Title: PSI 2019/2020 - Gravitational Physics - Lecture 2

Speakers: Ruth Gregory

Collection: PSI 2019/2020 - Gravitational Physics

Date: January 07, 2020 - 10:15 AM

URL: http://pirsa.org/20010046

LECTURE 2 Exterior derivative & forms

Recall, wrote $\underline{\omega} = \underset{\uparrow \atop \text{cpts}}{\omega_\mu} \, \underset{\uparrow \atop \text{coord basis for } T^*(\mathcal{M})}{dx^\mu}$

Dual basis satisfies $\langle dx^\mu | \frac{\partial}{\partial x^\nu} \rangle = \delta^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu}$

forms

x^μ

↖

coord basis

for $T^*(\mathcal{M})$

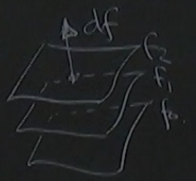
$$\frac{\partial}{\partial x^\nu} \rangle = \delta^\mu{}_\nu = \frac{\partial x^\mu}{\partial x^\nu}$$

Can extend to $C^\infty(\mathcal{M})$:

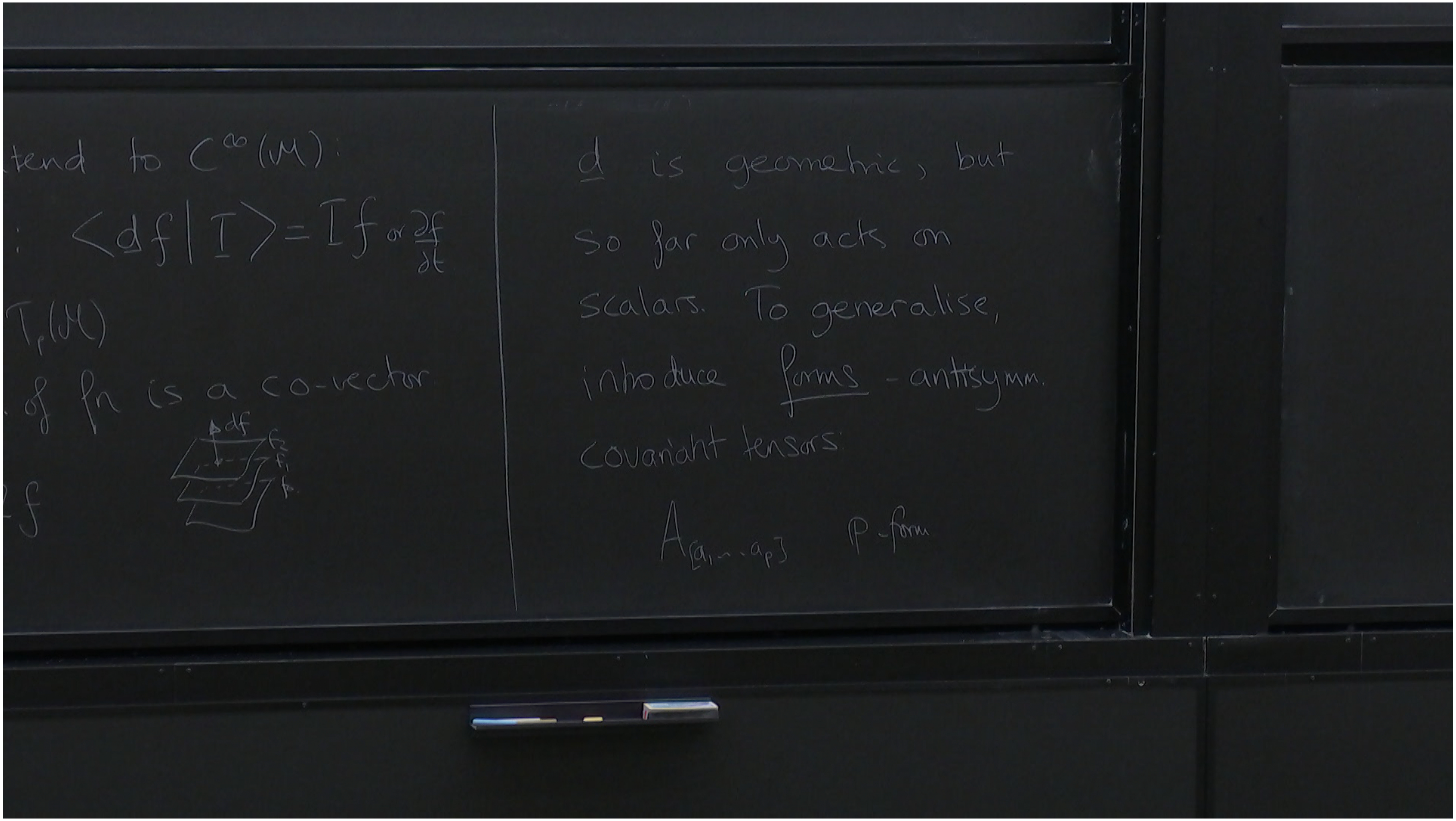
$$\underline{d}f : \langle \underline{d}f | \underline{T} \rangle = \underline{T}f \text{ or } \frac{\partial f}{\partial t}$$

$\forall\ \underline{T} \in T_p(\mathcal{M})$

Deriv. of fn is a co-vector



c.f. $\underline{\nabla} f$



tend to $C^\infty(\mathcal{M})$:

: $\langle \underline{d}f | \underline{T} \rangle = \underline{T}f$ or $\frac{\partial f}{\partial t}$

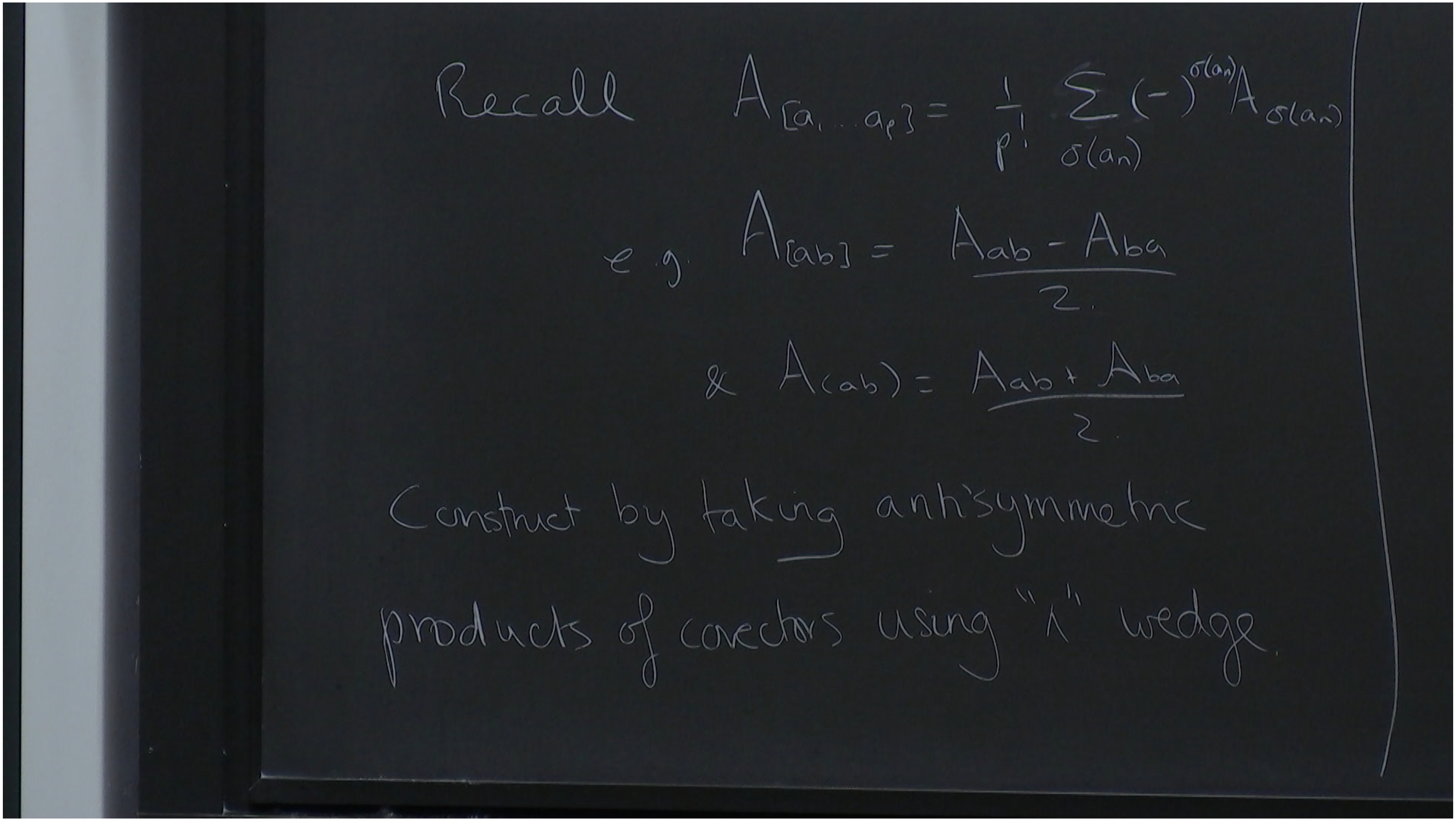
$T_p(\mathcal{M})$

of fn is a co-vector

f

$\underline{d}$ is geometric, but so far only acts on scalars. To generalise, introduce forms - antisymm. covariant tensors:

$A_{[a_1 \dots a_p]}$ p-form



Recall $A_{[a_1 \dots a_p]} = \frac{1}{p!} \sum_{\sigma(a_n)} (-)^{\sigma(a_n)} A_{\sigma(a_n)}$

e.g. $A_{[ab]} = \frac{A_{ab} - A_{ba}}{2}.$

& $A_{(ab)} = \frac{A_{ab} + A_{ba}}{2}.$

Construct by taking antisymmetric products of covectors using "$\wedge$" wedge.

