Title: PSI 2019/2020 - Gravitational Physics - Lecture 3

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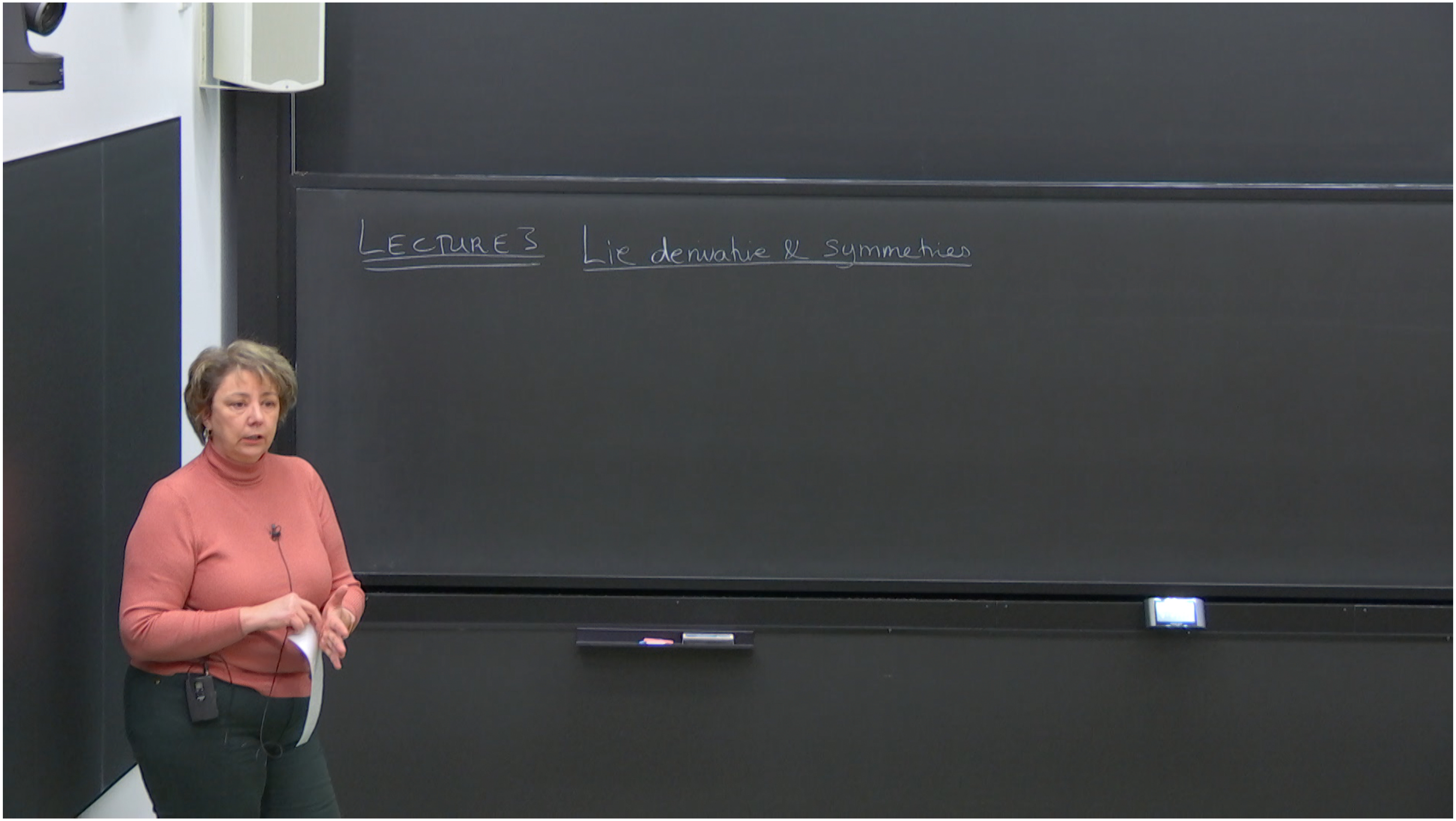

LECTURE 3
Lie derivative & symmetries

LECTURE 3 Lie derivative & symmetries

Recall the defn of derivative:

$$\text{“}\frac{dQ}{dt}\text{”} = \lim_{\delta t \to 0} \frac{Q_{t+\delta t} - Q_t}{\delta t}$$

On a manifold these are different tgt spaces.

