Title: PSI 2019/2020 - Gravitational Physics - Lecture 5

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Collection: PSI 2019/2020 - Gravitational Physics

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Lecture 5 Spherically Symmetric Solutions

Last time:

$$ds^2 = A^2 dt^2 - B^2 dr^2 - C^2 d\Omega_{II}^2$$

$$\underline{R}^{\hat{t}}{}_{\hat{r}} = \frac{1}{AB}\left(\frac{A'}{B}\right)' \underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{t}}$$

$$\underline{R}^{\hat{t}}{}_{\hat{\theta}} = -\frac{A'C'}{AB^2C} \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\theta}} \qquad \underline{R}^{\hat{t}}{}_{\hat{\varphi}} = -\frac{A'C'}{AB^2C} \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{\varphi}}$$

utions

$$\underline{R}^{\hat\theta}{}_{\hat r} = \frac{1}{CB}\left(\frac{C'}{B}\right)' \underline{\omega}^{\hat r} \wedge \underline{\omega}^{\hat\theta}$$

$$\underline{R}^{\hat\varphi}{}_{\hat r} = \frac{1}{CB}\left(\frac{C'}{B}\right)' \underline{\omega}^{\hat r} \wedge \underline{\omega}^{\hat\varphi}$$

$$\underline{R}^{\hat\varphi}{}_{\hat\theta} = \left(\frac{1}{C^2} - \frac{C'^2}{C^2B^2}\right) \underline{\omega}^{\hat\varphi} \wedge \underline{\omega}^{\hat\theta}$$

Can read off Riemann.

$$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$

$\wedge \underline{\omega}^{\hat\varphi}$

$\underline{\omega}^{\hat\theta}$

$\wedge\, \underline{\omega}^{\hat\varphi}$

$\hat\varphi \wedge \underline{\omega}^{\hat\theta}$

$\cdots cd\, \underline{\omega}^c \wedge \underline{\omega}^d$

$$R^{\hat t}{}_{\hat r \hat t \hat r} = -\frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

$$R^{\hat t}{}_{\hat\theta \hat t \hat\theta} = -\frac{1}{B^2}\,\frac{A'C'}{AC} = R^{\hat t}{}_{\hat\varphi \hat t \hat\varphi}$$

$$R^{\hat\theta}{}_{\hat r \hat\theta \hat r} = -\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\hat\varphi}{}_{\hat r \hat\varphi \hat r}$$

$$R^{\hat\varphi}{}_{\hat\theta \hat\varphi \hat\theta} = \frac{1}{C^2}\left(1 - \frac{C'^2}{B^2}\right)$$

$$\underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\varphi}}$$

$$\left(\frac{\quad}{B^2}\right) \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}$$

nann.

$$= \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{1}{B^2}\frac{A'C'}{AC} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}}$$

$$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R$$

$$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{C^2}\left(1 - \frac{C'^2}{B^2}\right)$$

— N.B. in o/n basis

Co-ord basis

$$\underline{R} = R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}\, \underline{e}_{\hat{a}}\, \underline{\omega}^{\hat{b}}\, \underline{\omega}^{\hat{c}}\, \underline{\omega}^{\hat{d}}$$

The o/n basis has components $\underline{e}_{\hat{a}} = e_{\hat{a}}{}^{\mu} \frac{\partial}{\partial x^{\mu}}$

$$\underline{\omega}^{\hat{a}} = \omega^{\hat{a}}{}_{\mu}\, dx^{\mu}$$

↳ for our metric $\underline{\omega}^{\hat{t}} = A\, dt$

only nonzero for "$\hat{a} = \mu$" $\quad \omega^{\hat{t}}{}_{t}$.