$(-)^{\sigma(a_n)} A_{\sigma(a_n)}$

n)

$\underline{ba}$

$\underline{A_{ba}}$

etric

wedge.

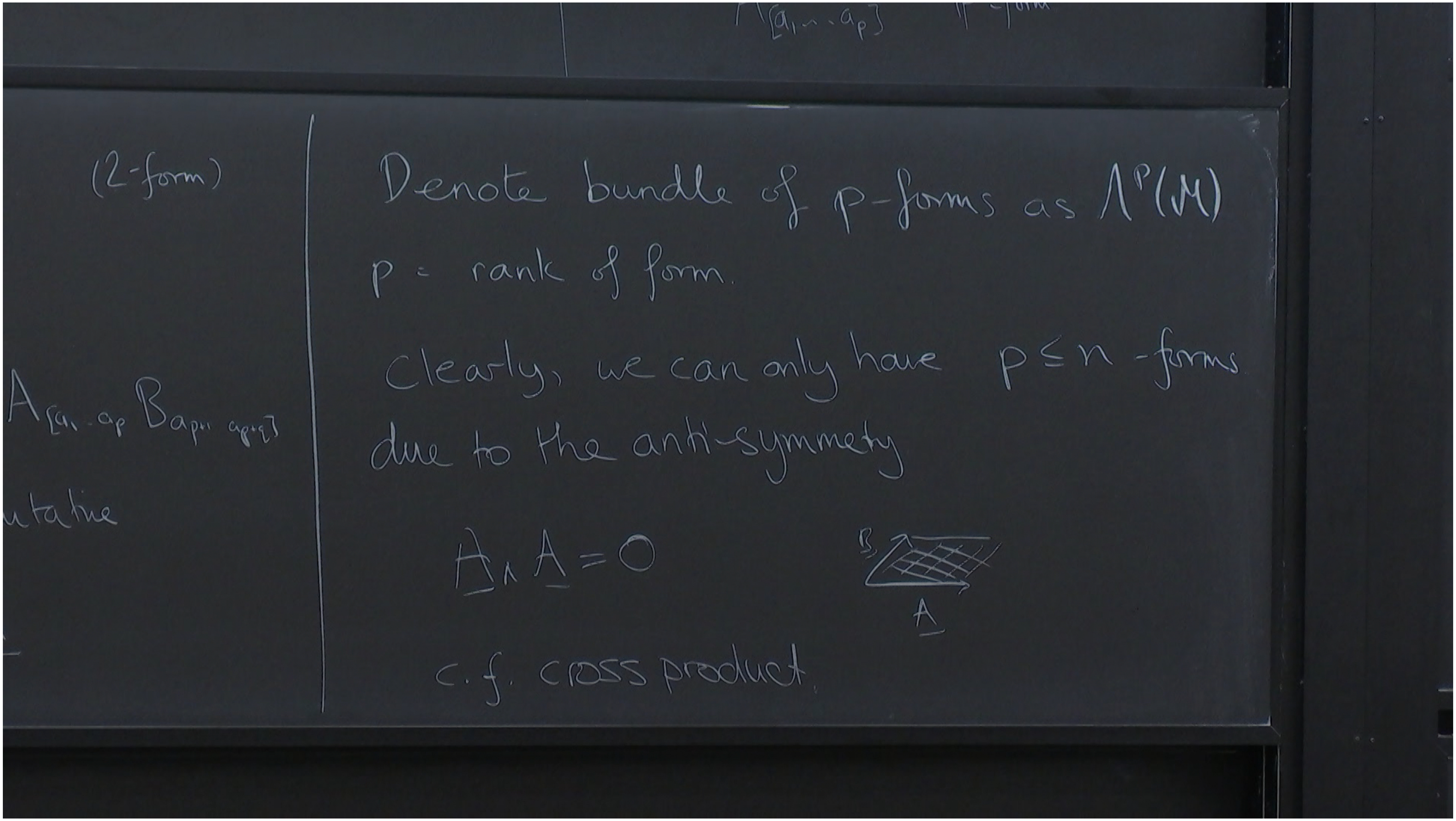
$$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A} \qquad (2\text{-form})$$

In general

$$[\underline{A}^{(p)} \wedge \underline{B}^{(q)}]_{a_1 \ldots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}$$

wedge linear but not commutative

$$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq} \underline{B} \wedge \underline{A}$$



$\Lambda_{[a_1 \dots a_p]}$ p-form

(2-form)

$A_{[a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}]}$

utative

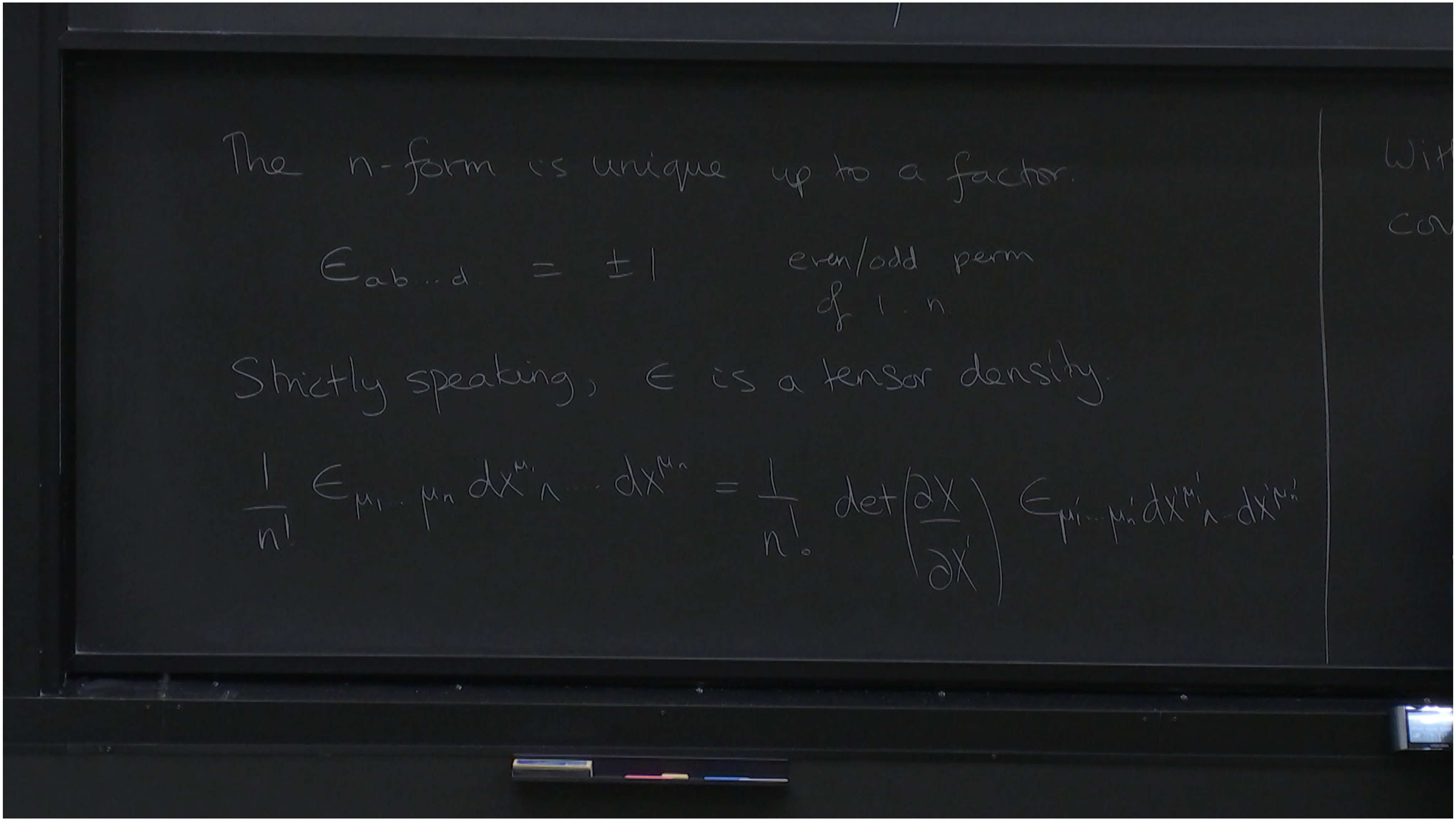
Denote bundle of p-forms as $\Lambda^p(\mathcal{M})$

p = rank of form.

Clearly, we can only have $p \leq n$ -forms due to the anti-symmetry

$\underline{A} \wedge \underline{A} = 0$

c.f. cross product.



The n-form is unique up to a factor.