?

what does this mean?

etries

:

On a
manifold
these are
different tgt
spaces.

Return to diffeos,

clearly form a group,

what are the generators?

$$X^\mu \to X^\mu + \delta X^\mu.$$

now look at δX^μ "small"

diffeos,

m a group,

he generator?

$X^\mu + \delta X^\mu$

t δX^μ "small"

Write $\delta X^\mu = \varepsilon \xi^\mu$.

small param

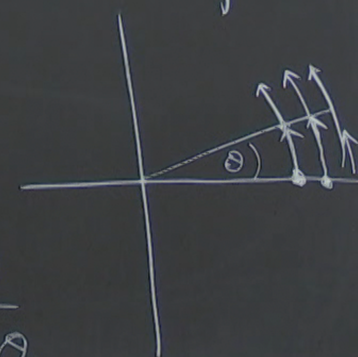
cpts of vector.

Vectors generate coord transfms

e.g. $\mathbb{R}^2$: $x' = x - \varepsilon y$

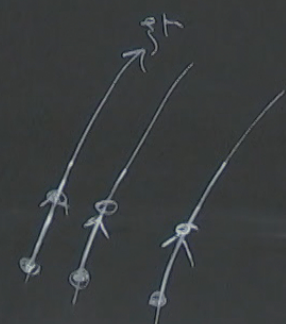
$y' = y + \varepsilon x$

$\xi^\mu = (-y, x) \leftrightarrow \frac{\partial}{\partial \theta}$.



The curves to which a vector field, ξ, is tangent are the <u>integral curves</u> of ξ

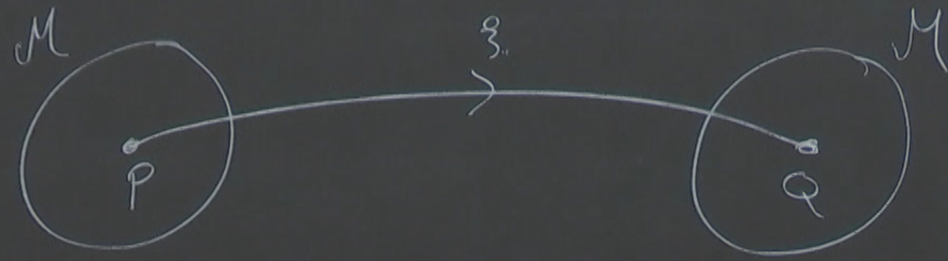
$$X^\mu_\xi = X^\mu_0 + \varepsilon\, \xi^\mu + O(\varepsilon^2)$$



eld, ξ,

of ξ

What are push forward & pull back?



$$P \leftrightarrow X^\mu \longrightarrow Q = X^\mu + \epsilon \xi^\mu$$

under this transformation

$\partial\theta.$

& pull back?

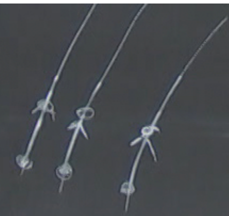
M

$x^\mu + \varepsilon \xi^\mu.$

$$V'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^\mu} V^\mu$$

$$= \left(\delta^{\mu'}_{\mu} + \varepsilon \xi^{\mu'}_{,\mu}\right) V^\mu$$

- push forward on V^μ.



under this transf

Take deriv of $\underline{V}$ along integral curve of $\underline{\xi}$

$$\text{"}\frac{dV}{dt}\text{"} = \lim_{\varepsilon \to 0} \frac{V(X')_{t+\delta t} - V_*}{\delta t}$$

$$= \text{"} \left[\cancel{V^\mu(X_0^\mu)} + \varepsilon \xi^\nu \partial_\nu V^\mu \big|_{x_0} - \left[\cancel{V^\mu}\big|_{x_0} + \varepsilon \xi^\mu{}_{,\nu} V^\nu \right] \right] / \varepsilon$$

$$= [\underline{\xi}, \underline{V}]^\mu$$

Can generalise to arb tensors;
denote this Lie Derivative by $\mathcal{L}_\xi$

$$(\mathcal{L}_\xi \underline{\omega})_\mu = \xi^\nu \partial_\nu \omega_\mu + \xi^\nu{}_{,\mu}\, \omega_\nu$$

$$(\mathcal{L}_\xi \underline{g})_{\mu\nu} = \xi^\sigma \partial_\sigma g_{\mu\nu} + \xi^\sigma{}_{,\mu}\, g_{\sigma\nu} + \xi^\sigma{}_{,\nu}\, g_{\mu\sigma}.$$

