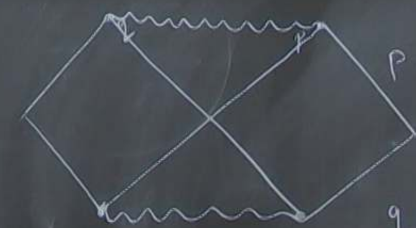
$(r=0)$



PENROSE-CARTER DIAGRAM

Maximal extension of Sch.

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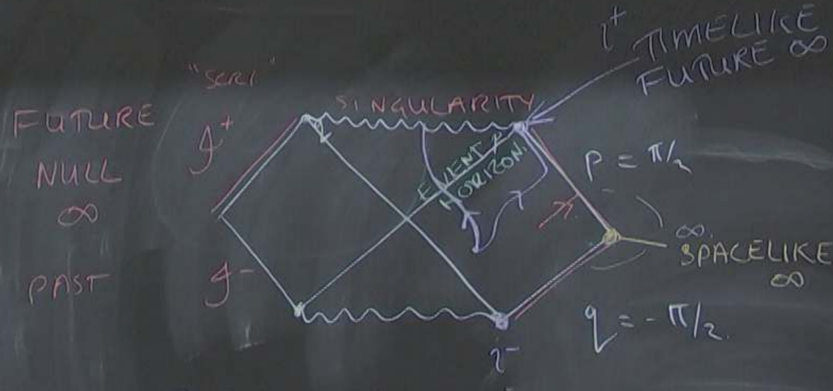
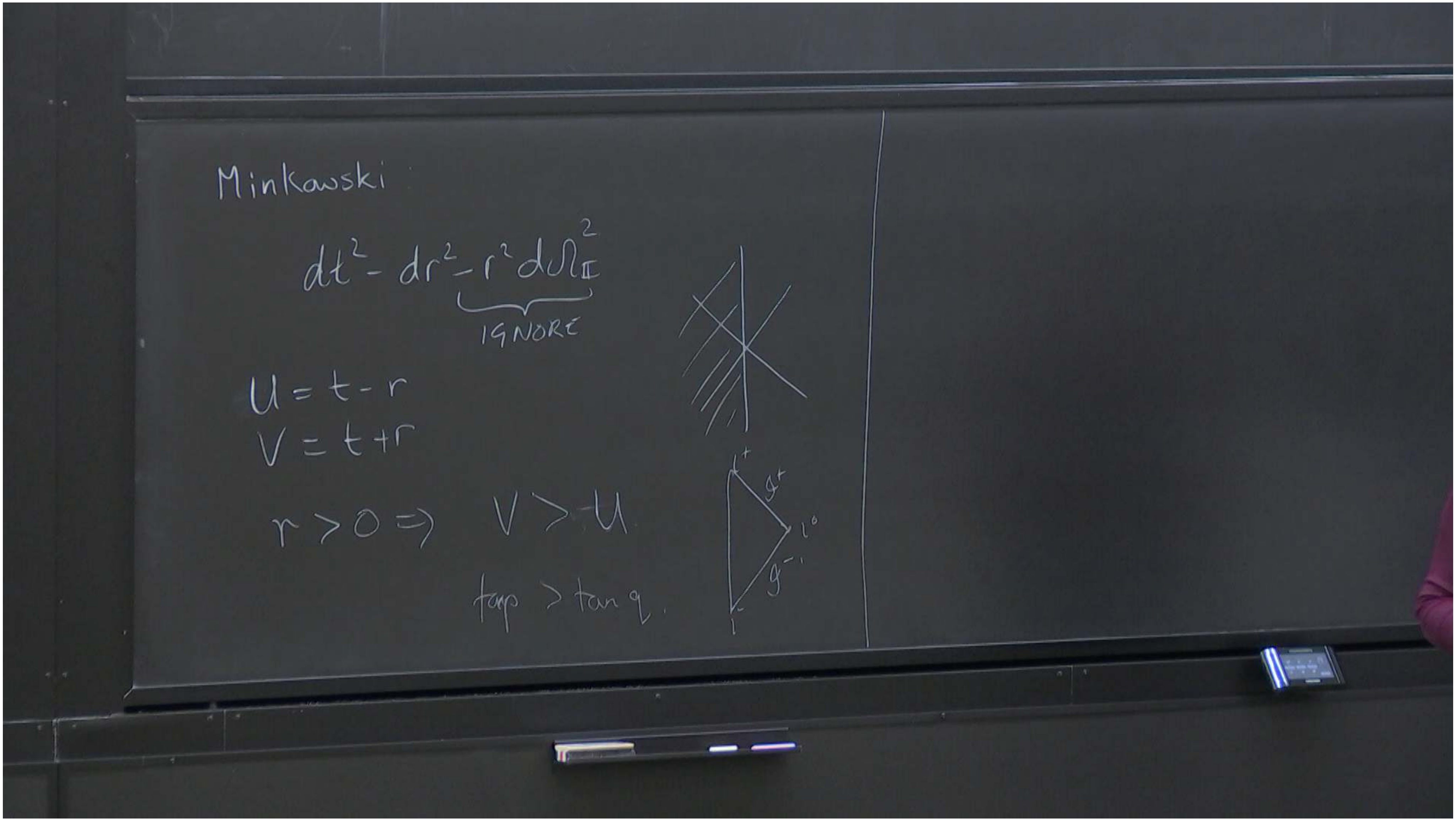