$$R^{\mu}{}_{\nu\lambda\sigma} = e^{\mu}{}_{a}\, \omega^{b}{}_{\nu}\, \omega^{c}{}_{\lambda}\, \omega^{d}{}_{\sigma}\, R^{a}{}_{bcd}$$

$$\Rightarrow R^{tr}{}_{tr} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

$$R^{\theta r}{}_{\theta r} = R^{\varphi r}{}_{\varphi r} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right)$$

$$R^{\varphi\theta}{}_{\varphi\theta} = \frac{1}{C^2}\left(\frac{C'^2}{B^2} - 1\right)$$

$$R^{t\theta}{}_{t\theta} = R^{t\varphi}{}_{t\varphi} = A'C' / AB^2C$$

$$e^{\mu}{}_{a}\,\omega^{b}{}_{\nu}\,\omega^{c}{}_{\lambda}\,\omega^{d}{}_{\sigma}\,R^{a}{}_{bcd}$$

$$\frac{1}{B^2}\left(\frac{A''}{A}-\frac{A'B'}{AB}\right)$$

$$R^{\varphi r}{}_{\varphi r}=\frac{1}{B^2}\left(\frac{C''}{C}-\frac{C'B'}{CB}\right)$$

$$\frac{1}{C^2}\left(\frac{C'^2}{B^2}-1\right)$$

$$R^{t}{}_{t\varphi}=A'C'/AB^2C$$

Take area gauge

$$C=r$$

$$R^{t}{}_{t}=\frac{1}{B^2}\left(\frac{A''}{A}-\frac{A'B'}{AB}+\frac{2A'}{Ar}\right)$$

$$R^{r}{}_{r}=\frac{1}{B^2}\left(\frac{A''}{A}-\frac{A'B'}{AB}-\frac{2B'}{Br}\right)$$

$$R^{\theta}{}_{\theta}=R^{\varphi}{}_{\varphi}=\frac{1}{B^2}\left(\frac{A'}{Ar}-\frac{B'}{Br}+\frac{1}{r^2}\right)-\frac{1}{r^2}$$

$$G^t_t = \frac{1}{r^2} - \left(\frac{r}{B^2}\right)' = 8\pi G T^0_0$$

$$B^{-2} = 1 - \frac{2G}{r}\int \underbrace{4\pi G r^2 T^0_0 \, dr}_{\text{"energy" between } r \,\&\, r+dr}$$

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$

dr.

between

dr.

Vacuum soln: M an integration const

$$B^{-2} = 1 - \frac{2GM}{r}.$$

dr.

between

dr.

Vacuum soln: M an integration const

$$B^{-2} = 1 - \frac{2GM}{r}.$$

$$R^t{}_t - R^r{}_r = \frac{2}{rB^2}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

$$\rightarrow \quad A^2 = 1/B^2 = 1 - \frac{2GM}{r} \Rightarrow \text{SCH}.$$

What about Λ? $\quad 8\pi G T_{ab} = \Lambda g_{ab}$

$$B^{-2} = 1 - \frac{1}{r}\int r^2 \Lambda \, dr.$$

$$= 1 - \frac{\Lambda}{3} r^2 \left(-\frac{2GM}{r}\right)$$

$$= A^2 \quad \text{as } R^t{}_t = R^r{}_r \text{ still}$$

What about Λ? $\quad 8\pi G T_{ab} = \Lambda g_{ab}$

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$$= 1 - \frac{\Lambda}{3} r^2 \left(-\frac{2GM}{r}\right)$$

$$= A^2 \quad \text{as } R^t{}_t = R^r{}_r \text{ still}$$

SCH (A)dS soln.

