Title: PSI 2019/2020 - Gravitational Physics - Lecture 4

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L4 Connections, Curvature & Cartan

Look at $\mathbb{R}^2$

$e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$ Standard basis

dual $\omega^1 = dx$, $\omega^2 = dy$

Same throughout $\mathbb{R}^2$

Have $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

could take ∇v to get a tensor. other coords?

$\{r, \theta\}$. Coord basis is.

$\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$ dual $\{dr, d\theta\}$

& Cartan

Standard basis

Have $\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

could take $\underline{\nabla}\underline{V}$ to get a tensor. Other coords?

$\{r, \theta\}$. Coord basis is.

$\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$ dual $\{dr, d\theta\}$

Also have o/n basis

$\underline{\hat{e}}_{\hat{r}} = \frac{\partial}{\partial r}$ $\quad \underline{\hat{e}}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}$.

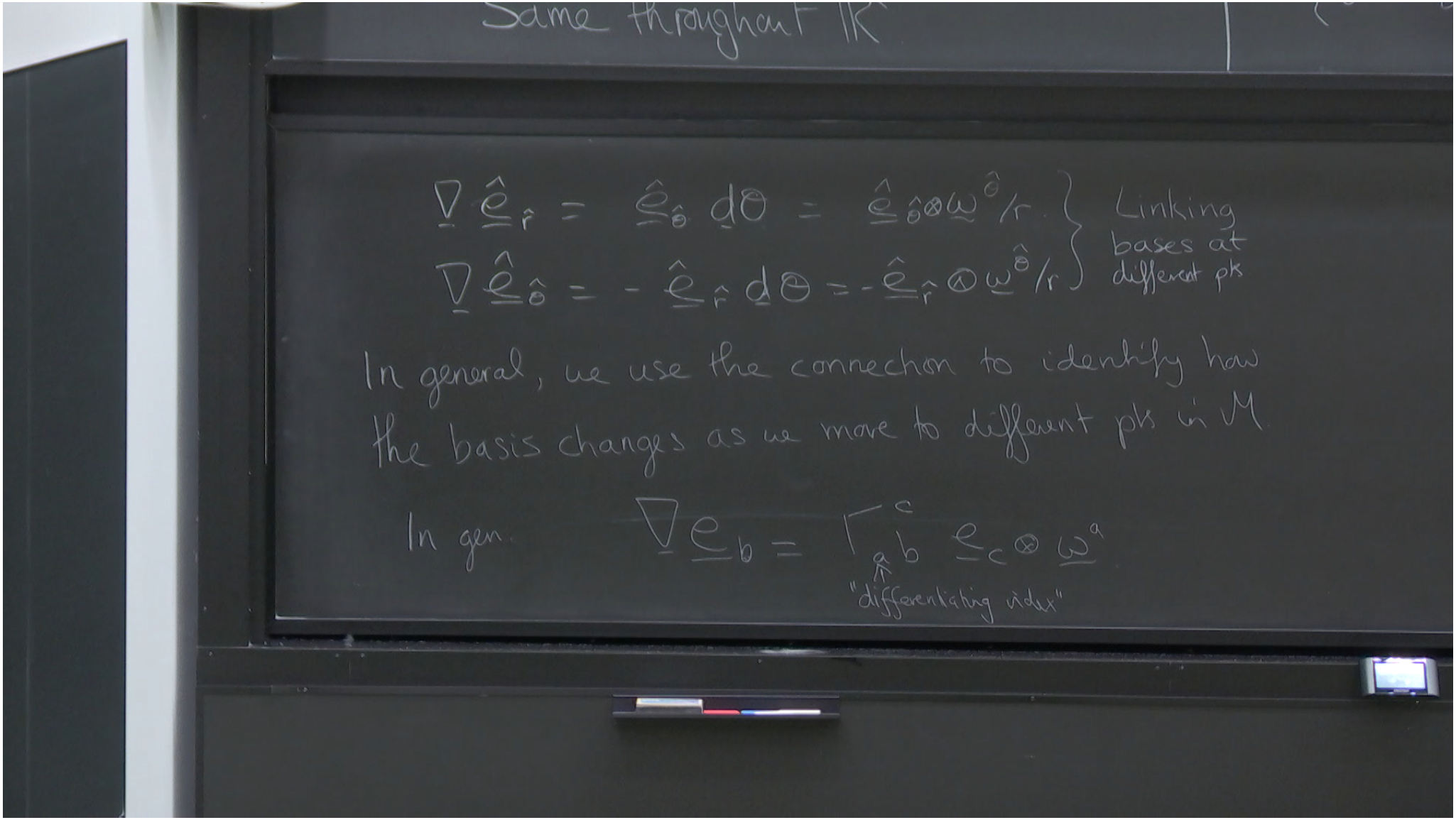
$\underline{\hat{\omega}}^{\hat{r}} = dr$ $\quad \underline{\hat{\omega}}^{\hat{\theta}} = r\,d\theta$

$\underline{\hat{e}}_{\hat{r}} = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$

$\underline{\hat{e}}_{\hat{\theta}} = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2$.

$\underline{\hat{\omega}}^{\hat{r}} = \cos\theta\, \underline{\omega}_1 + \sin\theta\, \underline{\omega}_2$.

$\underline{\hat{\omega}}^{\hat{\theta}} = -\sin\theta\, \underline{\omega}_1 + \cos\theta\, \underline{\omega}_2$.


same throughout R
$\nabla \hat{e}_{\hat{r}} = \hat{e}_{\hat{\theta}}\, d\theta = \hat{e}_{\hat{\theta}} \otimes \omega^{\hat{\theta}}/r$
$\nabla \hat{e}_{\hat{\theta}} = -\hat{e}_{\hat{r}}\, d\theta = -\hat{e}_{\hat{r}} \otimes \omega^{\hat{\theta}}/r$
Linking bases at different pts
In general, we use the connection to identify how the basis changes as we move to different pts in M.
In gen. $\nabla e_b = \Gamma^{c}_{a\,b}\, e_c \otimes \omega^a$
"differentiating index"

$\hat{\underline{\omega}}\hat{\theta} = -\sin\theta\,\underline{\omega}_1 + \cos\theta$

$\underline{e}_{\hat{\theta}} \otimes \underline{\omega}^{\hat{\theta}}/r$

$= -\underline{e}_{\hat{r}} \otimes \underline{\omega}^{\hat{\theta}}/r$

} Linking bases at different pts

onnection to identify how
move to different pts in M

$\Gamma^{c}_{ab}\, \underline{e}_c \otimes \underline{\omega}^a$

"differentiating index"

