Title: PSI 2019/2020 - Gravitational Physics - Lecture 3

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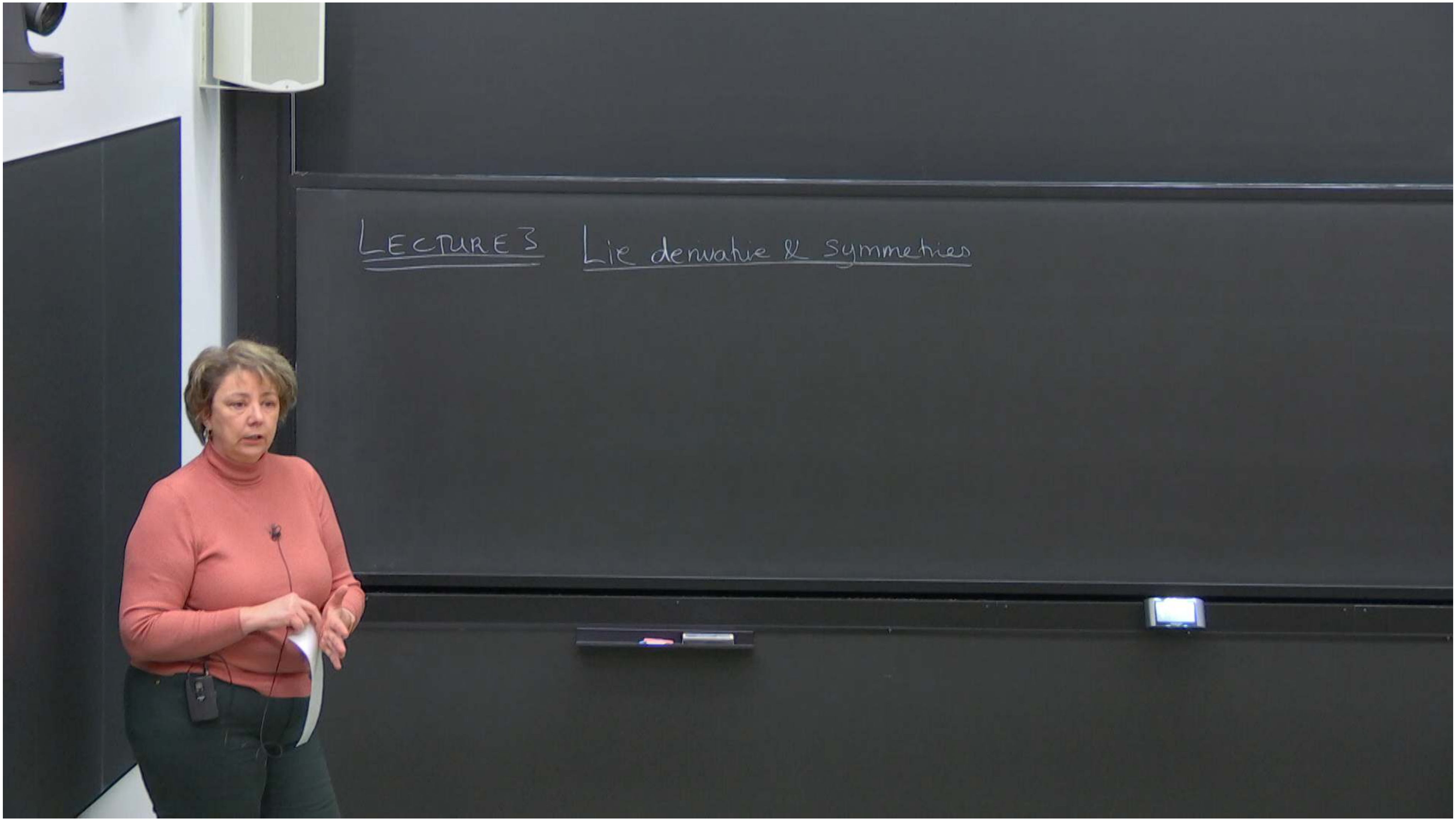

LECTURE 3
Lie derivative & symmetries

LECTURE 3 Lie derivative & symmetries

Recall the defn of derivative:

$$\underbrace{``\frac{dQ}{dt}"}_{?} = \lim_{\delta t \to 0} \frac{Q_{t+\delta t} - Q_t}{\delta t}$$

On a manifold these are different tgt spaces.

what does this mean?

etries

:

On a
manifold
these are
different tgt
spaces.