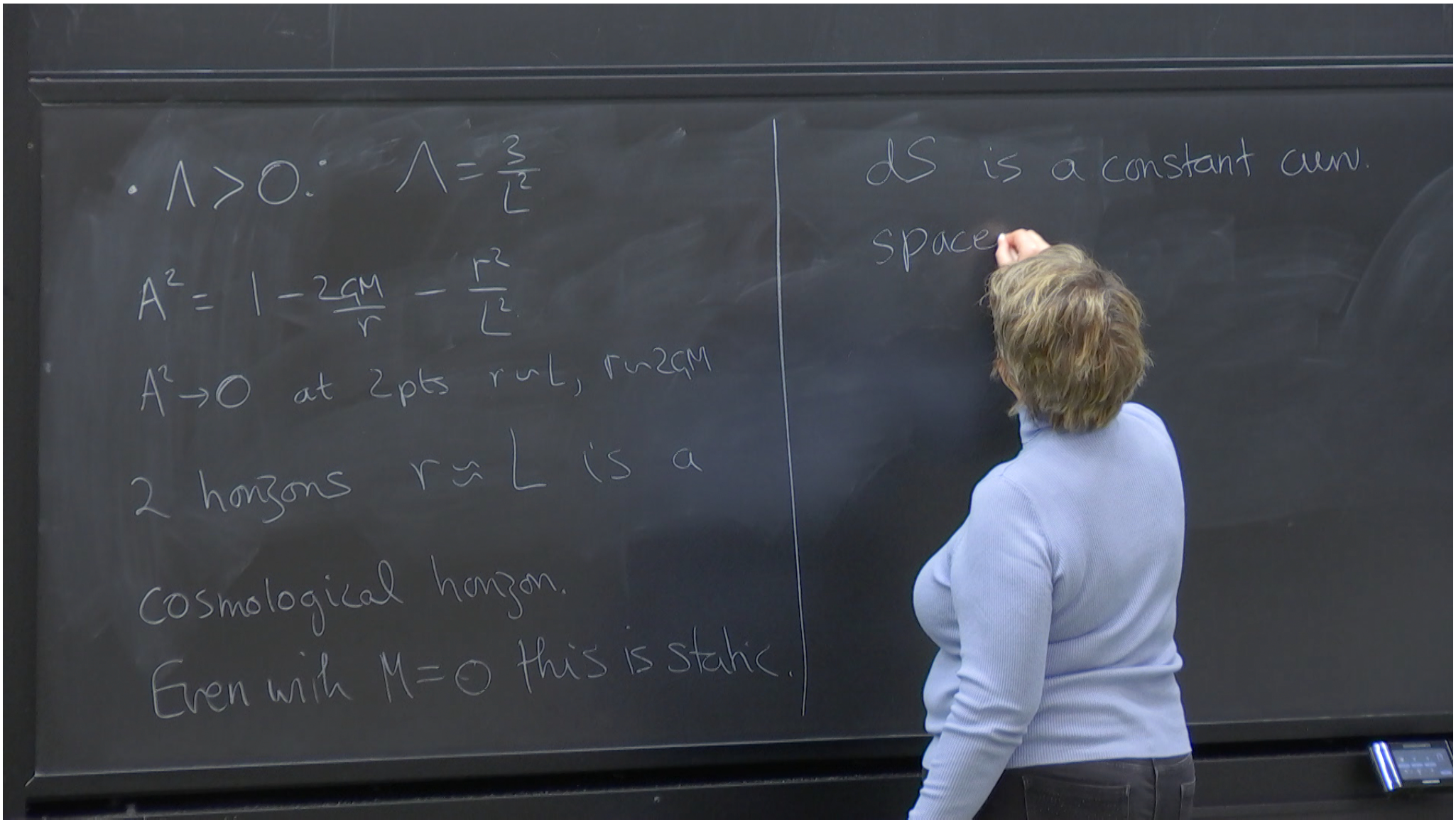
- $\Lambda > 0$: $\Lambda = \frac{3}{L^2}$

$$A^2 = 1 - \frac{2GM}{r} - \frac{r^2}{L^2}$$

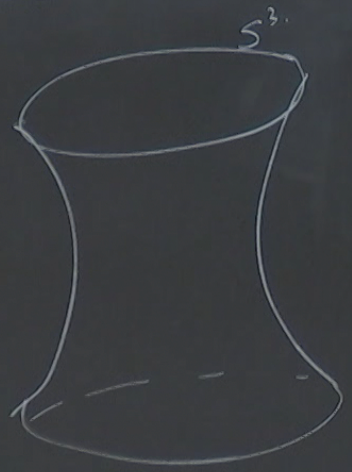
$A^2 \to 0$ at 2 pts $r \sim L$, $r \sim 2GM$

2 horizons $r \approx L$ is a

cosmological horizon.


• $\Lambda > 0$: $\Lambda = \frac{3}{L^2}$
$A^2 = 1 - \frac{2GM}{r} - \frac{r^2}{L^2}$
$A^2 \to 0$ at 2 pts $r \sim L$, $r \sim 2GM$
2 horizons $r \approx L$ is a
cosmological horizon.
Even with $M=0$ this is static.
dS is a constant curv.
space

dS is a constant curv. spacetime. Can represent as a hyperboloid in 5D flat spacetime. $X^2+Y^2+Z^2+U^2-T^2=L^2$.