$\underline{\nabla}$ is a derivation

(i) commutes with contractions

(ii) Leibnizian

(iii) Reduces to $\underline{d}$ on fns

$\hat{\omega}^{\hat{0}} = -\sin\theta\,\omega_1 + \cos\theta\,\omega_2$

...rking
...ases at
...fferent pts

...fy how
...s in M

∇ is a derivation

(i) commutes with contractions
(ii) Leibnitzian
(iii) Reduces to d on fns

Look at ∇ on a vector:

$$\nabla v = \nabla(v^a e_a)$$
$$= (\nabla v^a) e_a + v^a \nabla e_a$$
(∇v^a: SCALARS)
$$= (d v^a) e_a + v^a \Gamma^c_{ba}\, e_c \otimes \omega^b$$
$$= [\underbrace{v^a{}_{,b} + \Gamma^a_{bc} v^c}_{\nabla_b v^a \text{ is A.I.N.}}]\, e_a \otimes \omega^b$$

In gen. $\nabla \underline{e}_b = \Gamma^a{}_{bc}\, \underline{e}_c \otimes \underline{\omega}^a$

"differentiating index"

In GR we use the Levi-Civita connection, a metric-presering torsion-free connection.

i.e. $\nabla g = 0$

& $T : T(\underline{u}, \underline{v}) = \nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} - [\underline{u}, \underline{v}] = 0$