Return to diffeos,
clearly form a group,
what are the generators?

$$X^\mu \to X^\mu + \delta X^\mu.$$

now look at δX^μ "small"

diffeos,

m a group,

he generator?

$X^\mu + \delta X^\mu$

t δX^μ "small"

Write $\delta X^\mu = \varepsilon \, \xi^\mu$.

small param

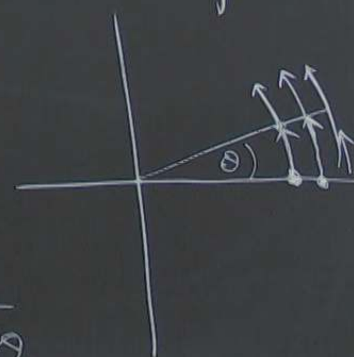
cpts of vector.

Vectors generate coord transfms

e.g. $\mathbb{R}^2$: $x' = x - \varepsilon y$

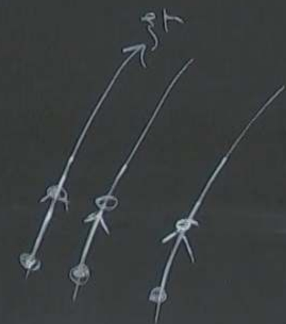
$y' = y + \varepsilon x$

$$\xi^\mu = (-y, x) \leftrightarrow \frac{\partial}{\partial \theta}.$$



The curves to which a vector field, ξ, is tangent are the <u>integral curves</u> of ξ

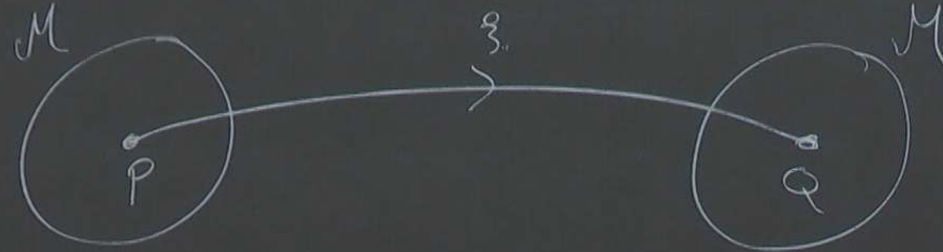
$$X^\mu_\xi = X^\mu_0 + \varepsilon\, \xi^\mu + O(\varepsilon^2)$$



eld, ξ,

of ξ

What are push forward & pull back?



$$p \leftrightarrow x^\mu \longrightarrow Q = x^\mu + \epsilon \xi^\mu$$

under this transformation

$\partial\theta$.

& pull back?