ien.
sent
flat
$^2 = L^2$

$$T = L\sqrt{1 - r^2/L^2}\,\sinh t/L$$

$$u = L\sqrt{1 - r^2/L^2}\,\cosh t/L$$

$$\underline{x} = r\,\underline{n}$$

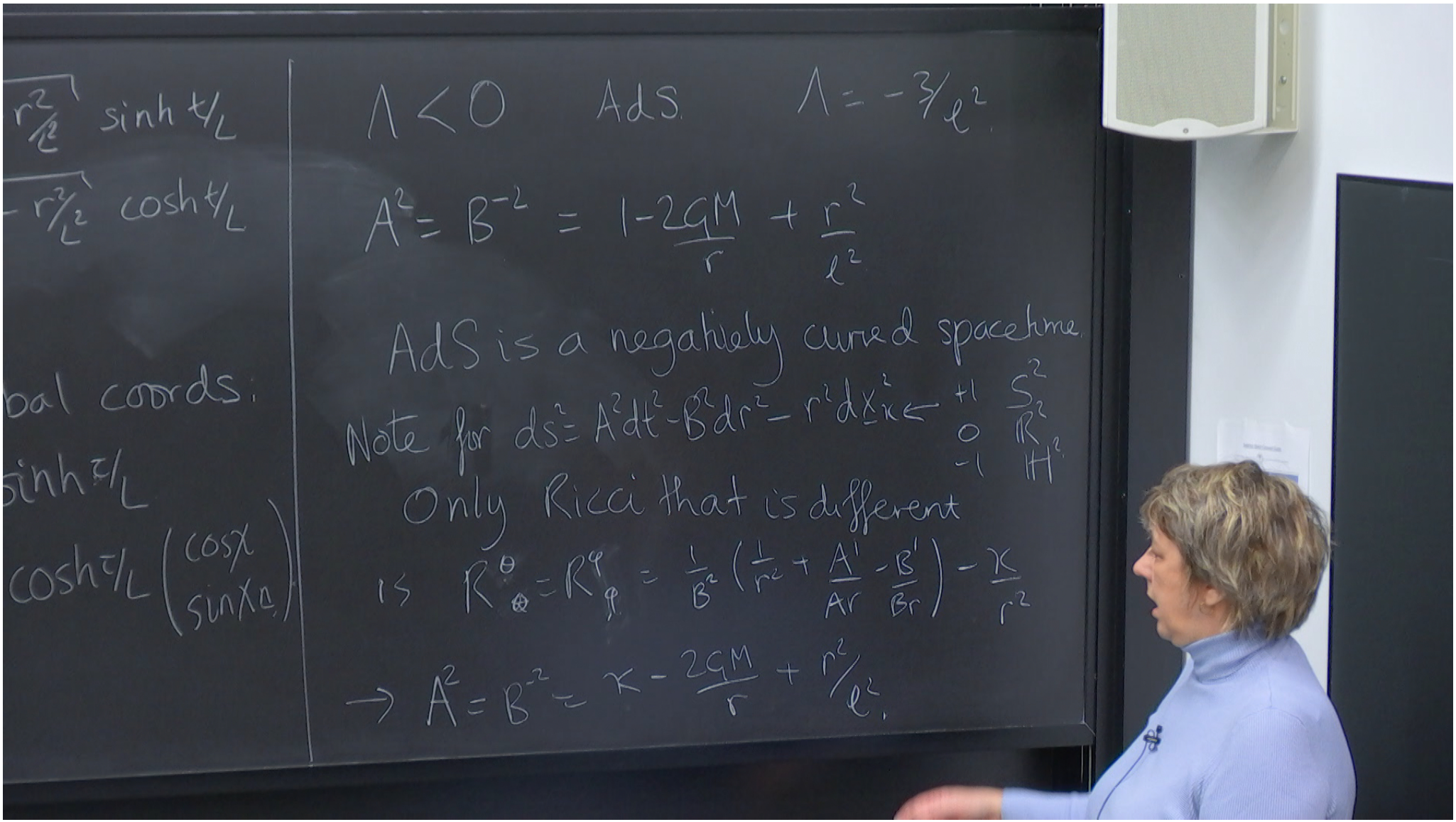
cos or global coords:

$$T = L \sinh \tau/L$$

$$\begin{matrix} u \\ \underline{x} \end{matrix} = L \cosh \tau/L \begin{pmatrix} \cos\chi \\ \sin\chi\,\underline{n} \end{pmatrix}$$