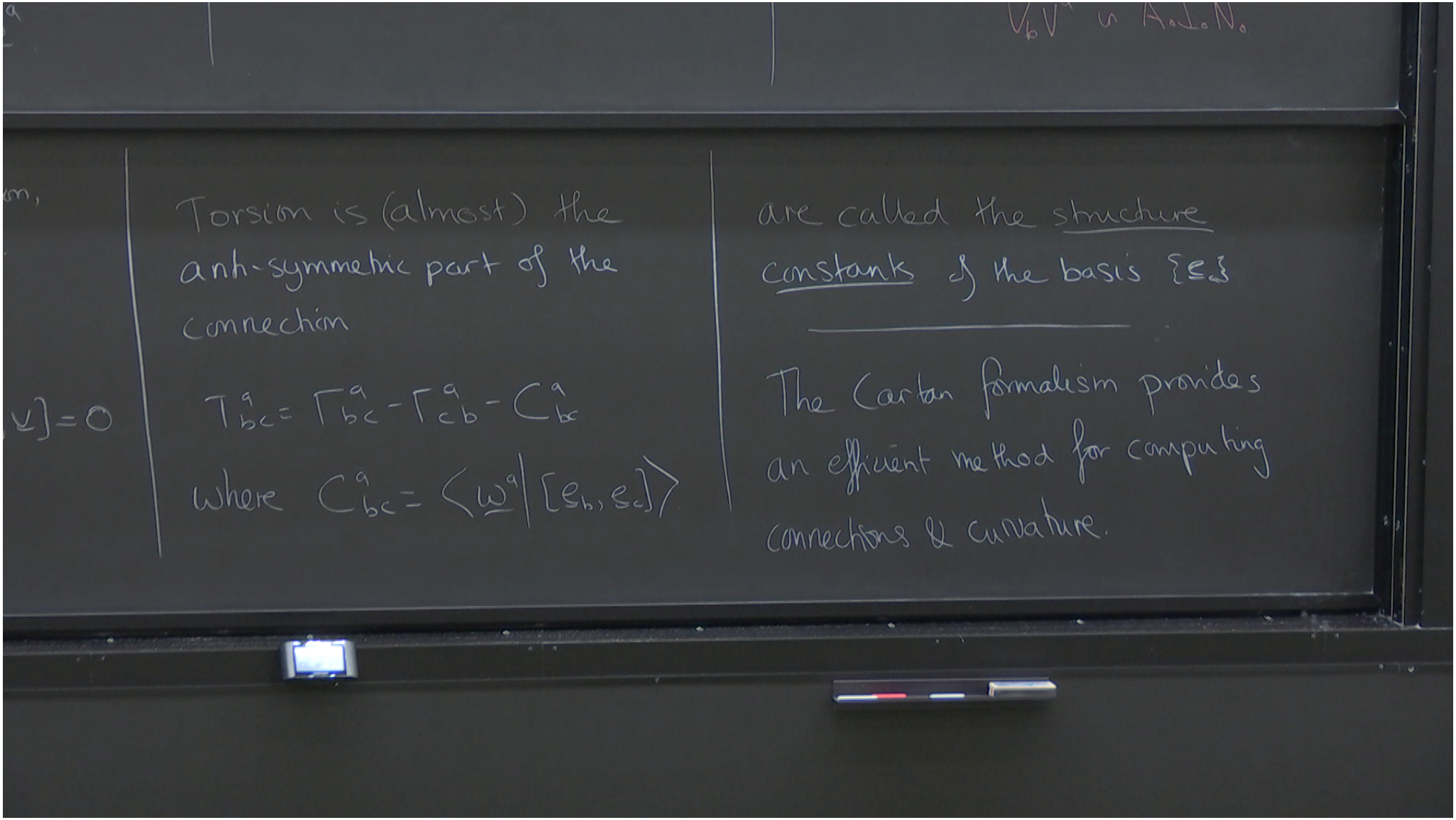
$\underline{u} \in \mathcal{V}$

$: T_p(M) \times T_p(M) \to T_p(M)$

Torsion is (almost) the anti-symmetric part of the connection

$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$

where $C^a_{bc} = \langle \underline{\omega}^a | [e_b, e_c] \rangle$

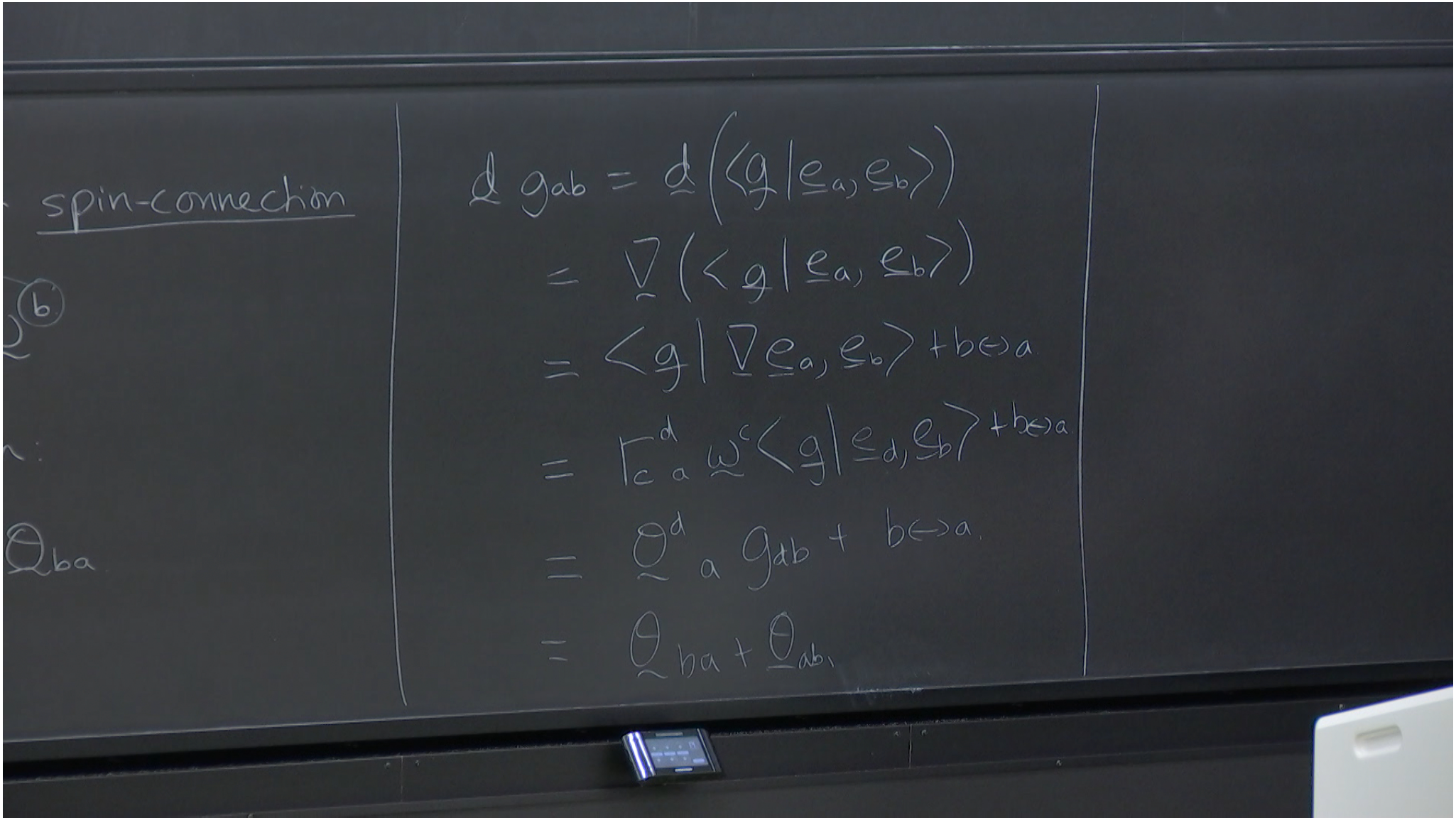

Torsion is (almost) the anti-symmetric part of the connection
$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$
where $C^a_{bc} = \langle \underline{\omega}^a | [e_b, e_c] \rangle$
are called the structure constants of the basis $\{e_a\}$
The Cartan formalism provides an efficient method for computing connections & curvature.

The connection 1-forms or spin-connection

$$\underset{\sim}{\Theta}{}^{a}{}_{c} = \Gamma_{(b)}{}^{a}{}_{c}\, \underset{\sim}{\omega}{}^{(b)}$$

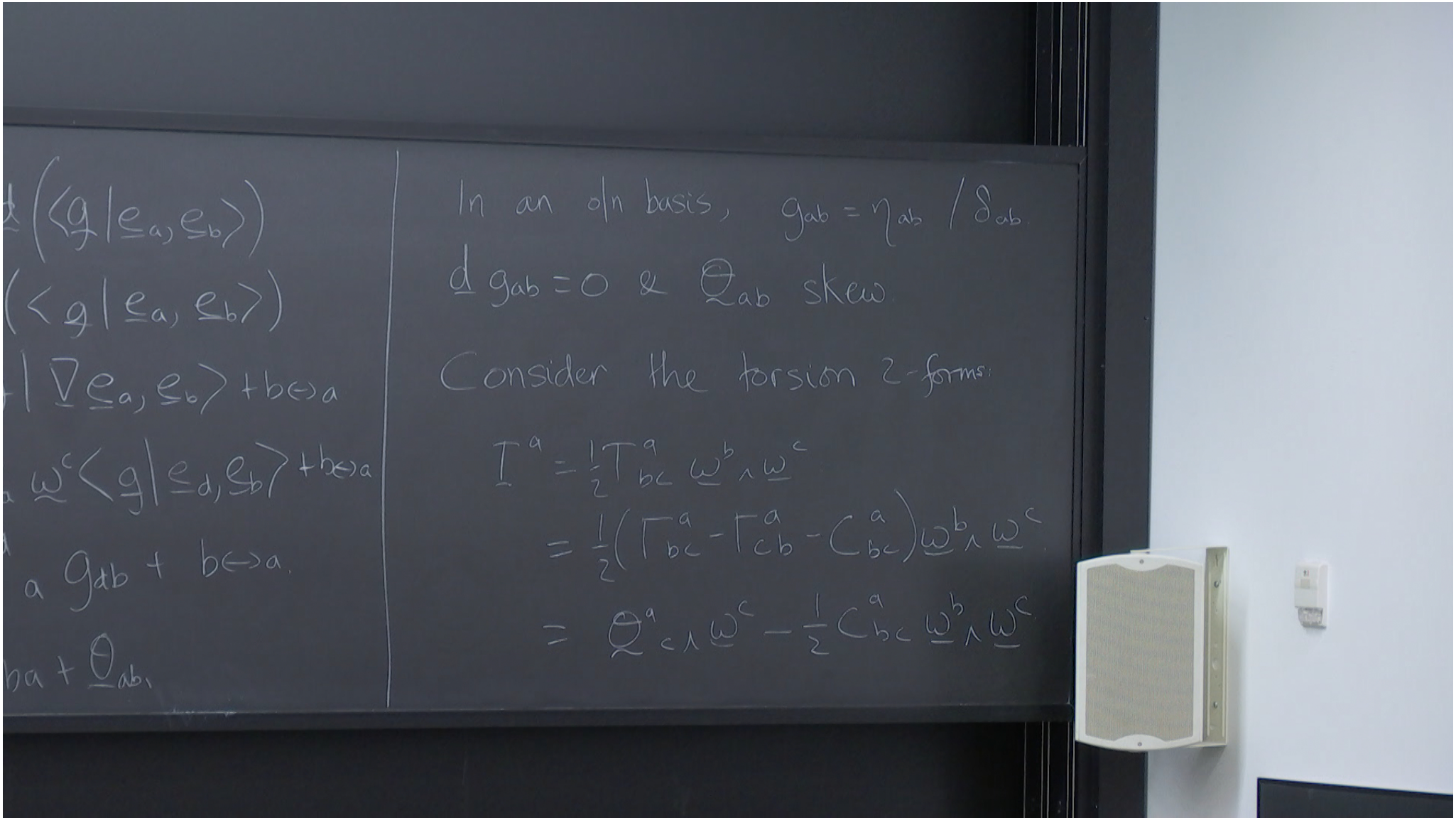
For the metric connection:

$$\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\Theta}_{ab} + \underset{\sim}{\Theta}_{ba}$$


spin-connection
$\underset{\sim}{\nabla} g_{ab} = \underset{\sim}{\nabla}\left(\langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle\right)$
$= \underset{\sim}{\nabla}\left(\langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle\right)$
$= \langle g | \underset{\sim}{\nabla} \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle + b \leftrightarrow a$
$= \Gamma^d_{ca}\, \omega^c \langle g | \underset{\sim}{e}_d, \underset{\sim}{e}_b \rangle + b \leftrightarrow a$
$= \underset{\sim}{\theta}^d{}_a\, g_{db} + b \leftrightarrow a$
$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$

In an o/n basis, $g_{ab} = \eta_{ab} / \delta_{ab}$.