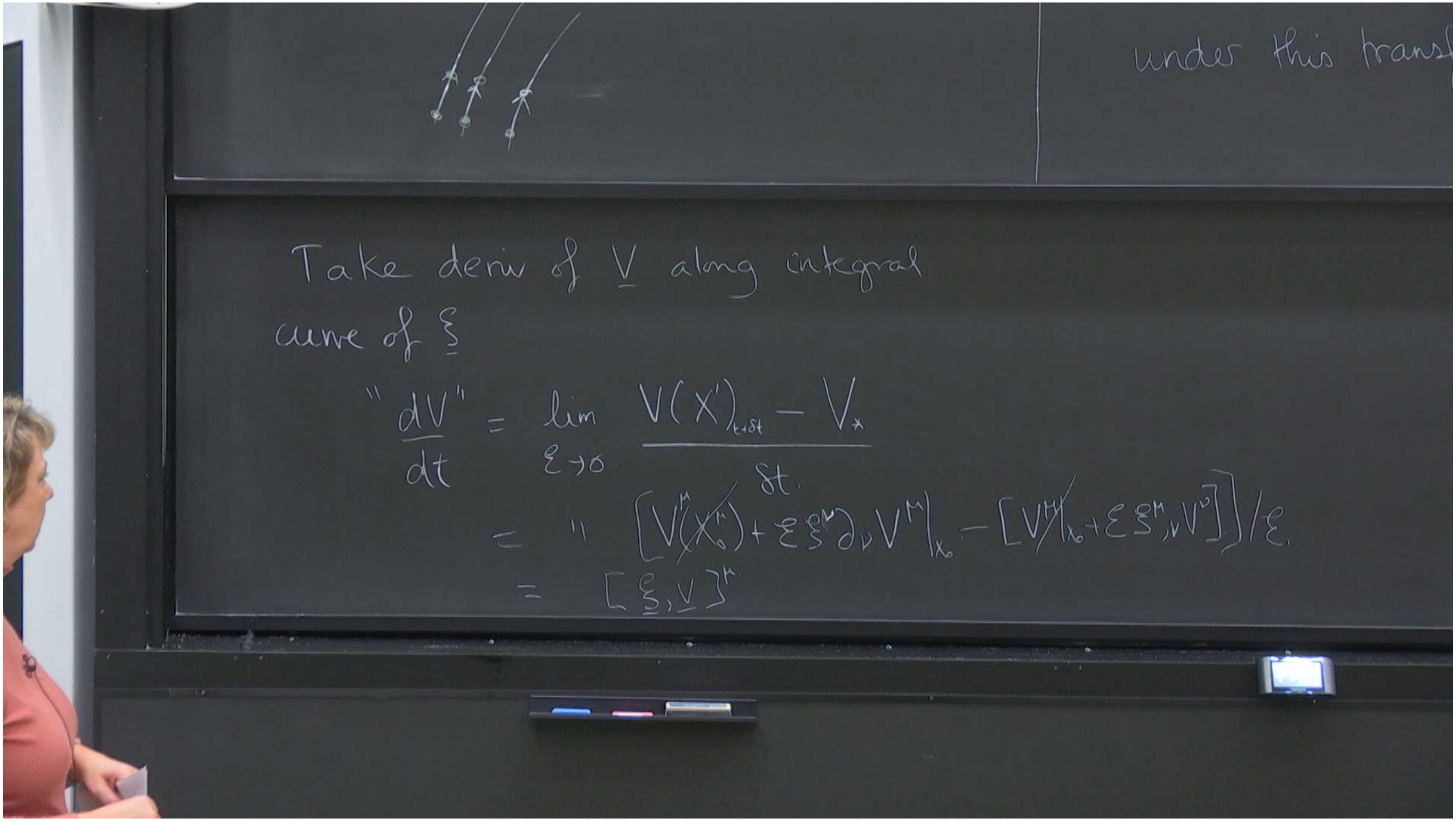
$\mathcal{M}$

$x^\mu + \varepsilon \xi^\mu$.

$$V'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^\mu} V^\mu$$

$$= \left(\delta^{\mu'}_{\mu} + \varepsilon \xi^{\mu'}_{,\mu}\right) V^\mu$$

- push forward on V^μ.



under this transf
Take deriv of $\underline{V}$ along integral
curve of $\underline{\xi}$
$$"\frac{dV}{dt}" = \lim_{\varepsilon \to 0} \frac{V(X')_{t+\delta t} - V_*}{\delta t}$$
$$= \;" \; \left[\cancel{V^\mu(X_0^\mu)} + \varepsilon \xi^\nu \partial_\nu V^\mu|_{X_0} - [\cancel{V^\mu}|_{X_0} + \varepsilon \xi^\mu{}_{,\nu} V^\nu]\right] / \varepsilon$$
$$= [\underline{\xi}, \underline{V}]^\mu$$