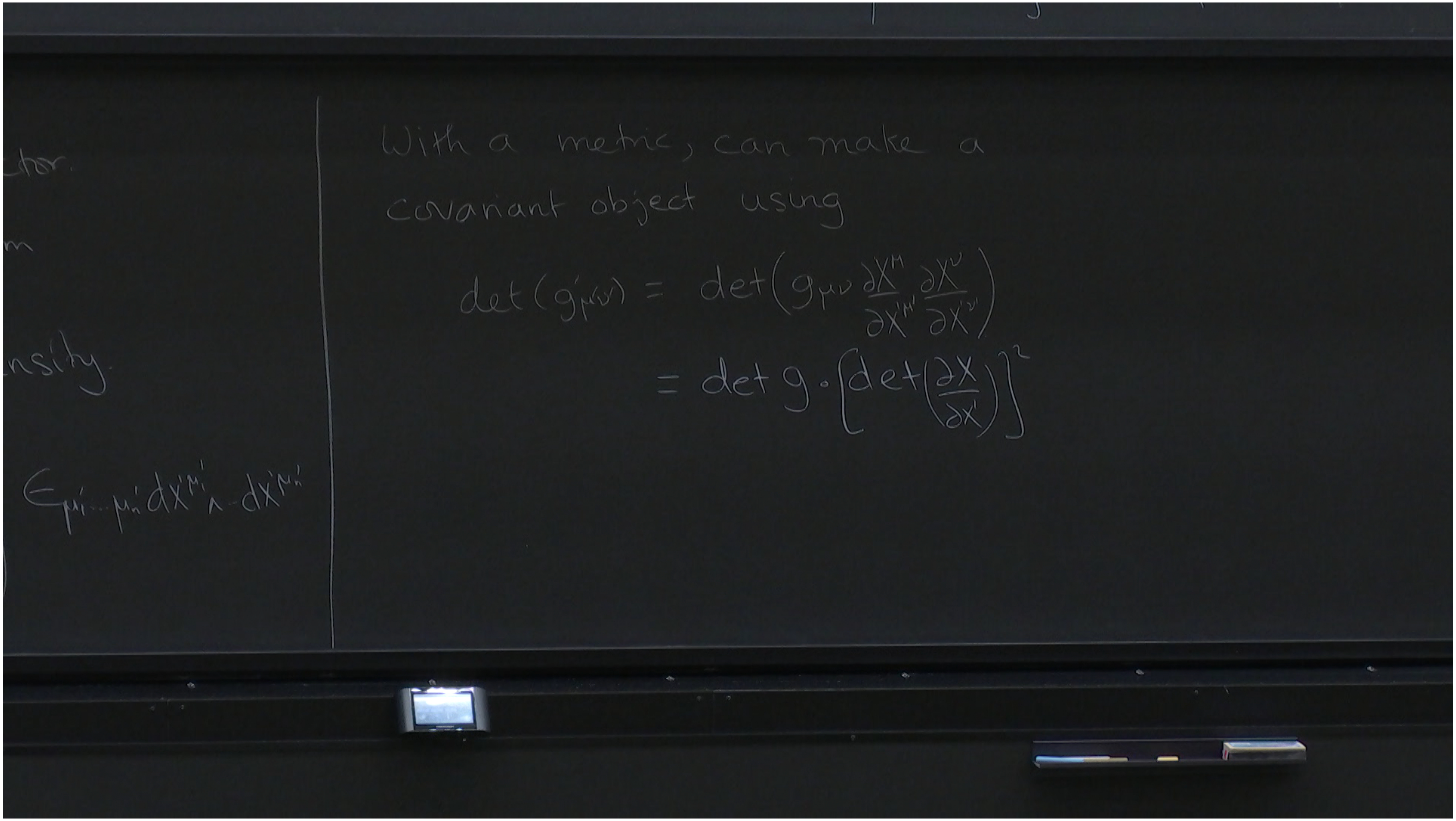
$\epsilon_{ab\ldots d} = \pm 1$ even/odd perm of $1 \ldots n$

Strictly speaking, ϵ is a tensor density.

$$\frac{1}{n!}\epsilon_{\mu_1\ldots\mu_n}dx^{\mu_1}\wedge\ldots dx^{\mu_n} = \frac{1}{n!}\det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\mu_1'\ldots\mu_n'}dx'^{\mu_1'}\wedge\ldots dx'^{\mu_n'}$$

Wit

cov



ctor.

m

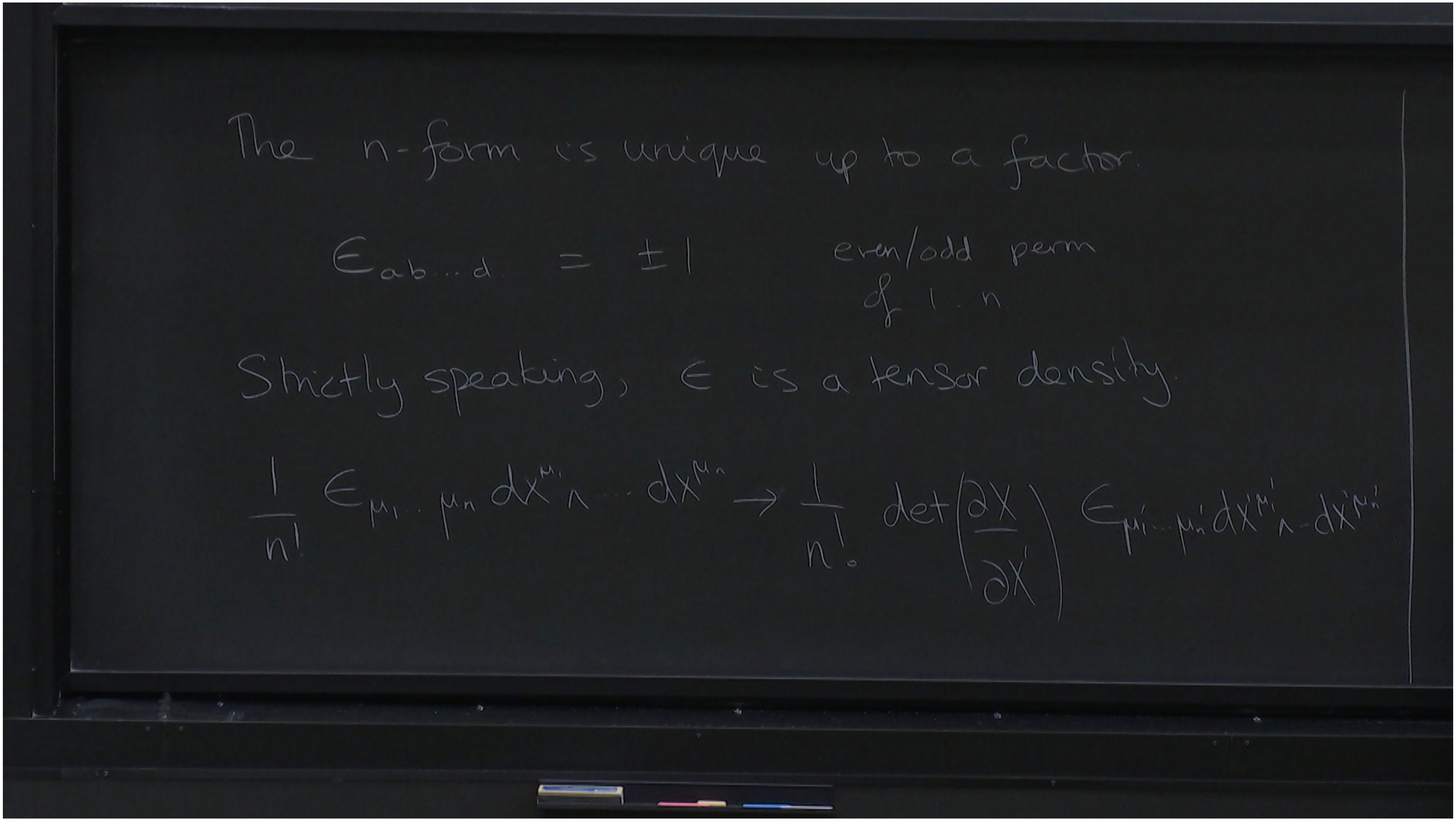
nsity.

$\epsilon_{\mu'_1 \dots \mu'_n} dx'^{\mu'_1} \wedge \dots dx'^{\mu'_n}$

With a metric, can make a covariant object using

$$\det(g'_{\mu'\nu'}) = \det\left(g_{\mu\nu}\frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\right)$$

$$= \det g \cdot \left[\det\left(\frac{\partial x}{\partial x'}\right)\right]^{2}$$

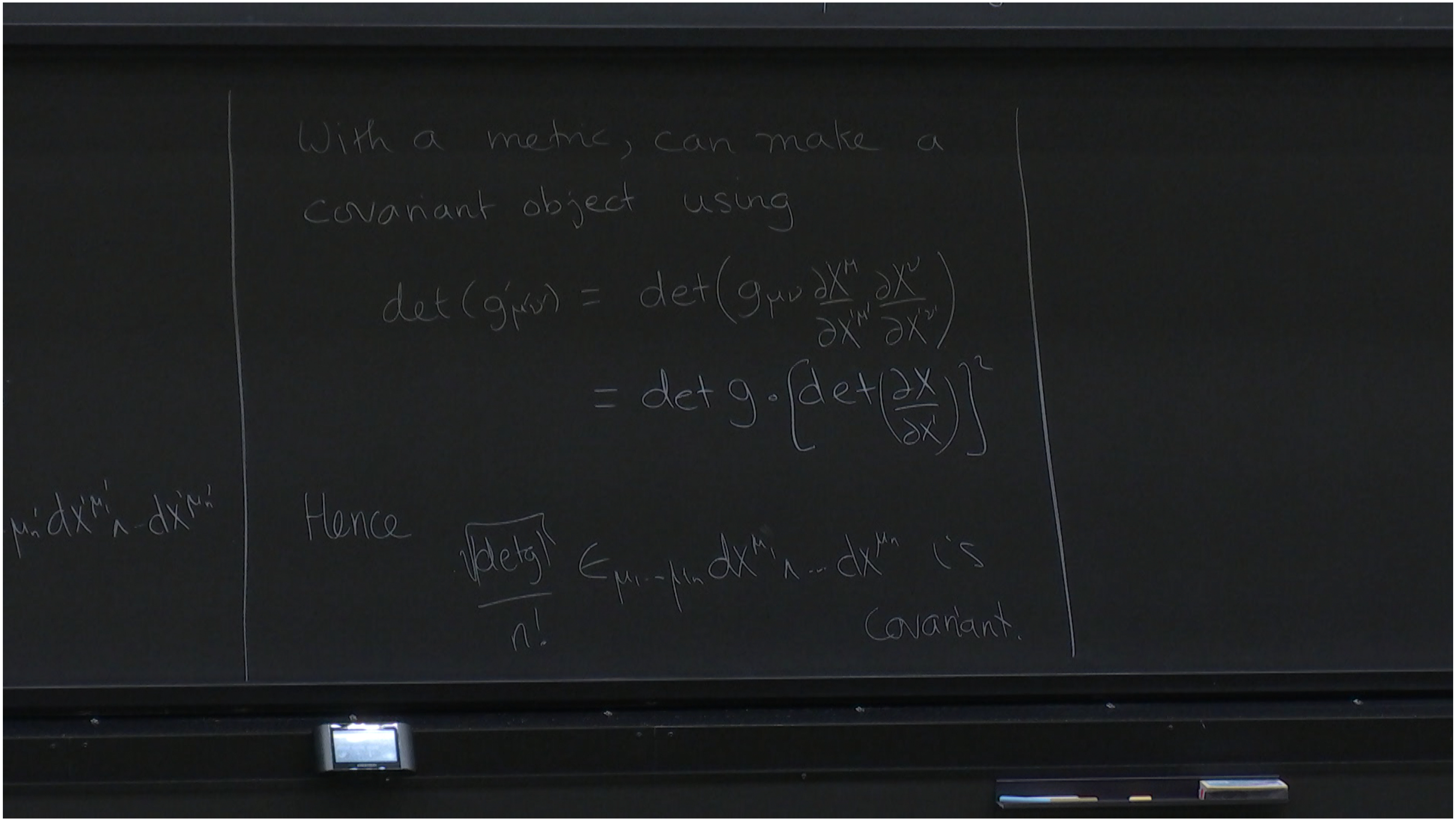


The n-form is unique up to a factor.