PENROSE-CARTER DIAGRAM

Encodes causal structure.


Minkowski
$dt^2 - dr^2 - \underbrace{r^2 d\Omega_{II}^2}_{\text{IGNORE}}$
$U = t - r$
$V = t + r$
$r > 0 \Rightarrow V > -U$
$\tan p > \tan q$.
i^+
$\mathscr{I}^+$
i^0
$\mathscr{I}^-$
i^-

Kerr Black Hole

$\Sigma = r^2 + a^2\cos^2\theta$

$\Delta = r^2 - 2GMr + a^2$

$a = J/M$

Astrophysical black holes rotate

$$ds^2 = \left(1 - \frac{2GMr}{\Sigma}\right)dt^2 + 4\frac{GMar\sin^2\theta}{\Sigma}\,dt\,d\varphi - \left[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\right]\frac{d\varphi^2}{\Sigma} - \frac{\Sigma}{\Delta}dr^2 - \Sigma\,d\theta^2$$

$$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\,d\varphi\right]^2 - \frac{\sin^2\theta}{\Sigma}\left[a\,dt - (r^2+a^2)\,d\varphi\right]^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma\,d\theta^2$$

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$$= \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\,d\varphi\right]^2 - \frac{\sin^2\theta}{\Sigma}\left[a\,dt - (r^2+a^2)\,d\varphi\right]^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma\,d\theta^2$$

No longer $SO(3)$ symmetric, but still have $\frac{\partial}{\partial t}$ & $\frac{\partial}{\partial\varphi}$ killing vectors.

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No longer $SO(3)$ symmetric, but still have $\frac{\partial}{\partial t}$ & $\frac{\partial}{\partial \varphi}$ killing vectors.

- $g_{tt} \to 0$ at $2GMr = \Sigma$
- $|g_{rr}| \to \infty$ at $\Delta = 0$

$$r = r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$$

- $g_{tt} \to 0$ at $2GMr = \Sigma$
- $|g_{rr}| \to \infty$ at $\Delta = 0$

$$r = r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2}$$

$g_{tt} \to 0$ at $R_{\pm} = GM \pm \sqrt{G^2M^2 - a^2\cos^2\theta}$

$$\boxed{R_+ \geq r_+}$$

$$R_- \leq r_-$$

$\overline{M^2 - a^2}$

$\cos^2\theta$

Event horizon: limit of "events we can see" i.e. limit of light geodesics escaping to ∞.

Explore geodesics in Kerr,

take $\theta = \pi/2$

For k^μ a Killing vector.

$$\frac{d}{d\tau}\left(K_\mu \dot{X}^\mu\right) = \dot{X}^\nu \nabla_\nu \left(k_\mu \dot{X}^\mu\right)$$

nit of "events
limit of light
∞.
Kerr,

$$\frac{d}{d\tau}\left(k_\mu \dot{X}^\mu\right) = \underbrace{\cancel{\dot{X}^\mu \dot{X}^\nu \nabla_{(\mu} k_{\nu)}}}_{\text{Killing vector}} + k_\mu \underbrace{\cancel{\left(\dot{X}^\nu \nabla_\nu \dot{X}^\mu\right)}}_{\text{Geodesic eqn.}} \quad (A|N)$$

$\rightarrow$ $k_\mu \dot{X}^\mu$ is a conserved quantity.

$$k = \frac{\partial}{\partial t} \Rightarrow E = g_{t\mu}\dot{X}^\mu$$

$$= \left(1 - \frac{2GMr}{\Sigma}\right)\dot{t} + \frac{2GMar}{\Sigma}\sin^2\theta\,\dot{\phi}$$

symmetric, but still have $\frac{\partial}{\partial t}$ & $\frac{\partial}{\partial \varphi}$ Killing vectors.

mit of "events

limit of light

to ∞.

s in Kerr,

ng vector.