Can generalise to arb tensors;
denote this Lie Derivative by $\mathcal{L}_\xi$

$$(\mathcal{L}_\xi \underline{\omega})_\mu = \xi^\nu \partial_\nu \omega_\mu + \xi^\nu{}_{,\mu}\, \omega_\nu$$

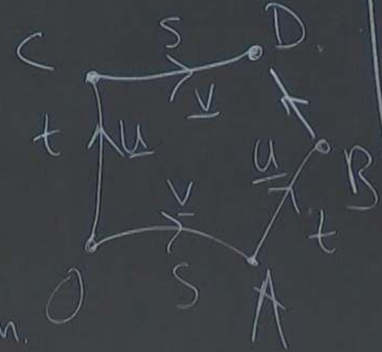
$$(\mathcal{L}_\xi \underline{g})_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + \xi^\sigma{}_{,\mu}\, g_{\sigma\nu} + \xi^\sigma{}_{,\nu}\, g_{\mu\sigma}.$$

Geometry of $\mathcal{L}$

Lie deriv is intrinsically linked to symmetries & coords.

Consider the commutator

$$[\underline{u}, \underline{v}].$$



Let A, B, C, D be as in diagram.

s, t sm

in local

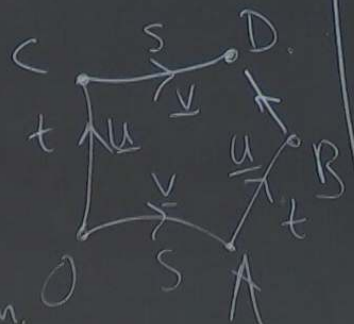
s, t small, perform analysis in local chart.

use
$$\begin{cases} X_A{}^\mu = X_0{}^\mu + s V^\mu + \frac{1}{2} s^2 V^\nu \partial_\nu V^\mu . \\ U_A^\mu = U_0{}^\mu + s V^\nu \partial_\nu U^\mu . \end{cases}$$

...sically linked

...ords.

...mutator



...s in diagram.

s,t small, perform analysis "$\exp[s\underline{V}]$"

in local chart.

use
$$\begin{cases} X_A^{\mu} = X_0^{\mu} + sV^{\mu} + \frac{1}{2}s^2 V^{\nu}\partial_{\nu}V^{\mu} \\ U_A^{\mu} = U_0^{\mu} + sV^{\nu}\partial_{\nu}U^{\mu} \end{cases}$$

$$X_B^{\mu} = X_A^{\mu} + tU^{\mu}\big|_A + \frac{1}{2}t^2 U^{\nu}\partial_{\nu}U^{\mu}\big|_A$$
$$= X_0^{\mu} + sV_0^{\mu} + \frac{1}{2}s^2 V^{\nu}\partial_{\nu}V_0^{\mu}$$
$$+ tU_0^{\mu} + st\,V^{\nu}\partial_{\nu}U_0^{\mu} + \frac{1}{2}t^2 U^{\nu}\partial_{\nu}U_0^{\mu}$$

Similarly

$X_D^{\mu} = X_0^{\mu} + \ldots$

$+ sV\ldots$

Discrepa...

lysis "exp[s$\underline{V}$]"

$\frac{1}{2} s^2 V^\nu \partial_\nu V^\mu$

$^\nu \partial_\nu U^\mu$

$t^2 U^\nu \partial_\nu U^\mu |_A$

$V^\nu \partial_\nu V^\mu_0$

$U^\mu_0 + \frac{1}{2} t^2 U^\nu \partial_\nu U^\mu_0$

Similarly

$$X_D^\mu = X_0^\mu + t\, U_0^\mu + \frac{1}{2} t^2 U^\nu \partial_\nu U^\mu_0 + s\, V_0^\mu + st\, U^\nu \partial_\nu V_0^\mu + \frac{1}{2} s^2 V^\nu \partial_\nu V^\mu_0$$

Discrepancy between D & B:

$$X_D^\mu - X_B^\mu = st\left(U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu U^\mu\right) = st\,[\underline{U}, \underline{V}]^\mu$$

metas $(\underline{U}\,\underline{V} - \underline{V}\,\underline{U}) f$

If $[\underline{u}, \underline{v}] \neq 0$, can't use (s,t) as coords.