$$\epsilon_{ab\ldots d} = \pm 1$$ even/odd perm of 1 .. n

Strictly speaking, ϵ is a tensor density.

$$\frac{1}{n!}\epsilon_{\mu_1\ldots\mu_n} dx^{\mu_1}\wedge\cdots dx^{\mu_n} \to \frac{1}{n!}\det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\mu_1'\ldots\mu_n'} dx'^{\mu_1'}\wedge\cdots dx'^{\mu_n'}$$



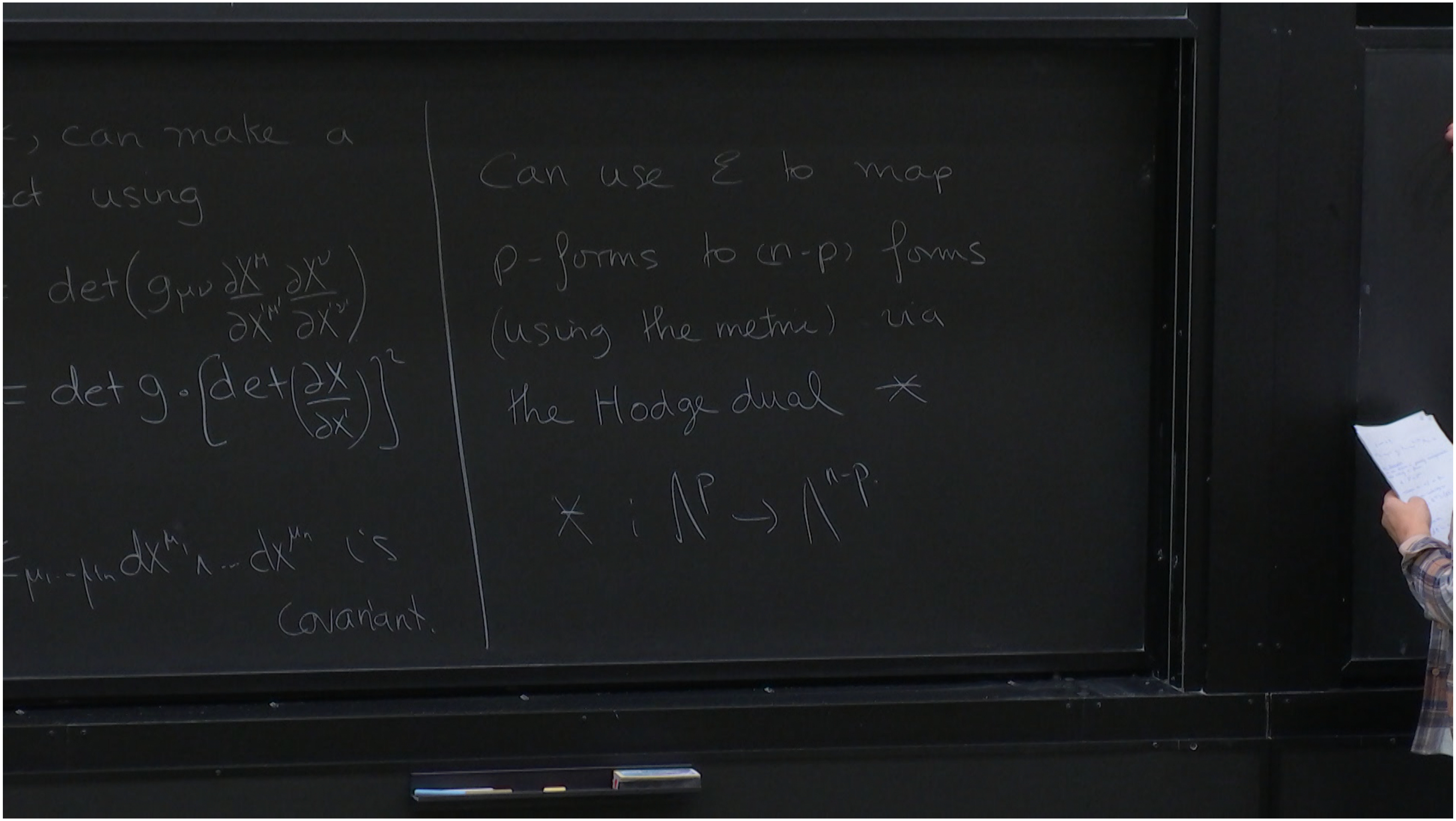
$\mu_n' dx^{\mu_1'} \wedge \dots dx^{\mu_n'}$

With a metric, can make a covariant object using

$$\det(g'_{\mu'\nu'}) = \det\left(g_{\mu\nu} \frac{\partial X^\mu}{\partial X'^{\mu'}} \frac{\partial X^\nu}{\partial X'^{\nu'}}\right)$$

$$= \det g \cdot \left[\det\left(\frac{\partial X}{\partial X'}\right)\right]^2$$

Hence $\frac{\sqrt{|\det g|}}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots dx^{\mu_n}$ is covariant.



…, can make a
…ct using

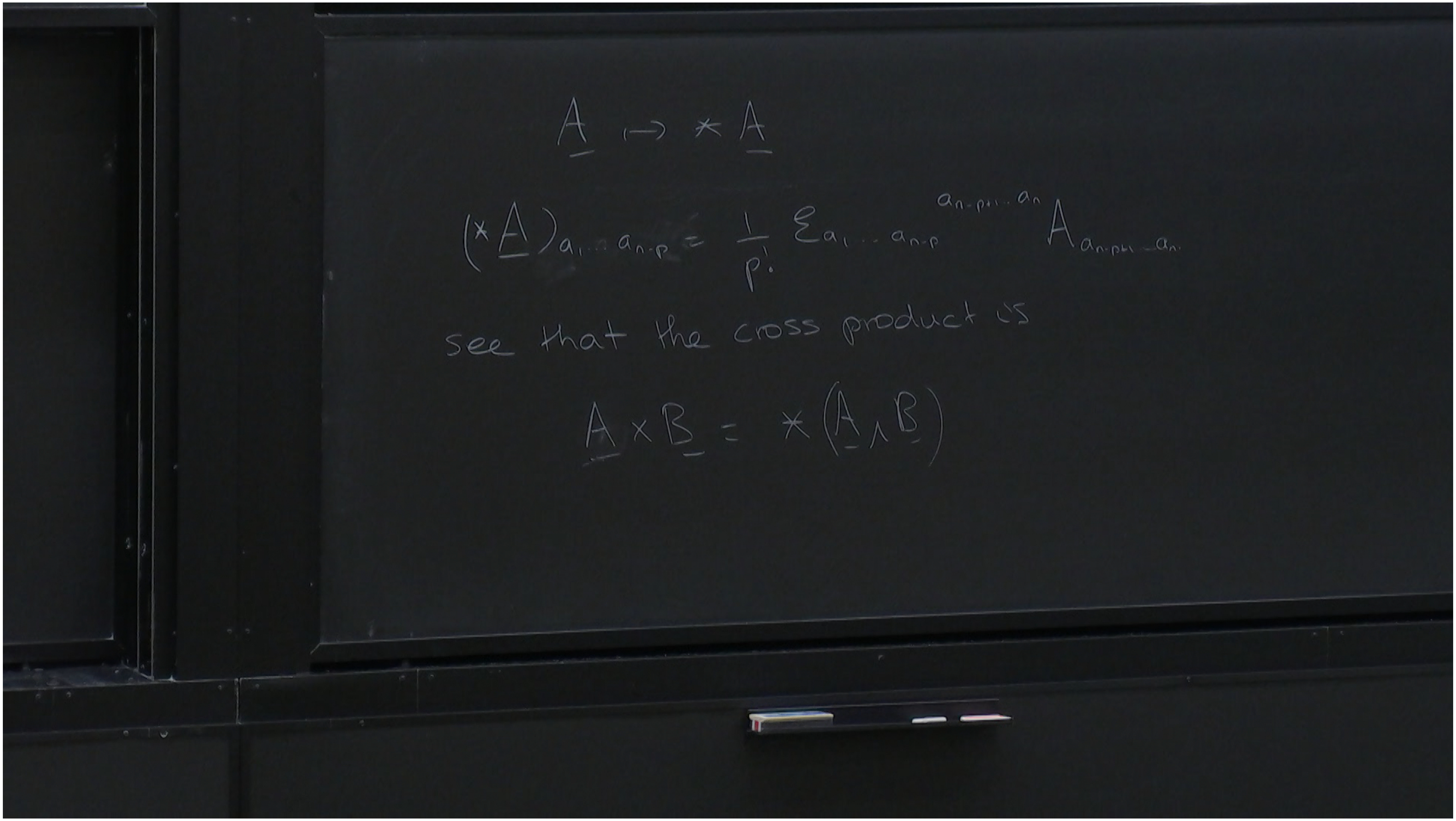
$$\ldots \det\left(g_{\mu\nu}\frac{\partial X^{\mu}}{\partial X'^{\mu'}}\frac{\partial X^{\nu}}{\partial X'^{\nu'}}\right)$$

$$= \det g \cdot \left[\det\left(\frac{\partial X}{\partial X'}\right)\right]^2$$

$\ldots_{\mu_1 \ldots \mu_n} dX^{\mu_1} \wedge \ldots dX^{\mu_n}$ is covariant.