$\dot X^\nu \nabla_\nu (k_\mu \dot X^\mu)$

$$\frac{d}{d\tau}\left(k_\mu \dot X^\mu\right) = \underbrace{\cancel{\dot X^\mu \dot X^\nu \nabla_\mu k_\nu}}_{\text{Killing vector}} + \underbrace{k_\mu\left(\cancel{\dot X^\nu \nabla_\nu \dot X^\mu}\right)}_{\text{Geodesic eqn.}} \quad (\text{A}|\text{N})$$

$\rightarrow$ $k_\mu \dot X^\mu$ is a conserved quantity.

- $k = \frac{\partial}{\partial t} \Rightarrow E = g_{t\mu}\dot X^\mu$

$$= \left(1-\frac{2GMr}{\Sigma}\right)\dot t + \frac{2GMar}{\Sigma}\sin^2\theta\,\dot\phi$$

$$\xrightarrow[\theta=\pi/2]{} \left(1-\frac{2GM}{r}\right)\dot t + \frac{2GMa}{r}\dot\varphi$$

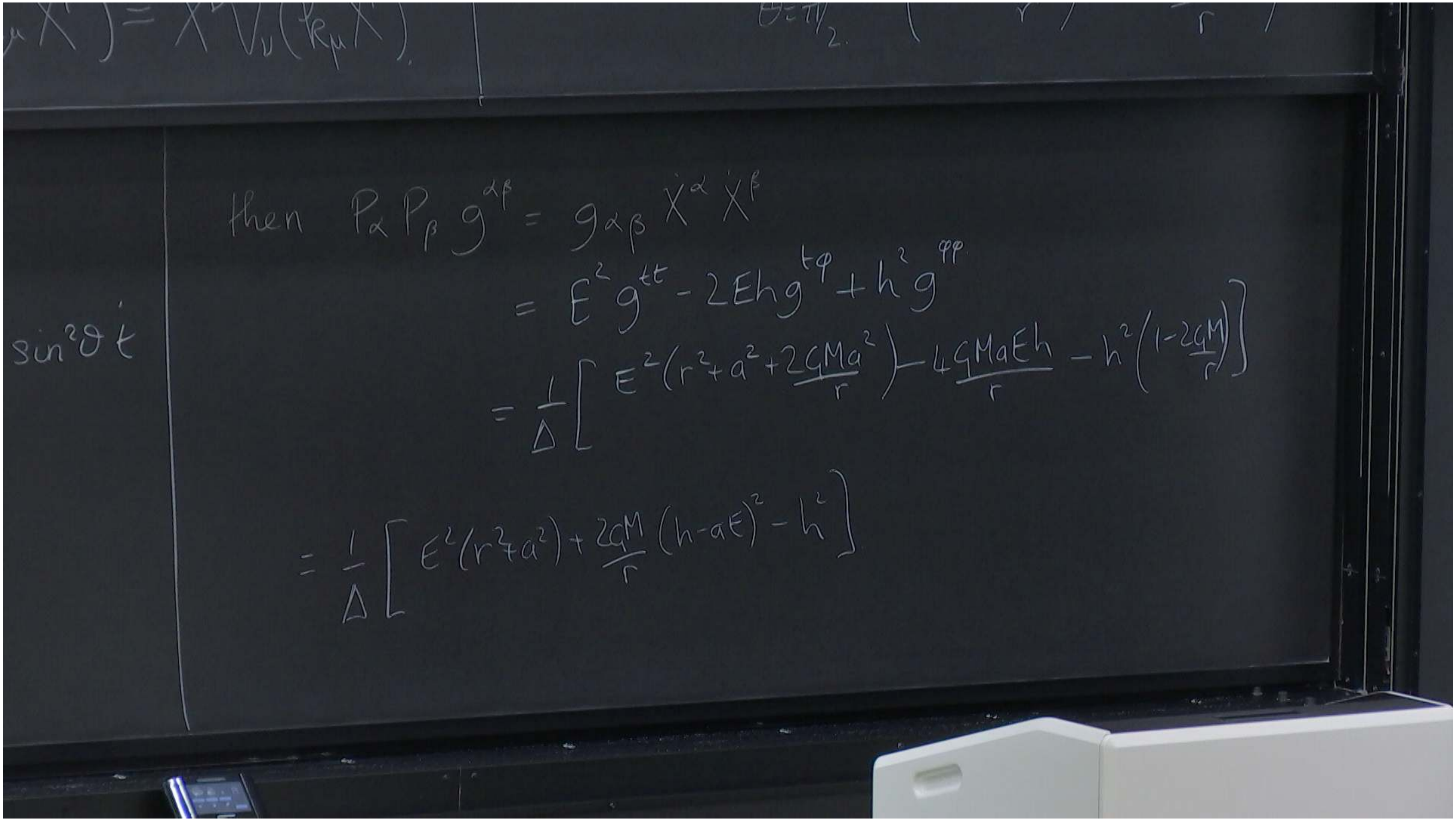
- $k = \frac{\partial}{\partial \varphi} \to h = -g_{\phi\mu}\dot{X}^{\mu}$