$\underline{d}\, g_{ab} = 0$ & $\underset{\sim}{\Theta}_{ab}$ skew.


In an o/n basis, $g_{ab} = \eta_{ab} / \delta_{ab}$
$\underline{d}\, g_{ab} = 0$ & $\underline{\Theta}_{ab}$ skew.
Consider the torsion 2-forms:
$\underline{T}^a = \frac{1}{2} T^a_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$
$= \frac{1}{2}(\Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc})\, \underline{\omega}^b \wedge \underline{\omega}^c$
$= \underline{\Theta}^a{}_c \wedge \underline{\omega}^c - \frac{1}{2} C^a_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$

Now consider $d\underline{\omega}^a$ - 2 form with cpts

$$(d\underline{\omega}^a)_{bc} = \langle d\underline{\omega}^a | \underline{e}_b, \underline{e}_c \rangle \qquad \delta^a_b$$

$$= \underline{e}_b(\langle \underline{\omega}^a | \underline{e}_c \rangle) - \underline{e}_c(\langle \underline{\omega}^a | \underline{e}_b \rangle)$$

$$- \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$$

$$= - C^a_{bc}$$

$$\Rightarrow d\underline{\omega}^a = -\frac{1}{2} C^a_{bc} \underline{\omega}^b \wedge \underline{\omega}^c$$

2 form with cpts

$|e_b, e_c\rangle$

δ^a_b

$|e_c\rangle) - e_c(\langle\langle \omega^a | e_b\rangle)$

$^a | [e_b, e_c]\rangle$

$\wedge \omega^c$

Hence

$$\boxed{T^a = d\omega^a + \partial^a{}_c \wedge \omega^c}$$

Cartan's 1st structural eqn

Curvature is defined as

a commutator of derivatives:

$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$

$= \underset{\sim}{\theta}^a{}_c \wedge \underline{\omega}^c - \frac{1}{2} C^a_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$

$\underline{\Omega}^a + \underline{\theta}^a{}_c \wedge \underline{\omega}^c$

st structural eqn

is defined as

utator of derivatives:

$$\underline{R}[\underline{u},\underline{v}]\underline{w} = \left(\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right)\underline{w}$$

map $T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$

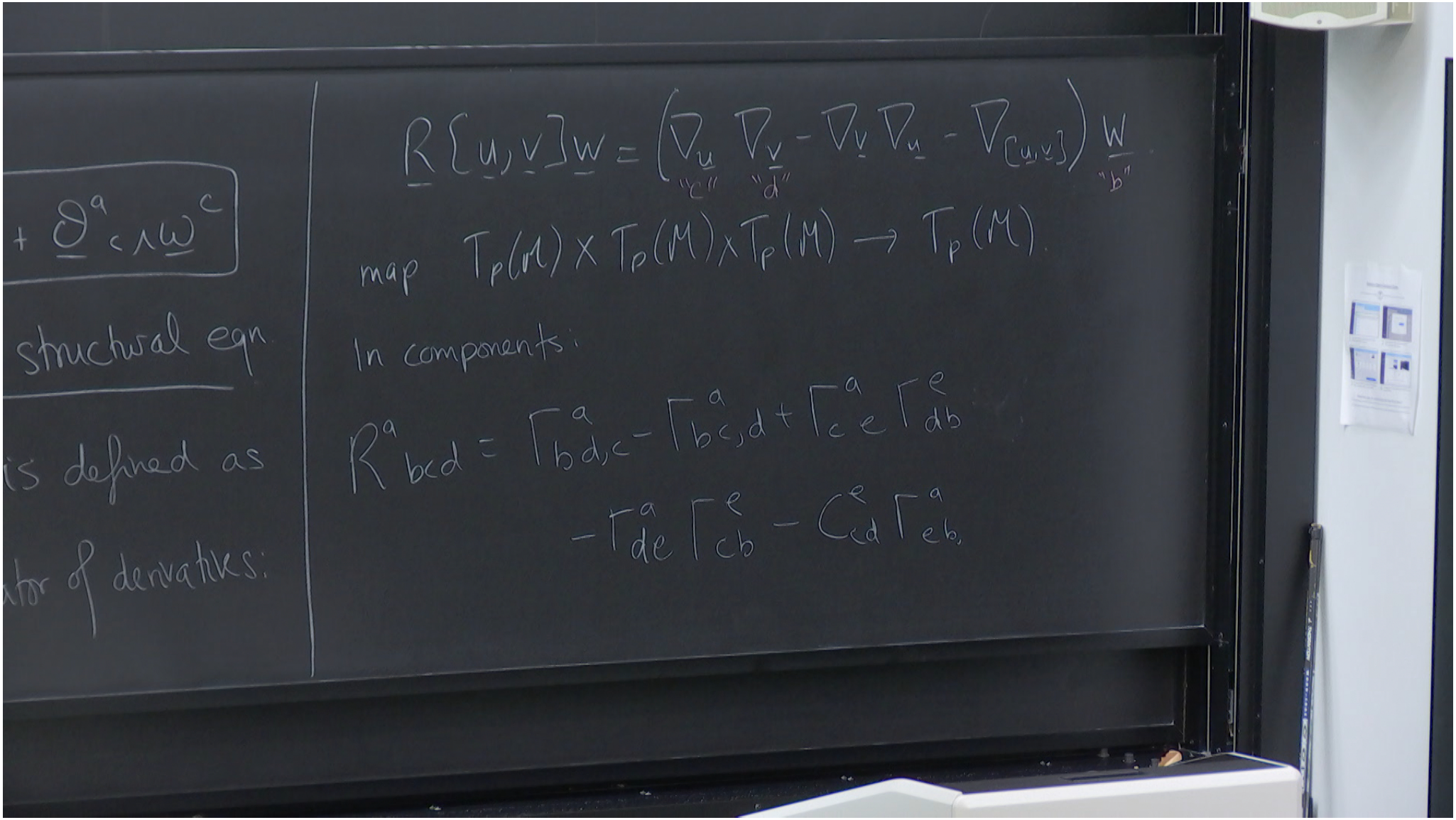
In components:

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb},$$

$\Rightarrow \underline{d}\underline{\omega} = -\frac{1}{2} C_{bc} \underline{\omega}^b \wedge \underline{\omega}^c$