$\Lambda < 0$ AdS. $\Lambda = -3/\ell^2$.

$$A^2 = B^{-2} = 1 - \frac{2GM}{r} + \frac{r^2}{\ell^2}$$


$\sqrt{r^2/L^2}\ \sinh t/L$
$\sqrt{-r^2/L^2}\ \cosh t/L$
bal coords.
$\sinh \tau/L$
$\cosh \tau/L \begin{pmatrix} \cos\chi \\ \sin\chi\, \underline{n} \end{pmatrix}$
$\Lambda < 0$ AdS $\Lambda = -3/\ell^2$
$A^2 = B^{-2} = 1 - \frac{2GM}{r} + \frac{r^2}{\ell^2}$
AdS is a negatively curved spacetime.
Note for $ds^2 = A^2 dt^2 - B^2 dr^2 - r^2 d\underline{x}^2_\kappa \leftarrow$ $+1$ S^2, 0 $\mathbb{R}^2$, -1 $\mathbb{H}^2$
Only Ricci that is different
is $R^\theta{}_\theta = R^\varphi{}_\varphi = \frac{1}{B^2}\left(\frac{1}{r^2} + \frac{A'}{Ar} - \frac{B'}{Br}\right) - \frac{\kappa}{r^2}$
$\rightarrow A^2 = B^{-2} = \kappa - \frac{2GM}{r} + r^2/\ell^2$

In AdS, planar & hyperbolic black holes are possible.

Star?

In AdS, planar & hyperbolic black holes are possible.

<u>Star?</u>

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$

$$-8\pi G p_r = \frac{1}{r^2} - \left(1 - \frac{2GM(r)}{r}\right)\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right)$$

In AdS, planar & hyperbolic black holes are possible.

<u>Star?</u>

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$

$$-8\pi G p_r = \frac{1}{r^2} - \left(1 - \frac{2GM(r)}{r}\right)\left(\frac{2A'}{Ar} + \frac{1}{r^2}\right)$$

$$\hookrightarrow \frac{(A^2)'}{A^2} = \frac{2GM(r) + 8\pi G p_r r^3}{r(r - 2GM(r))}$$

— Assume

- Assume perfect isotropic fluid

use e-m consvn.

$$0 = \nabla_a T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$$

- Assume perfect isotropic fluid

use e-m consm:

$$0 = \nabla_a T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$$

$$= \frac{p'}{B} +$$

$$\underline{\theta}^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB}\,\underline{\omega}^{\hat{r}}$$