Can use ε to map p-forms to (n-p) forms (using the metric) via the Hodge dual $*$

$$* : \Lambda^p \to \Lambda^{n-p}.$$

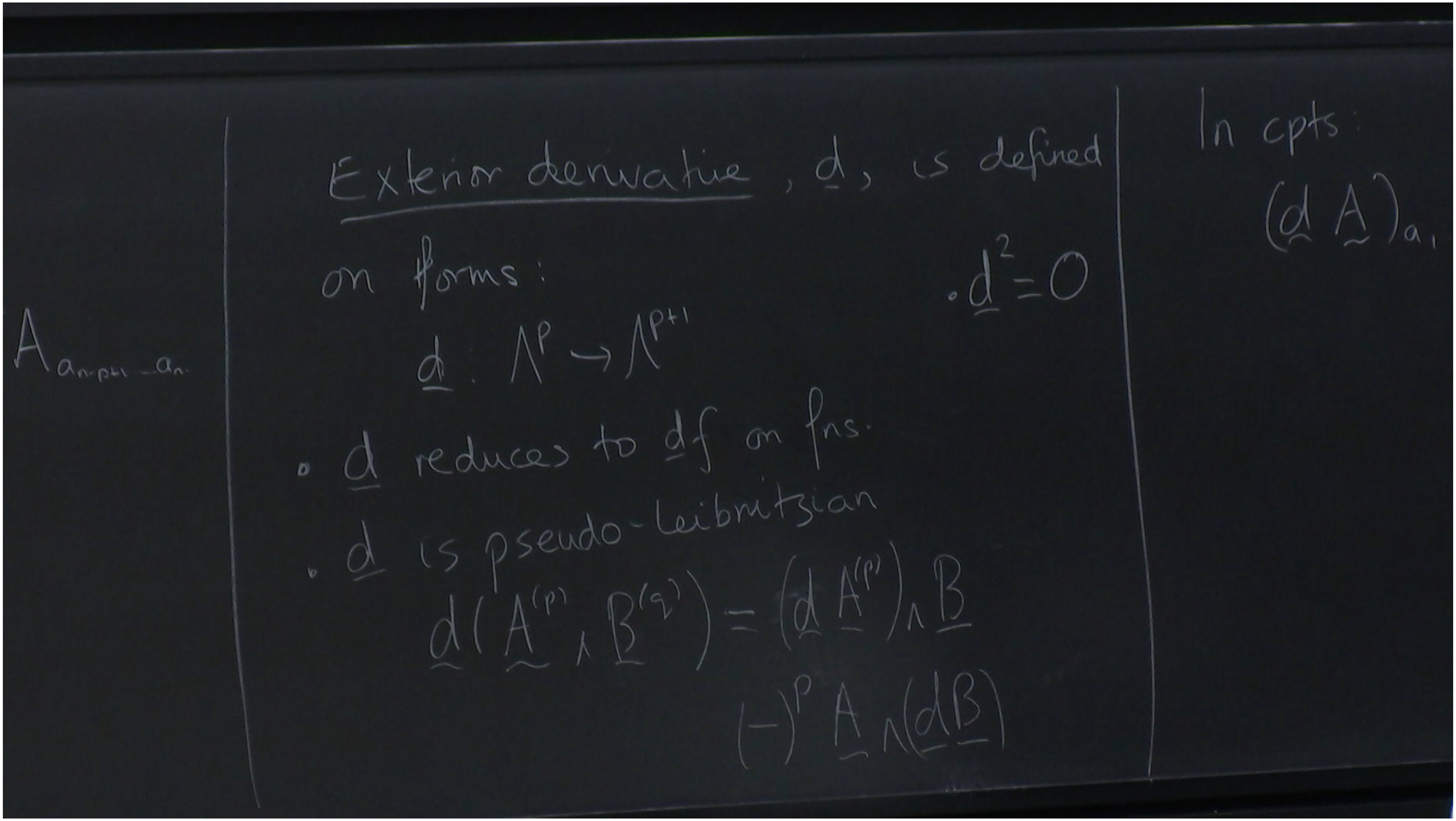


$$\underline{A} \mapsto * \underline{A}$$

$$(*\underline{A})_{a_1 \ldots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \ldots a_{n-p}}{}^{a_{n-p+1} \ldots a_n} A_{a_{n-p+1} \ldots a_n}$$

see that the cross product is

$$\underline{A} \times \underline{B} = *(\underline{A} \wedge \underline{B})$$



$A_{a_{n-p+1}\ldots a_n}$

<u>Exterior derivative</u>, $\underset{\sim}{d}$, is defined on forms:

$$\underset{\sim}{d} : \Lambda^p \to \Lambda^{p+1}$$

- $\underset{\sim}{d}^2 = 0$

- $\underset{\sim}{d}$ reduces to $\underset{\sim}{d}f$ on fns.
- $\underset{\sim}{d}$ is pseudo-Leibnitzian

$$\underset{\sim}{d}(\underset{\sim}{A}^{(p)} \wedge \underset{\sim}{B}^{(q)}) = (\underset{\sim}{d}\,\underset{\sim}{A}^{(p)}) \wedge \underset{\sim}{B}$$
$$(-)^p \underset{\sim}{A} \wedge (\underset{\sim}{d}\underset{\sim}{B})$$

In cpts:

$(\underset{\sim}{d}\,\underset{\sim}{A})_{a_1}$

is defined

$\cdot\, \underline{d}^2 = 0$

$p')\wedge \underline{B}$

$\wedge(\underline{d}\underline{B})$

In cpts:

$$(\underset{\sim}{d}\,\underset{\sim}{A})_{a_1\dots a_{p+1}} = \frac{(p+1)!}{p!}\,\partial_{[a_1}A_{a_2\dots a_{p+1}]}$$

With a metric can also define

$$\delta = *d* \qquad \Lambda^p \to \Lambda^{p-1}$$

$$(\delta A)_{a_1\dots a_{p-1}} = (-)^p\,\nabla^{a_0} A_{a_0\dots a_p}$$

e.g. Electromagnetism:

gauge pot'l $A_\mu = (\phi, -\underline{A})$

$$\underline{A} = A_\mu dx^\mu .$$

Maxwell tensor

$$\underline{F} = \underline{d}\,\underline{A} \rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

agnetism:

$$A_\mu = (\phi, -\underline{A})$$

μ

$$\longrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

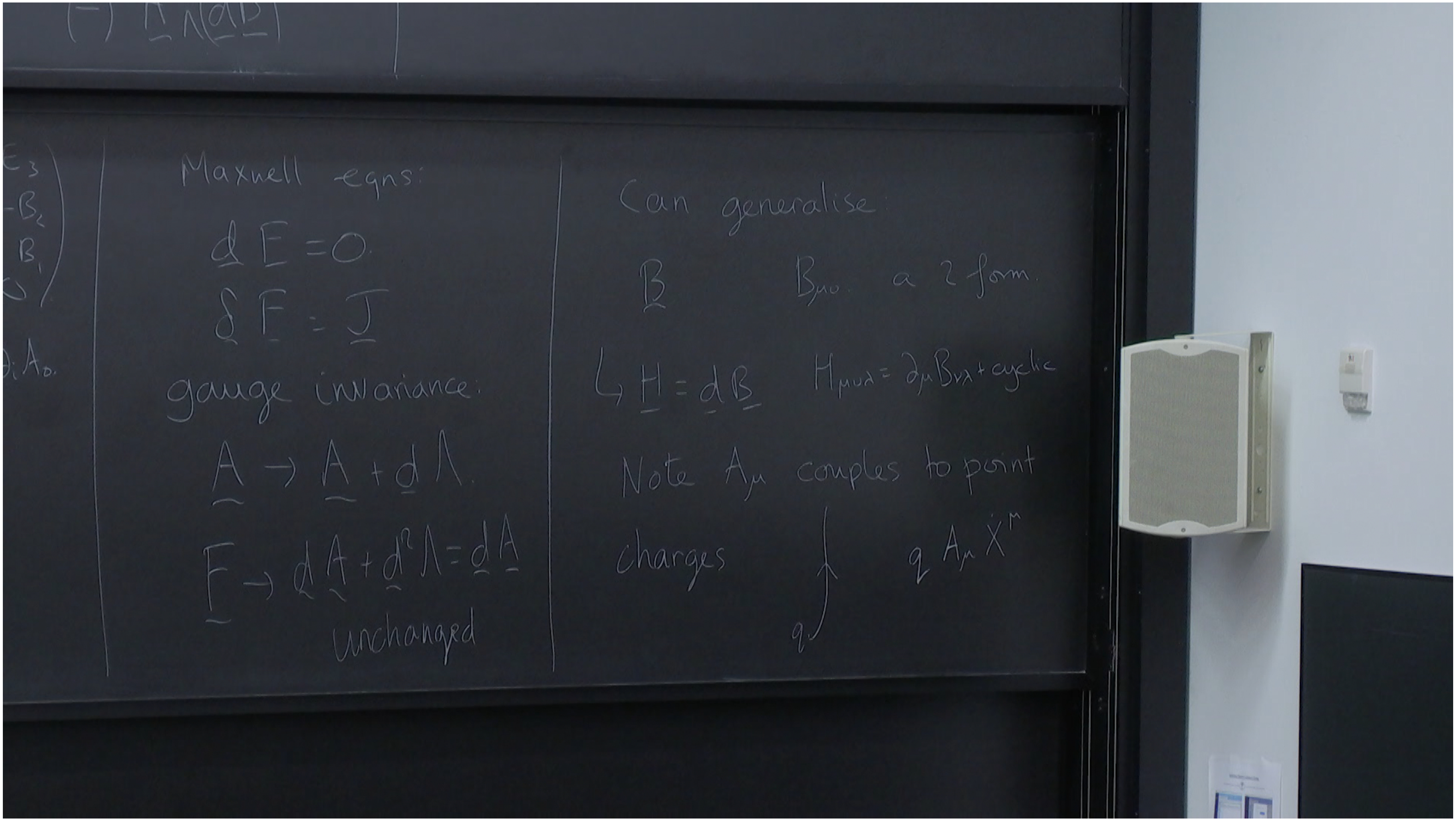
$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\underline{E} = F_{0i} = \partial_0 A_i - \partial_i A_0 .$$

$$= -\dot{\underline{A}} - \underline{\nabla}\phi$$

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

M



Maxwell eqns:

$$\underset{\sim}{d}\,\underset{\sim}{F} = 0.$$

$$\delta\,\underset{\sim}{F} = \underset{\sim}{J}$$

gauge invariance:

$$\underset{\sim}{A} \to \underset{\sim}{A} + \underset{\sim}{d}\,\Lambda.$$

$$\underset{\sim}{F} \to \underset{\sim}{d}\,\underset{\sim}{A} + \underset{\sim}{d}^2\Lambda = \underset{\sim}{d}\,\underset{\sim}{A}$$

unchanged

Can generalise:

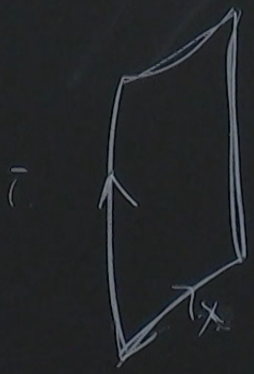
$\underset{\sim}{B}$ $\qquad B_{\mu\nu}.$ a 2 form.

$\hookrightarrow \underset{\sim}{H} = \underset{\sim}{d}\,\underset{\sim}{B}$ $\qquad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \text{cyclic}$

Note A_μ couples to point charges

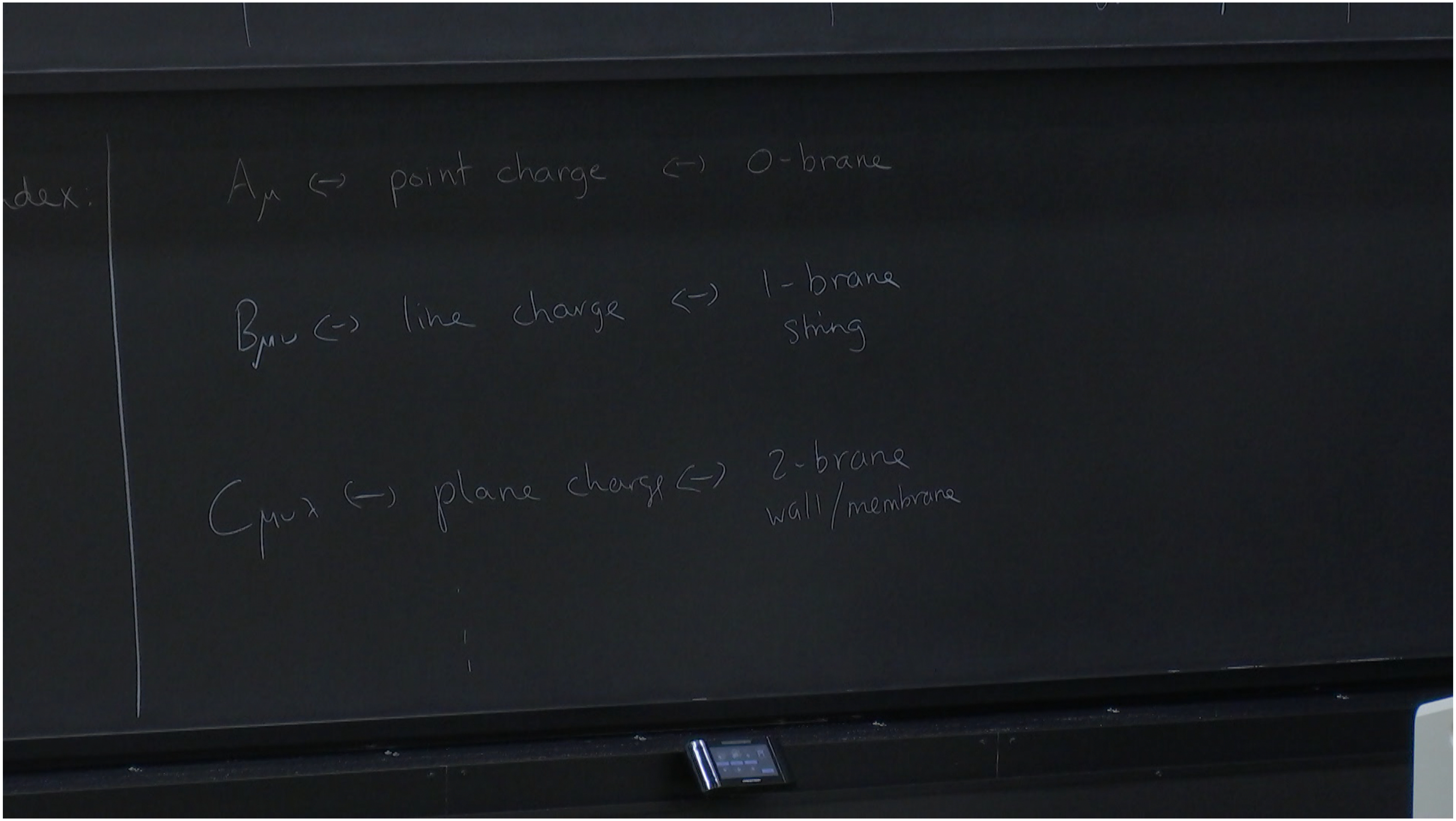
$$\int_q q\, A_\mu \dot{X}^\mu$$

$B_{\mu\nu}$ requires an additional index:



$$q \, B_{\mu\nu} \dot{X}^{\mu} X'^{\nu} .$$

World sheet



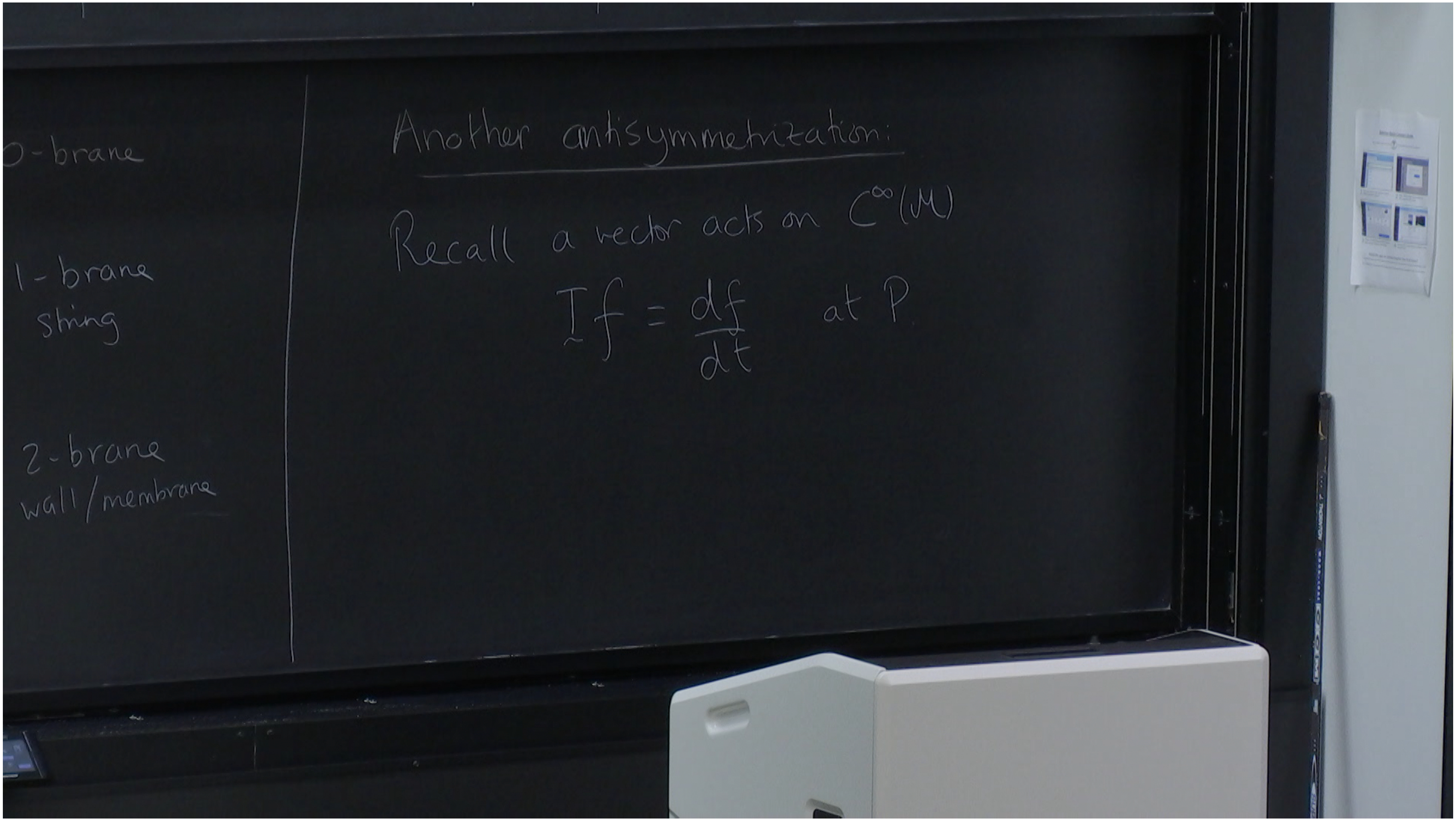
dex:

$A_\mu \leftrightarrow$ point charge $\leftrightarrow$ 0-brane

$B_{\mu\nu} \leftrightarrow$ line charge $\leftrightarrow$ 1-brane
string

$C_{\mu\nu\lambda} \leftrightarrow$ plane charge $\leftrightarrow$ 2-brane
wall/membrane