We then define curvature 2-forms

$$\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$


+ $\underline{\theta}^a{}_c \wedge \underline{\omega}^c{}_b$
structural eqn
is defined as
tor of derivatives:
$\underline{R}(\underline{u},\underline{v})\underline{w} = (\nabla_{\underline{u}} \nabla_{\underline{v}} - \nabla_{\underline{v}} \nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]})\underline{w}$
"c" "d" "b"
map $T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$
In components:
$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ce}\Gamma^e_{db}$
$- \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$

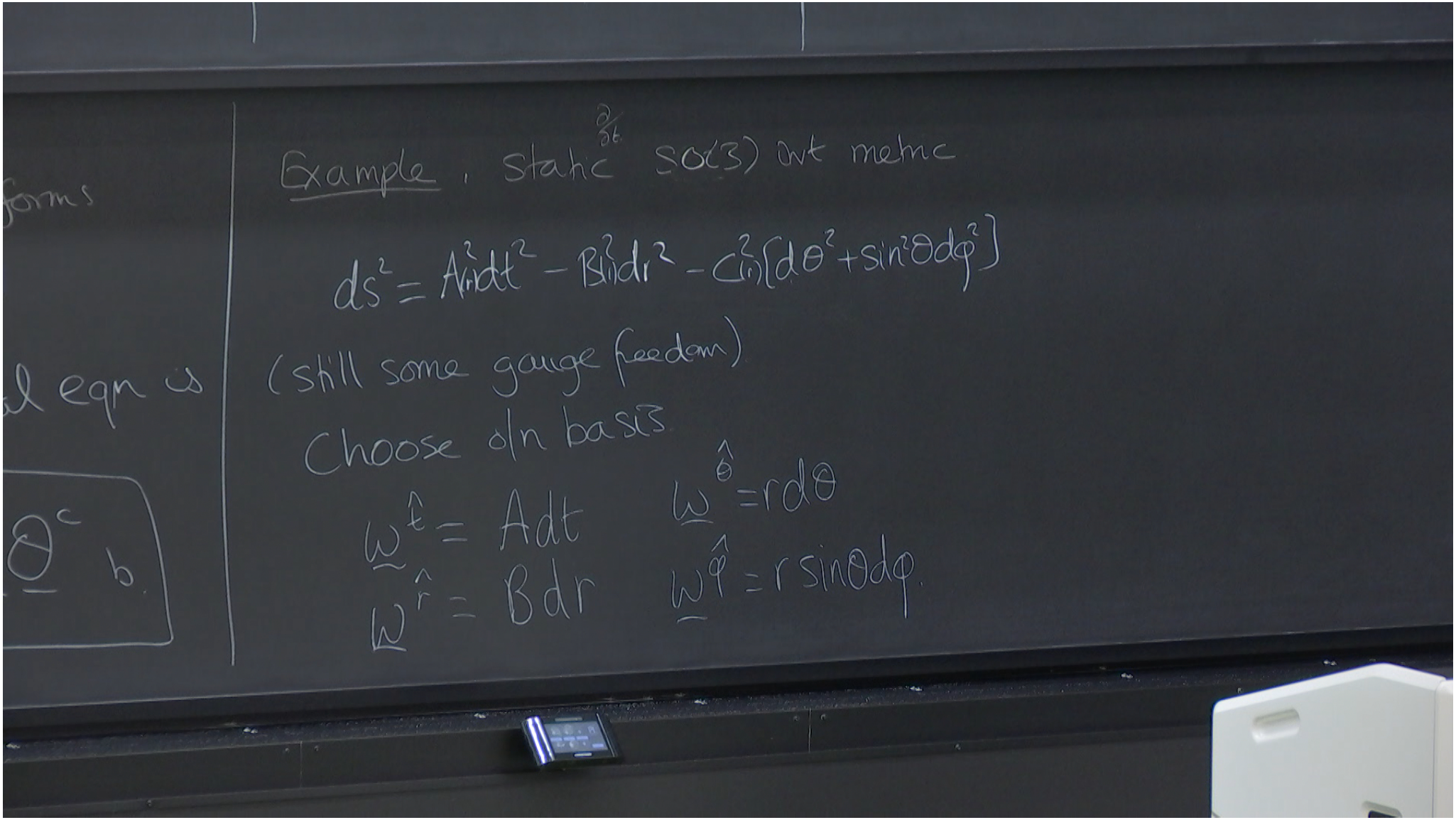
$\Rightarrow \underline{d}\underline{\omega}^{a} = -\frac{1}{2} C^{a}{}_{bc}\, \underline{\omega}^{b} \wedge \underline{\omega}^{c}$

We then define curvature 2-forms

$$\underline{R}^{a}{}_{b} = \frac{1}{2} R^{a}{}_{bcd}\, \underline{\omega}^{c} \wedge \underline{\omega}^{d}$$

Then Cartan's 2nd structural eqn is

$$\boxed{\underline{R}^{a}{}_{b} = \underline{d}\underline{\Theta}^{a}{}_{b} + \underline{\Theta}^{a}{}_{c} \wedge \underline{\Theta}^{c}{}_{b}}$$


forms
l eqn is
Example. Static SO(3) int metric
$ds^2 = A_{(r)}^2 dt^2 - B_{(r)}^2 dr^2 - C_{(r)}^2 [d\theta^2 + \sin^2\theta d\varphi^2]$
(still some gauge freedom)
Choose o/n basis
$\omega^{\hat{t}} = A dt$
$\omega^{\hat{r}} = B dr$
$\omega^{\hat{\theta}} = r d\theta$
$\omega^{\hat{\varphi}} = r \sin\theta d\varphi$.