$$= \left(r^2 + a^2 + \frac{2GMr}{\Sigma} a^2 \sin^2\theta\right)\dot{\phi} - \frac{2GMar}{\Sigma}\sin^2\theta\, \dot{t}$$

$$\to \left(r^2 + a^2 + \frac{2GMa^2}{r}\right)\dot{\phi} - \frac{2GMa}{r}\dot{t}$$

Collect: $P_\alpha = (E, -h) = g_{\alpha\beta}\dot{X}^{\beta}$

$\alpha, \beta = t, \varphi$


$\theta = \pi/2$
$\sin^2\vartheta\,\dot t$
then $P_\alpha P_\beta g^{\alpha\beta} = g_{\alpha\beta}\dot X^\alpha \dot X^\beta$
$= E^2 g^{tt} - 2Ehg^{t\varphi} + h^2 g^{\varphi\varphi}$
$= \frac{1}{\Delta}\left[E^2\left(r^2+a^2+\frac{2GMa^2}{r}\right) - \frac{4GMaEh}{r} - h^2\left(1-\frac{2GM}{r}\right)\right]$
$= \frac{1}{\Delta}\left[E^2(r^2+a^2) + \frac{2GM}{r}(h-aE)^2 - h^2\right]$

$$\Rightarrow ds^2 = \frac{4r_+}{r} e^{-r/r_+} dU\, dV - r^2 d\Omega_{II}^2$$

Maximal exten

Geodesic eqn

$$P^2 + g_{rr}\dot{r}^2 = \eta \begin{cases} 1 & \text{TIMELIKE} \\ 0 & \text{NULL} \end{cases}$$

OR $\dot{r}^2 + V_{eff}(r) = 0$

Light: $\dot{r}^2 = P^2/{-g_{rr}} = \frac{P^2\Delta}{\Sigma} \to \frac{P^2\Delta}{r^2}$

$$\Delta P^2 = E^2(r^2+a^2) + \frac{2GM}{r}($$

$$\Delta p^2 = E^2(r^2+a^2) + \frac{2GM}{r}(h-aE)^2 - h^2 \cdot$$

- rescale affine param to set $E=1$.
- "extreme" Kerr black

$$\Delta P^2 = E^2(r^2+a^2) + \frac{2GM}{r}(h-aE)^2 - h^2$$

- rescale affine param to set $E=1$.
- "extreme" Kerr black $a = GM$

$$\Delta P^2 \to r^2 - (h^2-a^2) + \frac{2a}{r}(h-a)^2$$

$$\frac{1}{r}\left(r-(h-a)\right)(r-r_1)(r-r_2)$$

$$r_{1,2} = \frac{1}{2}(a -$$

1 TIMELIKE

0 NULL

$$P^2\Delta \over \Sigma \rightarrow {P^2 \Delta \over r^2}$$

$$\Delta P^2 = E^2(r^2+a^2) + \frac{2GM}{r}(h-aE)^2 - h^2$$

- rescale affine param to set $E=1$.
- "extreme" Kerr black $a = GM$

$$\Delta P^2 \rightarrow r^2 - (h^2-a^2) + \frac{2a}{r}(h-a)^2.$$

$$\frac{1}{r}\underbrace{\left(r-(h-a)\right)}_{\text{outermost.}}\left(r-r_1\right)\left(r-r_2\right)$$

$aE)^2 - h^2 \ldots$

set $E = 1$.