$$\Rightarrow \quad \Gamma^{\hat{t}}_{\hat{t}\hat{r}} = \frac{A'}{AB}$$

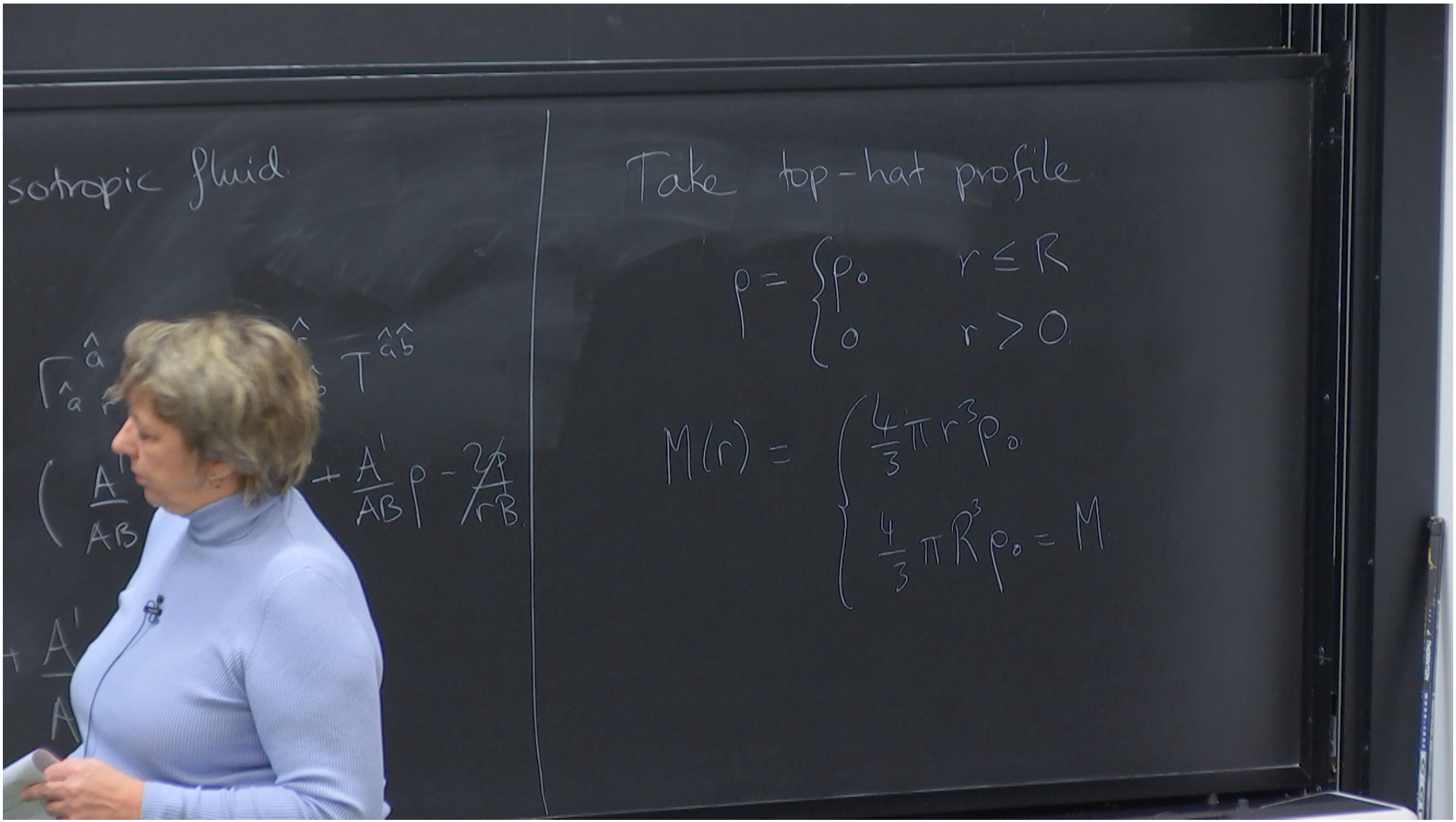
– Assume perfect isotropic fluid

use e-m consvn:

$$0 = \nabla_a T^{\hat{a}\hat{r}} = \partial_{\hat{r}} p_r + \Gamma^{\hat{a}}_{\hat{a}\hat{r}} p_r + \Gamma^{\hat{r}}_{\hat{a}\hat{b}} T^{\hat{a}\hat{b}}$$

$$= \frac{p'}{B} + \left(\frac{A'}{AB} + \cancel{\frac{2}{rB}}\right) p + \frac{A'}{AB}\rho - \cancel{\frac{2\rho}{rB}}$$

$$= \frac{1}{B}\left(p' + \frac{A'}{A}(\rho + p)\right)$$


sotropic fluid
Take top-hat profile
$\rho = \begin{cases} \rho_0 & r \le R \\ 0 & r > 0 \end{cases}$
$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 \\ \frac{4}{3}\pi R^3 \rho_0 = M \end{cases}$

$$A^2 \qquad r(r-2GM(r))$$

Inside star

$$P' = -(P+\rho_0)\,\frac{[GM(r)+4\pi G P r^3]}{r(r-2GM(r))}$$

Inside star

$$p' = -(p+\rho_0)\,\frac{\{GM(r)+4\pi G p r^3\}}{r(r-2GM(r))}$$

$$= -(p+\rho_0)\,\frac{4\pi G r^3}{3}\,\frac{(3p+\rho_0)}{r\left(r-\frac{8\pi G \rho_0 r^3}{3}\right)}$$

$$= -(p+\rho_0)\cdot\frac{4\pi G r(3p+\rho_0)}{3-8\pi G r^2 \rho_0}$$

TOLMAN
OPPENHEIMER
VOLKOFF

Solution:

$$P = \rho_0 \left[\frac{\sqrt{1-\frac{2GM}{R}} - \sqrt{1-\frac{2GMr^2}{R^3}}}{\sqrt{1-\frac{2GMr^2}{R^3}} - 3\sqrt{1-\frac{2GM}{R}}} \right]$$

MAN
ENHEIMER
KOFF