forms

l eqn is

θ^c b

Example. Static $\overset{2}{S}t.$ SO(3) wt metric

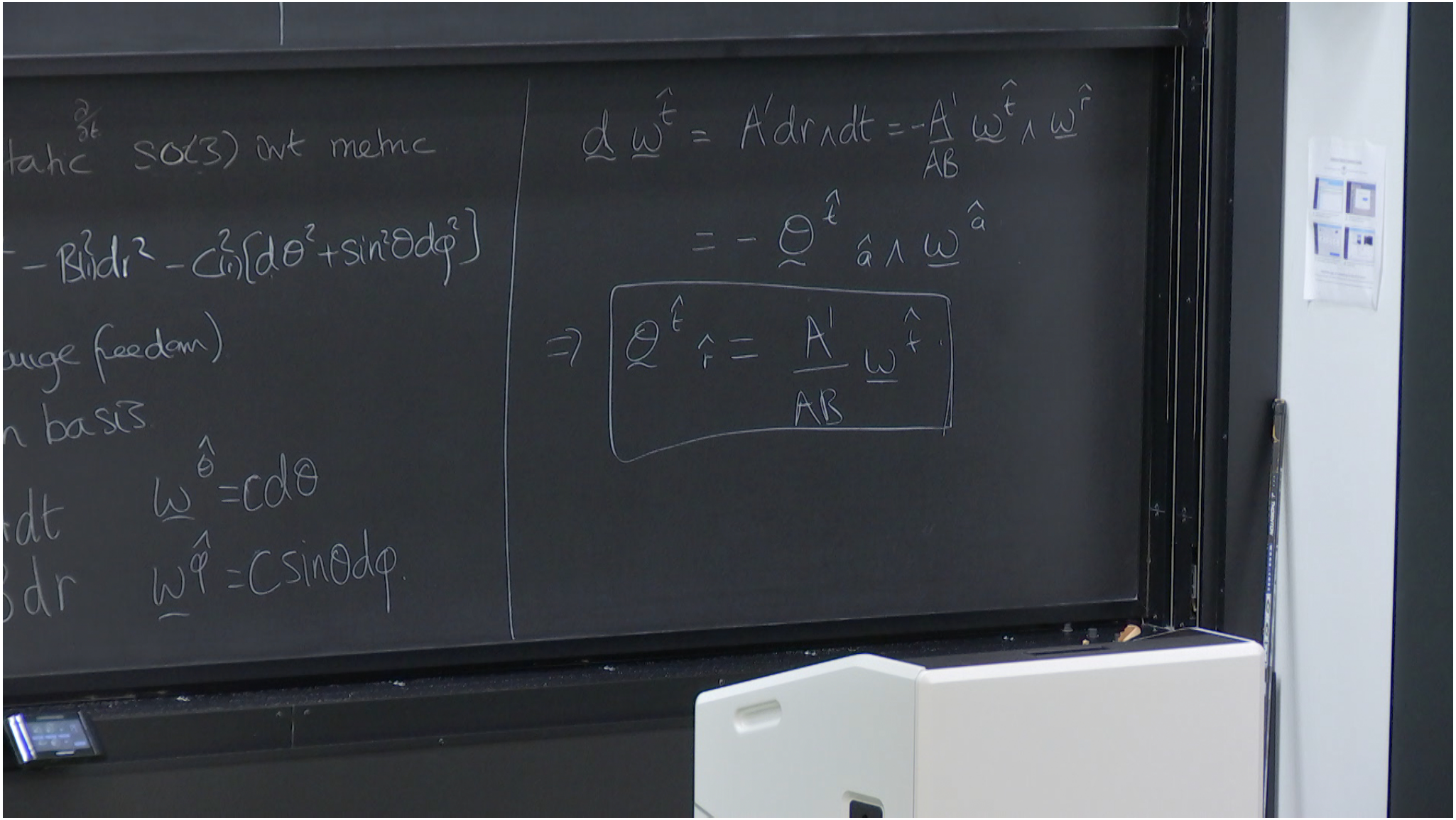
$$ds^2 = A_{(r)}^2 dt^2 - B_{(r)}^2 dr^2 - C_{(r)}^2 [d\theta^2 + \sin^2\theta d\varphi^2]$$

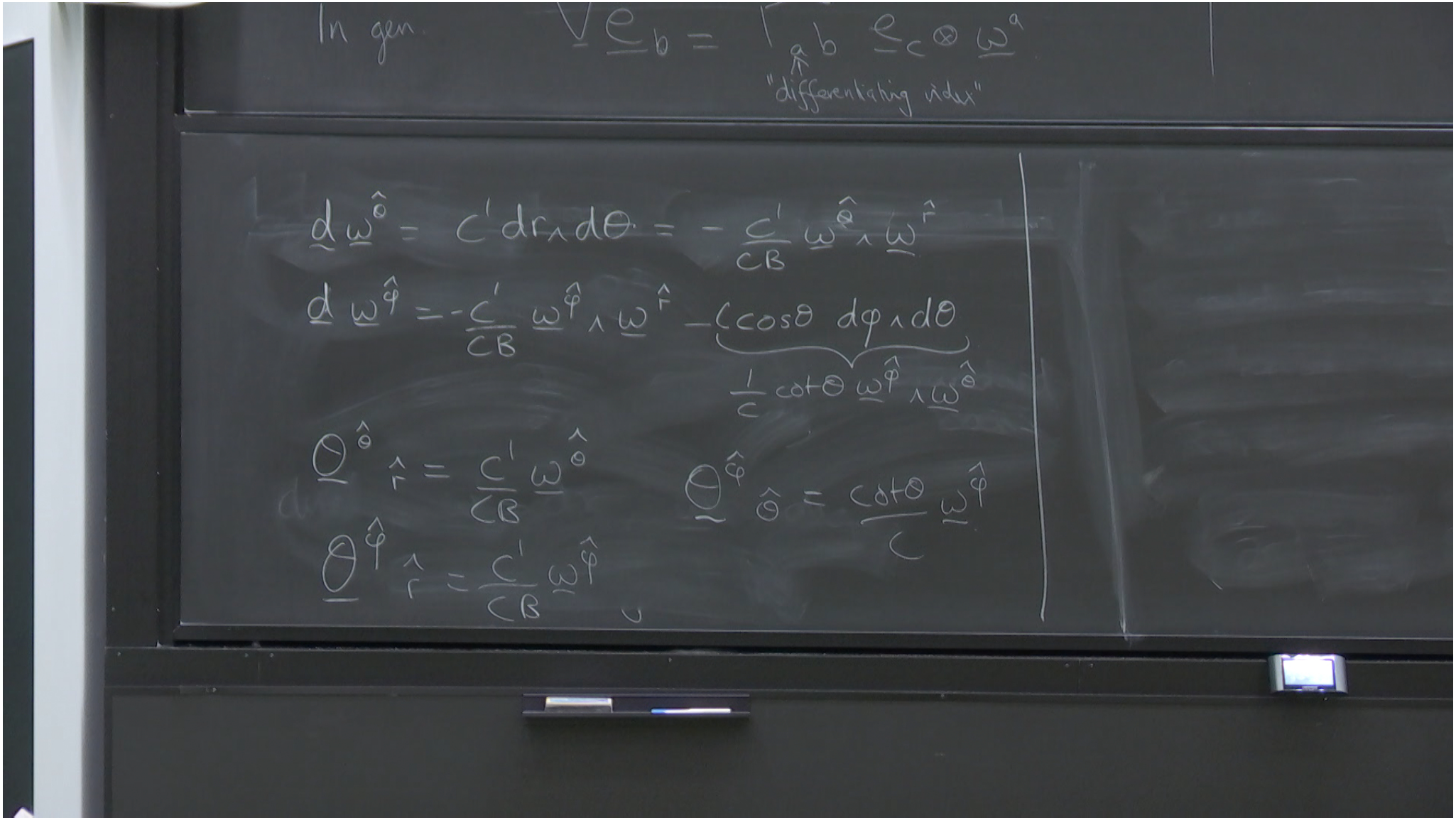
(still some gauge freedom)