$\vdots$



0-brane

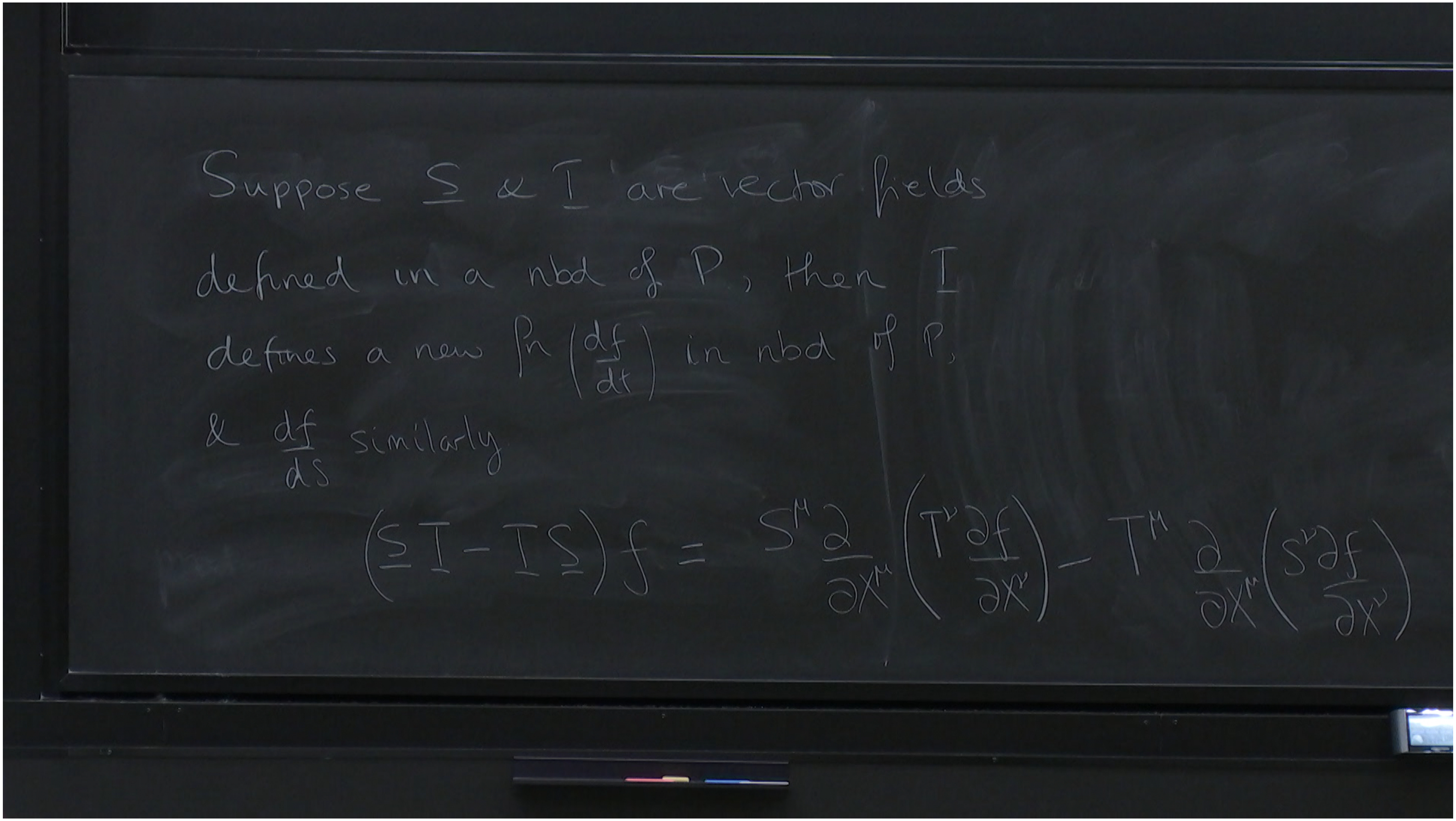
1-brane
string

2-brane
wall/membrane

Another antisymmetrization:

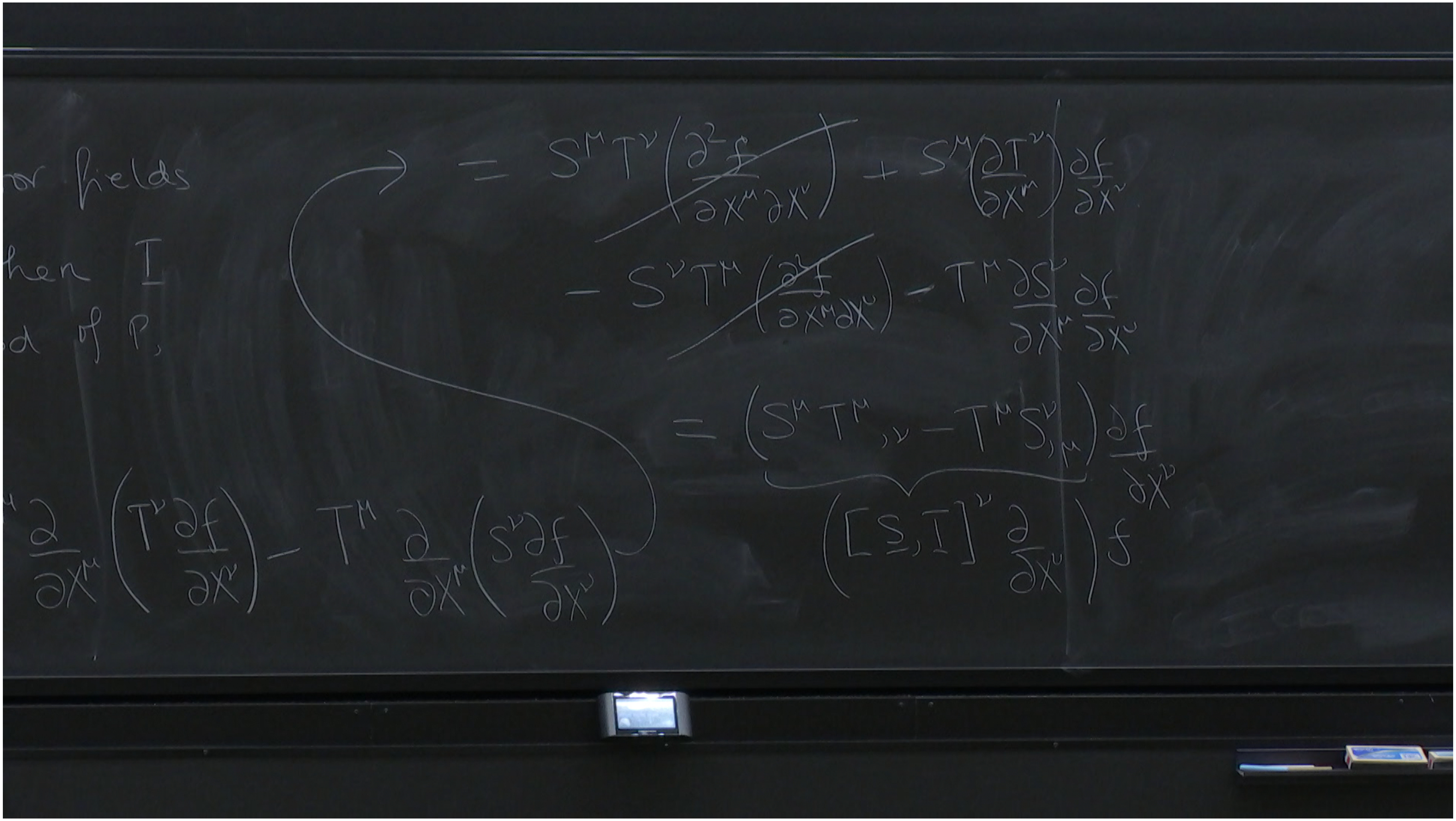
Recall a vector acts on $C^\infty(\mathcal{M})$

$$\underset{\sim}{I} f = \frac{df}{dt} \quad \text{at } P$$



Suppose $\underline{S}$ & $\underline{T}$ are vector fields
defined in a nbd of P, then $\underline{T}$
defines a new fn $\left(\frac{df}{dt}\right)$ in nbd of P,
& $\frac{df}{dS}$ similarly

$$(\underline{S}\,\underline{T}-\underline{T}\,\underline{S})f = S^{\mu}\frac{\partial}{\partial x^{\mu}}\left(T^{\nu}\frac{\partial f}{\partial x^{\nu}}\right) - T^{\mu}\frac{\partial}{\partial x^{\mu}}\left(S^{\nu}\frac{\partial f}{\partial x^{\nu}}\right)$$



or fields

hen I

d of P,

$$^{\mu}\frac{\partial}{\partial x^{\mu}}\left(T^{\nu}\frac{\partial f}{\partial x^{\nu}}\right) - T^{\mu}\frac{\partial}{\partial x^{\mu}}\left(S^{\nu}\frac{\partial f}{\partial x^{\nu}}\right)$$

$$= \cancel{S^{\mu}T^{\nu}\left(\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}}\right)} + S^{\mu}\left(\frac{\partial T^{\nu}}{\partial x^{\mu}}\right)\frac{\partial f}{\partial x^{\nu}}$$

$$- \cancel{S^{\nu}T^{\mu}\left(\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}}\right)} - T^{\mu}\frac{\partial S^{\nu}}{\partial x^{\mu}}\frac{\partial f}{\partial x^{\nu}}$$

$$= \underbrace{\left(S^{\mu}T^{\mu}{}_{,\nu} - T^{\mu}S^{\nu}{}_{,\mu}\right)}_{\left([\underline{S},\underline{T}]^{\nu}\frac{\partial}{\partial x^{\nu}}\right)f}\frac{\partial f}{\partial x^{\nu}}$$

$$T^{\nu}\cancel{\left(\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}}\right)} + S^{\mu}\left(\frac{\partial T^{\nu}}{\partial x^{\mu}}\right)\frac{\partial f}{\partial x^{\nu}}$$

$$S^{\nu}T^{\mu}\cancel{\left(\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}}\right)} - T^{\mu}\frac{\partial S^{\nu}}{\partial x^{\mu}}\frac{\partial f}{\partial x^{\nu}}$$

$$= \underbrace{\left(S^{\mu}T^{\mu}{}_{,\nu} - T^{\mu}S^{\nu}{}_{,\mu}\right)}_{\left([\underline{S},\underline{T}]^{\nu}\frac{\partial}{\partial x^{\nu}}\right)f}\frac{\partial f}{\partial x^{\nu}}$$