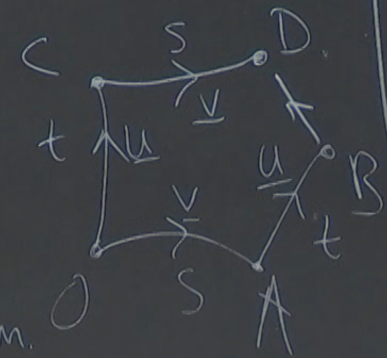
Geometry of $\mathcal{L}$

Lie deriv is intrinsically linked

to symmetries & coords.

Consider the commutator

$$[\underline{u}, \underline{v}].$$



Let A, B, C, D be as in diagram.

s,t sm

in local

s,t small, perform analysis

in local chart.

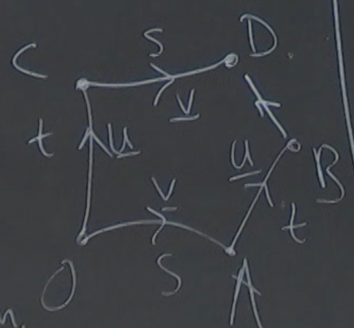
use
$$\begin{cases} X_A{}^{\mu} = X_0{}^{\mu} + sV^{\mu} + \frac{1}{2}s^2 V^{\nu}\partial_{\nu}V^{\mu} \\ U_A^{\mu} = U_0{}^{\mu} + sV^{\nu}\partial_{\nu}U^{\mu} \end{cases}$$

£

...sically linked

...ords.

...mutator



...s in diagram.

s,t small, perform analysis in local chart. "exp[s$\underline{V}$]"

use $\begin{cases} X_A^{\mu} = X_0^{\mu} + sV^{\mu} + \frac{1}{2} s^2 V^{\nu} \partial_{\nu} V^{\mu} \\ U_A^{\mu} = U_0^{\mu} + s V^{\nu} \partial_{\nu} U^{\mu} \end{cases}$

$$X_B^{\mu} = X_A^{\mu} + t U^{\mu}\big|_A + \frac{1}{2} t^2 U^{\nu} \partial_{\nu} U^{\mu}\big|_A$$

$$= X_0^{\mu} + sV_0^{\mu} + \frac{1}{2} s^2 V^{\nu} \partial_{\nu} V_0^{\mu}$$

$$+ t U_0^{\mu} + st V^{\nu} \partial_{\nu} U_0^{\mu} + \frac{1}{2} t^2 U^{\nu} \partial_{\nu} U_0^{\mu}$$

Similarly

$X_D^{\mu} = X_0^{\mu} + \ldots$

$+ s V \ldots$

Discrepa...

lysis "$\exp[s\underline{V}]$"

$\frac{1}{2} s^2 V^\nu \partial_\nu V^\mu$

$^\nu \partial_\nu U^\mu$

$t^2 U^\nu \partial_\nu U^\mu \big|_A$

$V^\nu \partial_\nu V^\mu_0$

$U_0^\mu + \frac{1}{2} t^2 U^\nu \partial_\nu U^\mu_0$

Similarly

$$X_D^\mu = X_0^\mu + t U_0^\mu + \frac{1}{2} t^2 U^\nu \partial_\nu U^\mu_0$$
$$+ s V_0^\mu + st U^\nu \partial_\nu V_0^\mu + \frac{1}{2} s^2 V^\nu \partial_\nu V^\mu_0$$

Discrepancy between D & B:

$$X_D^\mu - X_B^\mu = st\left(U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu U^\mu\right)$$
$$= st\,[\underline{U}, \underline{V}]^\mu$$

metas $\left(\underline{U}\,\underline{V} - \underline{V}\,\underline{U}\right) f$

If $[\underline{u}, \underline{v}] \neq 0$, can't use (s,t) as coords.