Choose o/n basis

$\underset{\sim}{\omega}^{\hat{t}} = A dt$ $\quad \underset{\sim}{\omega}^{\hat{\theta}} = C d\theta$

$\underset{\sim}{\omega}^{\hat{r}} = B dr$ $\quad \underset{\sim}{\omega}^{\hat{\varphi}} = C \sin\theta d\varphi.$


∂_t
SO(3) int metric
$-B^2 dr^2 - C^2[d\theta^2 + \sin^2\theta d\varphi^2]$
gauge freedom)
basis
$\omega^{\hat{\theta}} = C d\theta$
$\omega^{\hat{\varphi}} = C \sin\theta d\varphi$
$d\omega^{\hat{t}} = A' dr \wedge dt = -\frac{A'}{AB} \omega^{\hat{t}} \wedge \omega^{\hat{r}}$
$= -\Theta^{\hat{t}}{}_{\hat{a}} \wedge \omega^{\hat{a}}$
$\Rightarrow \boxed{\Theta^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB} \omega^{\hat{t}}}$


In gen. $\nabla e_b = \Gamma^a{}_b \, e_c \otimes \omega^a$
"differentiating index"
$d\omega^{\hat{0}} = c' dr \wedge d\theta = -\frac{c'}{cB} \omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$
$d\omega^{\hat{\varphi}} = -\frac{c'}{cB} \omega^{\hat{\varphi}} \wedge \omega^{\hat{r}} - \underbrace{c \cos\theta \, d\varphi \wedge d\theta}_{\frac{1}{c} \cot\theta \, \omega^{\hat{\varphi}} \wedge \omega^{\hat{\theta}}}$
$\Theta^{\hat{0}}{}_{\hat{r}} = \frac{c'}{cB} \omega^{\hat{0}}$
$\Theta^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{c \cot\theta}{c} \omega^{\hat{\varphi}}$
$\Theta^{\hat{\varphi}}{}_{\hat{r}} = \frac{c'}{cB} \omega^{\hat{\varphi}}$

e.g. To get $\Theta^{\hat{\theta}}{}_{\hat{\varphi}} = \eta^{\hat{\theta}\hat{\alpha}}\,\Theta_{\hat{\alpha}\hat{\varphi}}$

$= -\eta^{\hat{\theta}\hat{\alpha}}\,\Theta_{\hat{\varphi}\hat{\alpha}}$

$= -\eta^{\hat{\theta}\hat{\alpha}}\,\eta_{\hat{\varphi}\hat{b}}\,\Theta^{\hat{b}}{}_{\hat{\alpha}}$

$= -\Theta^{\hat{\varphi}}{}_{\hat{\theta}}$.

$\Theta_{\hat{\theta}\hat{3}1}$

$\Theta_{\hat{\varphi}\hat{3}1}$

is a derivation

commutes with contractions

Leibnizian

) Reduces to $\underline{d}$ on fns

k at $\underline{\nabla}$ on a vector:

$$\underline{\nabla}\,\underline{v} = \underline{\nabla}(V^a \underline{e}_a)$$

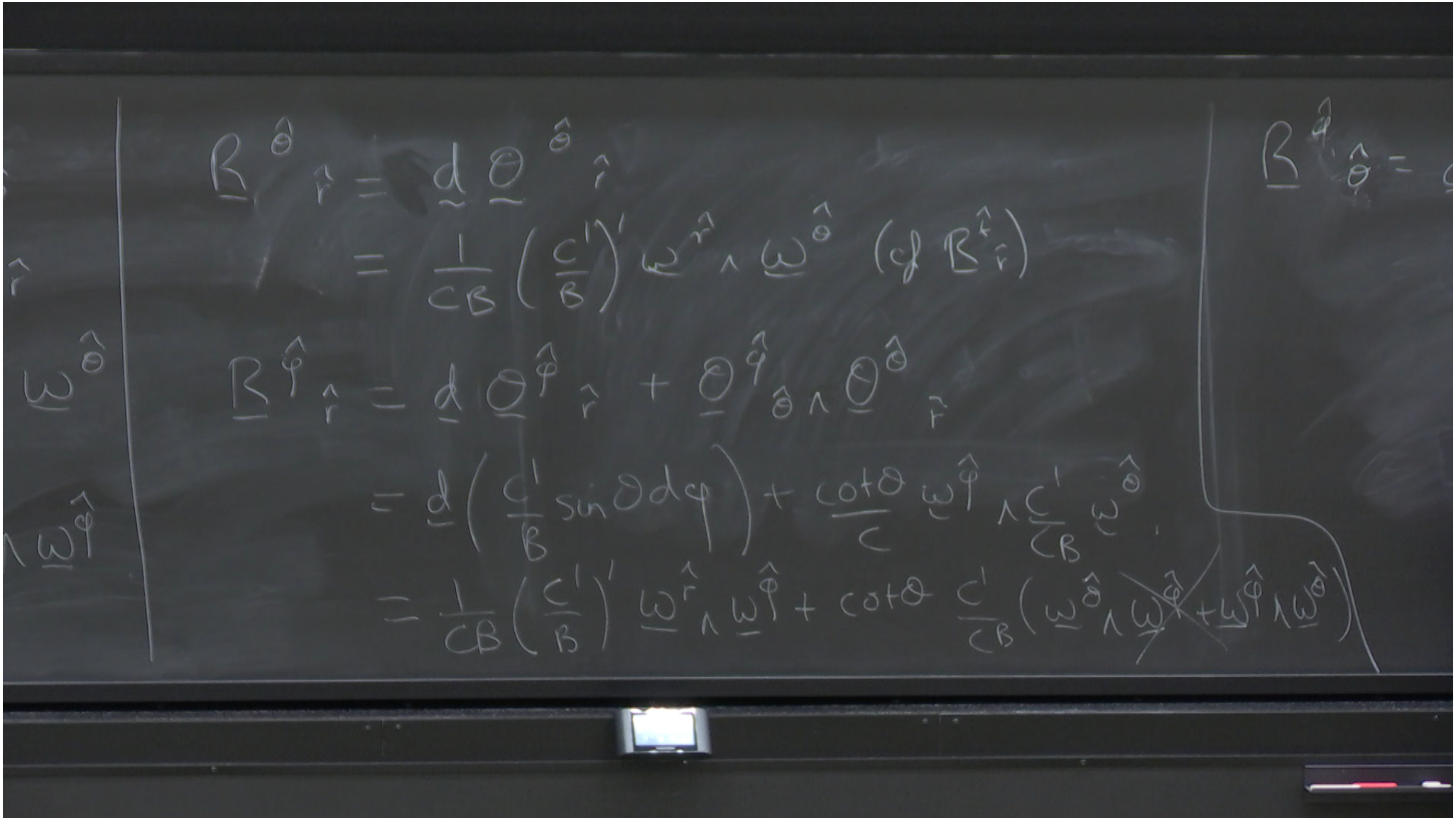
$$= (\underline{\nabla} V^a)\underline{e}_a + V^a \underline{\nabla}\underline{e}_a$$