If $\mathcal{L}_{\xi} Q = 0$, then Q is unchanged as we Lie drag along integral curves of $\underline{\xi}$.

s coords.

ged as

ξ.

al curves

ce.

<u>Defn</u> A <u>Killing vector</u> is a vector field along which the metric is invariant:

$$\xi^\sigma \partial_\sigma g_{\mu\nu} + \xi^\sigma{}_{,\mu} g_{\sigma\nu} + \xi^\sigma{}_{,\nu} g_{\mu\sigma} = 0.$$

$$\mathcal{L}_3\, g = 0 \;\Rightarrow\; g_{\mu\nu,\varphi} = 0 \qquad \text{no } \varphi\text{-dep.}$$

$$\mathcal{L}_{1\,2} \quad \underset{\cos}{\sin}\varphi\; \underline{g_{\mu\nu,\theta}} + \underset{-\sin}{\cos}\varphi \left(g_{\theta\nu}\,\delta^{\varphi}_{\mu} + g_{\theta\mu}\,\delta^{\varphi}_{\nu} \right)$$

$$- \csc^2\theta\, \underset{-\sin}{\cos}\varphi \left(g_{\varphi\nu}\,\delta^{\theta}_{\mu} + g_{\varphi\mu}\,\delta^{\theta}_{\nu} \right)$$

$$- \cot\theta\, \underset{\cos}{\sin}\varphi \left(\underline{g_{\varphi\nu}\,\delta^{\varphi}_{\mu}} + g_{\varphi\mu}\,\delta^{\varphi}_{\nu} \right)$$

$\xi_3 = \partial_\varphi$.

φ - dep.

$$\begin{cases} g_{\mu\nu,\theta} = \cot\theta \left(g_{\varphi\nu}\,\delta^{\varphi}_{\mu} + g_{\varphi\mu}\,\delta^{\varphi}_{\nu} \right) \\ g_{\theta\nu}\,\delta^{\varphi}_{\mu} + g_{\theta\mu}\,\delta^{\varphi}_{\nu} = \csc^2\theta \left(g_{\varphi\nu}\,\delta^{\theta}_{\mu} + g_{\theta\mu}\,\delta^{\varphi}_{\nu} \right) \end{cases}$$

(sin) (cos)

e.g. $\mu = \varphi$, $\nu = t$. $\rightarrow g_{\theta t} = 0$. also, $g_{t\varphi} = g_{r\theta} = g_{r\varphi} = 0$

$\mu = \nu = \theta$: $g_{\theta\theta,\theta} = 0$

$\mu = \nu = \varphi$: $g_{\varphi\varphi,\theta} = 2\cot\theta\, g_{\varphi\varphi} \rightarrow g_{\varphi\varphi} \propto \sin^2\theta$.

$2 g_{\theta\varphi} = 0$

$\mu = \theta$ $\nu = \varphi$.

$g_{\theta\theta} = g_{\varphi\varphi} \csc^2\theta$.

under this trans

Put together:

$$ds^2 = \underbrace{\gamma^{(t,r)}_{\mu\nu}\, dx^\mu dx^\nu}_{\mu,\nu = t,r} - C^2(t,r)\,\Big[\underbrace{d\theta^2 + \sin^2\theta\, d\phi^2}_{\text{metric on a unit } S^2}\Big]$$

under this transformation

A useful relation between $\mathcal{L}$ & d:

$$\langle \underline{d}\underline{\omega} \mid [\underline{u},\underline{v}] \rangle = \underline{u}(\langle \underline{\omega} | \underline{v} \rangle) - \underline{v}(\langle \underline{\omega} | \underline{u} \rangle) - \langle \underline{\omega} \mid [\underline{u},\underline{v}] \rangle$$