If $\mathcal{L}_\xi Q = 0$, then Q is unchanged as we Lie drag along integral curves of ξ.

s coords.

ged as

ξ.

al curves

ce.

<u>Defn</u> A <u>Killing vector</u> is a vector field along which the metric is invariant:

$$\xi^\sigma \partial_\sigma g_{\mu\nu} + \xi^\sigma{}_{,\mu}\, g_{\sigma\nu} + \xi^\sigma{}_{,\nu}\, g_{\mu\sigma} = 0.$$

$$\mathcal{L}_3\, g = 0 \;\Rightarrow\; g_{\mu\nu,\varphi} = 0 \qquad \text{no } \varphi\text{-dep.}$$

$$\mathcal{L}_{1,2} \quad \underset{\cos}{\sin\varphi}\; \underline{g_{\mu\nu,\theta}} + \underset{-\sin}{\cos\varphi}\left(g_{\theta\nu}\,\delta^{\varphi}_{\mu} + g_{\theta\mu}\,\delta^{\varphi}_{\nu}\right)$$

$$-\csc^2\theta\, \underset{-\sin}{\cos\varphi}\left(g_{\varphi\nu}\,\delta^{\theta}_{\mu} + g_{\varphi\mu}\,\delta^{\theta}_{\nu}\right)$$

$$-\cot\theta\, \underset{\cos}{\sin\varphi}\left(\underline{g_{\varphi\nu}\,\delta^{\varphi}_{\mu}} + g_{\varphi\mu}\,\delta^{\varphi}_{\nu}\right)$$

$\xi_3 = \partial_\varphi$.

φ-dep.

$$\begin{cases} g_{\mu\nu,\theta} = \cot\theta\,(g_{\varphi\nu}\,\delta^\varphi_\mu + g_{\varphi\mu}\,\delta^\varphi_\nu) \\ g_{\theta\nu}\,\delta^\varphi_\mu + g_{\theta\mu}\,\delta^\varphi_\nu = \csc^2\theta\,(g_{\varphi\nu}\,\delta^\theta_\mu + g_{\theta\mu}\,\delta^\varphi_\nu) \end{cases}$$

(sin) (cos)

e.g. $\mu=\varphi$, $\nu=t$. $\rightarrow$ $g_{\theta t}=0$. also, $g_{t\varphi} = g_{r\theta} = g_{r\varphi} = 0$

$\mu=\nu=\theta$: $g_{\theta\theta,\theta} = 0$.

$\mu=\nu=\varphi$: $g_{\varphi\varphi,\theta} = 2\cot\theta\, g_{\varphi\varphi} \rightarrow g_{\varphi\varphi} \propto \sin^2\theta$.

$2g_{\theta\varphi} = 0$.

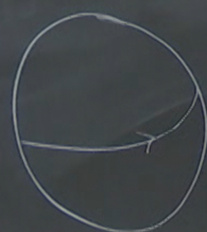
$\mu=\theta$ $\nu=\varphi$.

$g_{\theta\theta} = g_{\varphi\varphi}\csc^2\theta$.

under this trans

Put together:

$$ds^2 = \underbrace{\gamma^{(t,r)}_{\mu\nu} dx^\mu dx^\nu}_{\mu,\nu = t,r} - C^2(t,r)\left[\underbrace{d\theta^2 + \sin^2\theta \, d\phi^2}_{\text{metric on a unit } S^2}\right]$$



under this transformation

A useful relation between $\mathcal{L}$ & d:

$$\langle \underline{d}\,\underline{\omega} \,|\, [\underline{u},\underline{v}] \rangle = \underline{u}\left(\langle \underline{\omega}|\underline{v}\rangle\right) - \underline{v}\left(\langle \underline{\omega}|\underline{u}\rangle\right) - \langle \underline{\omega} \,|\, [\underline{u},\underline{v}] \rangle$$