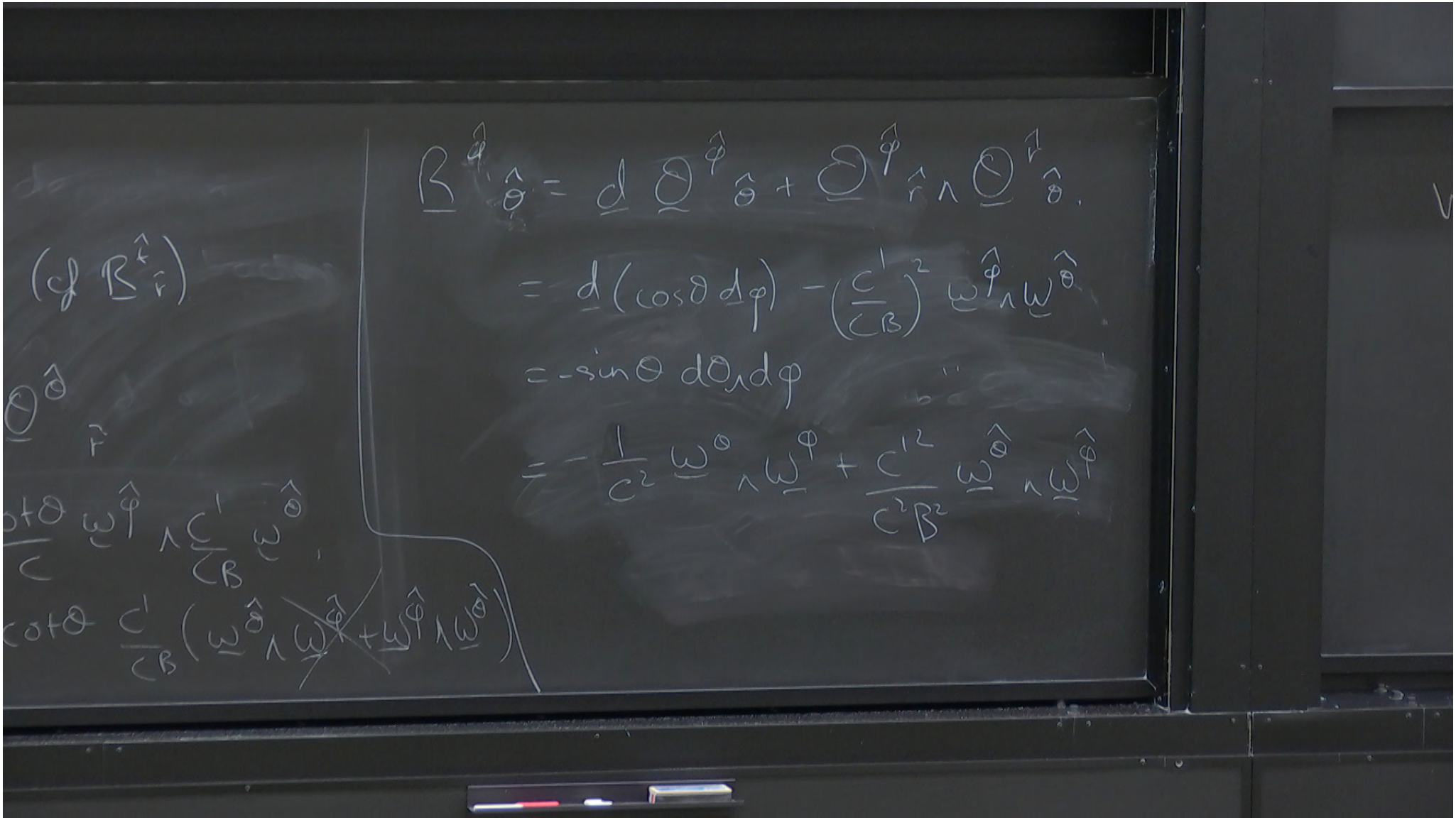
(SCALARS)

$$= (\underline{d} V^a)\underline{e}_a + V^a \Gamma^c_{ba}\, \underline{e}_c \otimes \underline{\omega}^b$$

$$= [\underbrace{V^a{}_{,b} + \Gamma^a_{bc} V^c}]\, \underline{e}_a \otimes \underline{\omega}^b$$

$\nabla_b V^a$ is A.J.N.


$\underline{R}^{\hat{\theta}}{}_{\hat{r}} = \underline{d}\,\underline{\Theta}^{\hat{\theta}}{}_{\hat{r}}$
$= \frac{1}{CB}\left(\frac{C'}{B}\right)' \underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\theta}}$ $(\text{of } \underline{R}^{\hat{r}}{}_{\hat{r}})$
$\underline{R}^{\hat{\varphi}}{}_{\hat{r}} = \underline{d}\,\underline{\Theta}^{\hat{\varphi}}{}_{\hat{r}} + \underline{\Theta}^{\hat{\varphi}}{}_{\hat{\theta}} \wedge \underline{\Theta}^{\hat{\theta}}{}_{\hat{r}}$
$= \underline{d}\left(\frac{C'}{B}\sin\theta\, d\varphi\right) + \frac{\cot\theta}{C}\,\underline{\omega}^{\hat{\varphi}} \wedge \frac{C'}{CB}\,\underline{\omega}^{\hat{\theta}}$
$= \frac{1}{CB}\left(\frac{C'}{B}\right)' \underline{\omega}^{\hat{r}} \wedge \underline{\omega}^{\hat{\varphi}} + \cot\theta\, \frac{C'}{CB}\left(\underline{\omega}^{\hat{\theta}} \wedge \underline{\omega}^{\hat{\varphi}} + \underline{\omega}^{\hat{\varphi}} \wedge \underline{\omega}^{\hat{\theta}}\right)$
$\underline{R}^{\hat{\varphi}}{}_{\hat{\theta}} =$


$R^{\hat{\theta}}{}_{\hat{\theta}} = d\,\Theta^{\hat{\theta}}{}_{\hat{\theta}} + \Theta^{\hat{\theta}}{}_{\hat{r}} \wedge \Theta^{\hat{r}}{}_{\hat{\theta}}$
$= d(\cos\theta\, d\varphi) - \left(\frac{c'}{cB}\right)^2 \omega^{\hat{\varphi}} \wedge \omega^{\hat{\theta}}$
$= -\sin\theta\, d\theta \wedge d\varphi$
$= -\frac{1}{c^2}\omega^{\hat{\theta}} \wedge \omega^{\hat{\varphi}} + \frac{c'^2}{c^2 B^2}\omega^{\hat{\theta}} \wedge \omega^{\hat{\varphi}}$