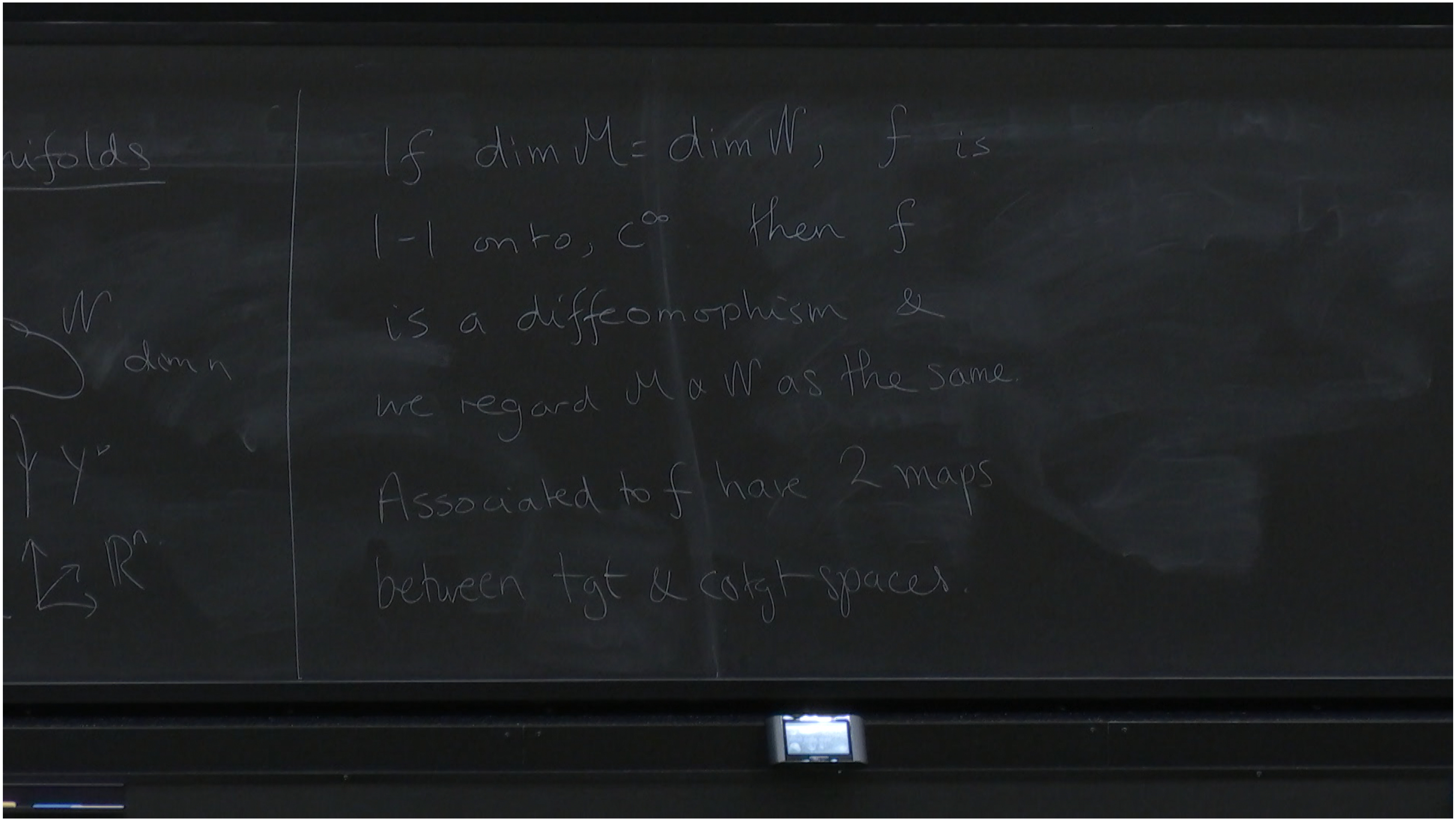
Since this is true $\forall f \in C_{\infty}(\mathcal{M})$

we deduce

$[\underline{S},\underline{T}]$ is a vector

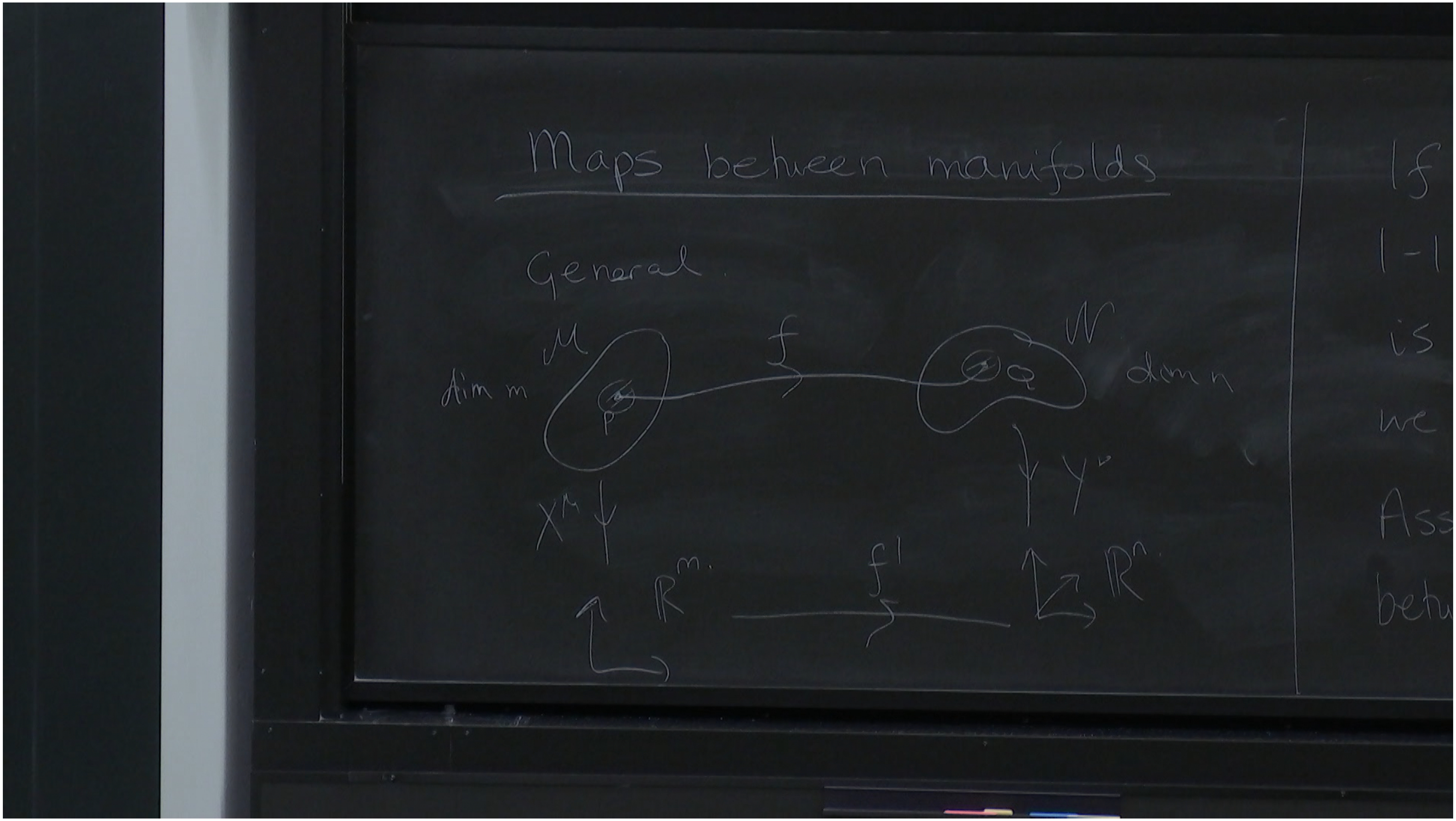
Lie bracket of $\underline{S}$ & $\underline{T}$.

Can we generalise?



If dim $\mathcal{M}$ = dim $\mathcal{N}$, f is 1-1 onto, C^∞ then f is a diffeomophism & we regard $\mathcal{M}$ & $\mathcal{N}$ as the same.

Associated to f have 2 maps between tgt & cotgt spaces.



Maps between manifolds

General.

If

1-1

is

we

Ass

betw

is

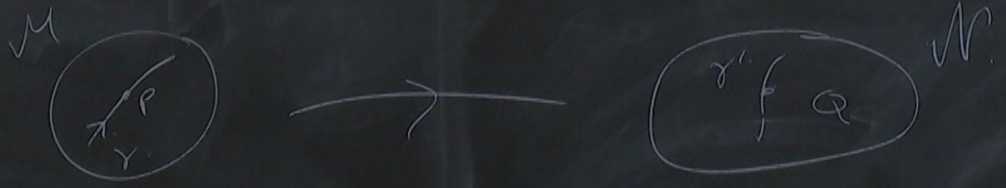
same.

maps

es.

Push-forward

$f_*: T_p(\mathcal{M}) \to T_Q(\mathcal{N})$



$\underline{S} = \frac{d}{dt}$ along γ $\to$ $\underline{T} = \frac{d}{dt}$ along γ'

Pull-back $f^*: T^*_Q(\mathcal{N}) \to T^*_P(\mathcal{M})$

via $\langle f^*(\underline{\omega}) \,|\, \underline{\xi} \rangle = \langle \underline{\omega} \,|\, f_*(\underline{\xi}) \rangle$