GM

$\frac{2a}{r}(h-a)^2$.

$-r_1)(r-r_2)$

$$r_{1,2} = \frac{1}{2}\left(a - h \pm \sqrt{h^2 + 6ah - 7a^2}\right) < h-a.$$

If $\Delta^2 P > 0$, can choose $\oplus$ root for $\dot{r}$,) have an outgoing null geodesic. If $r \geq h-a$, no further root, so escapes.

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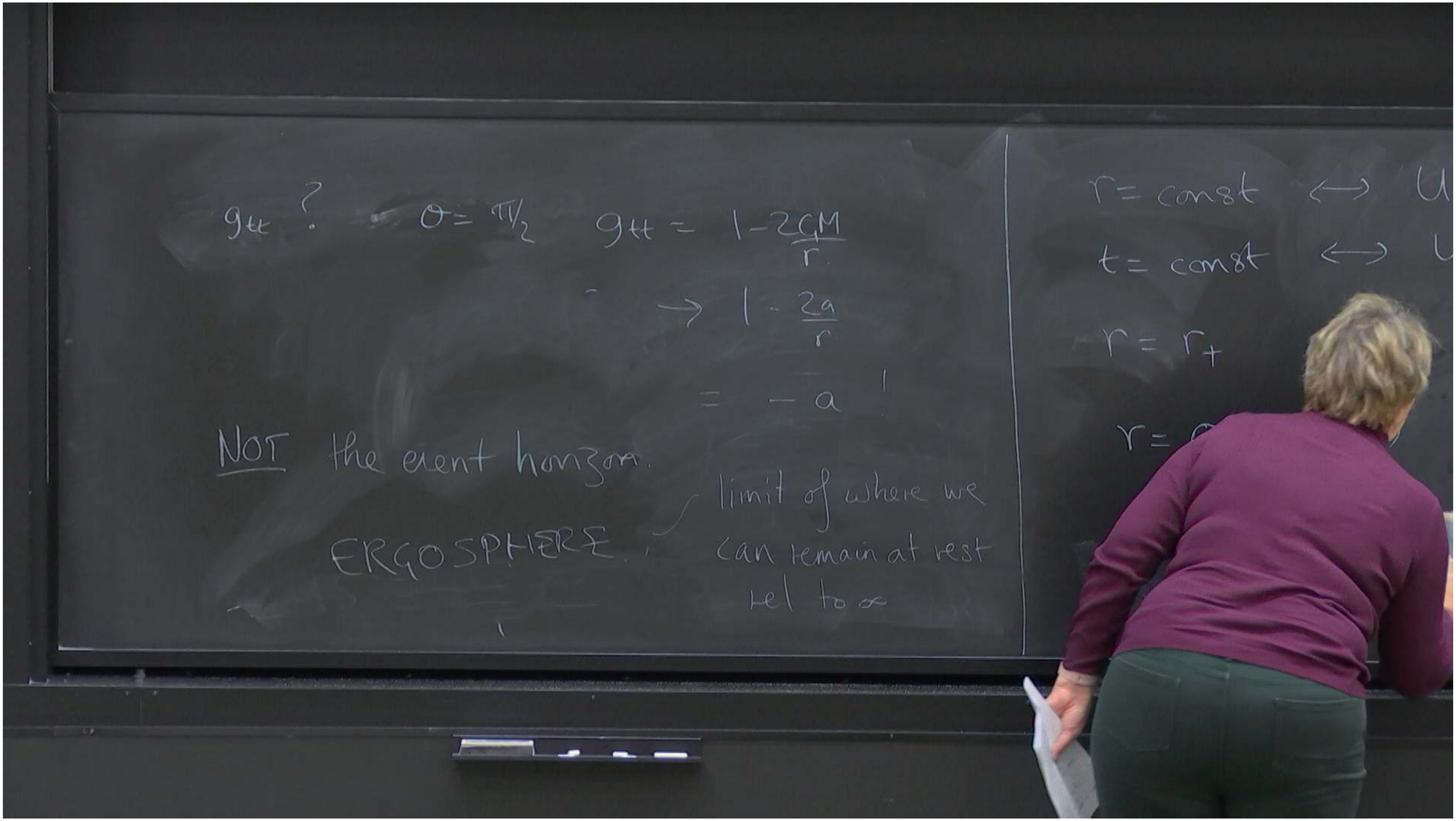
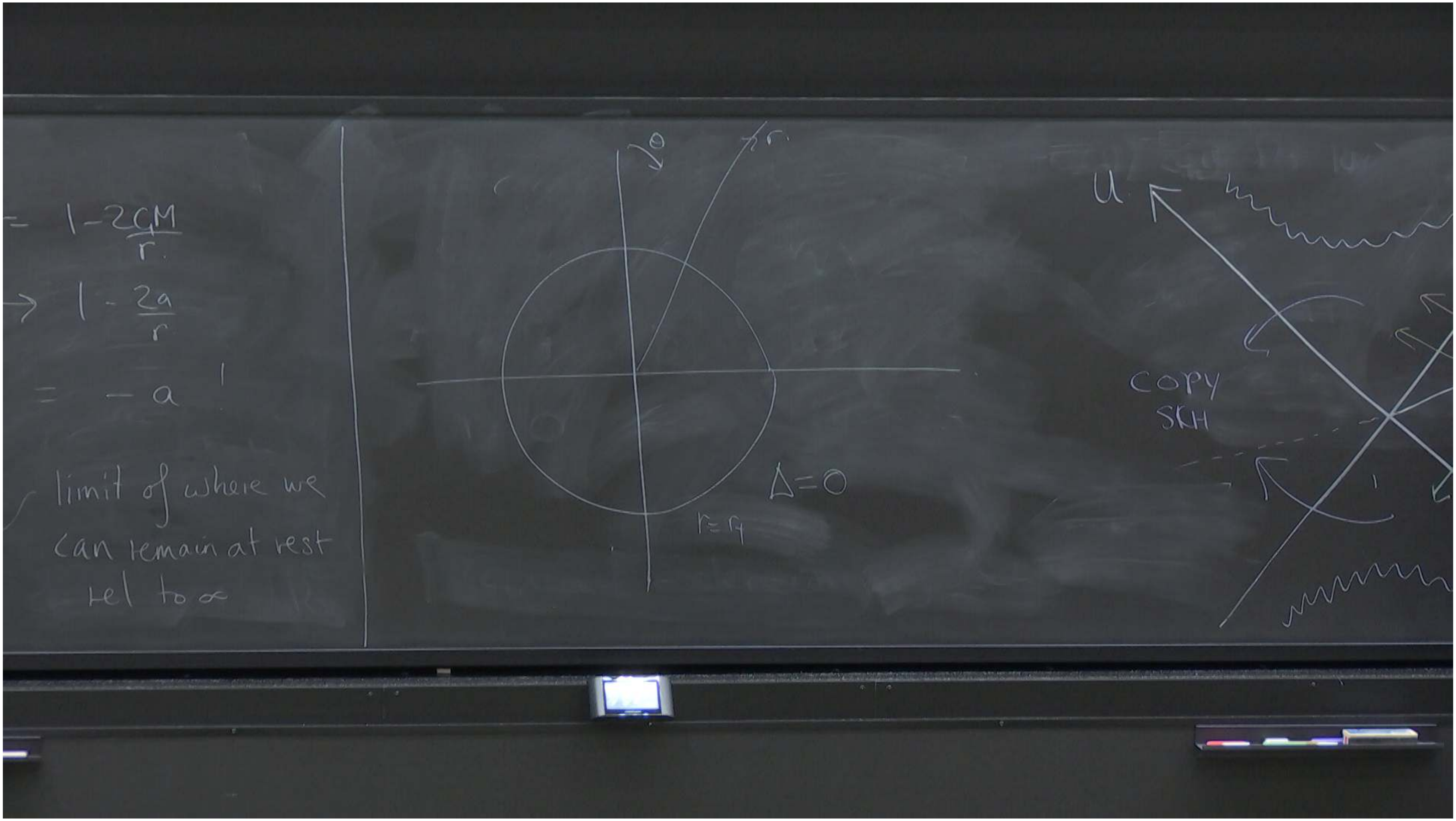
$h \to 2a$ $r_0 \to a$ & geodesics from $a+\varepsilon$ escape is. $r=a$ is $\underline{\text{EVENT}}$ horizon.


gtt ? θ = π/2 gtt = 1 − 2GM/r
→ 1 − 2a/r
= − a !
NOT the event horizon.
ERGOSPHERE
limit of where we can remain at rest rel to ∞
r = const ⟷
t = const ⟷
r = r+
r =


= 1-2GM/r.
→ 1 - 2a/r
= -a !
limit of where we
can remain at rest
rel to ∞
θ
r
Δ=0
U
COPY
SKH