

Title: PSI 2019/2020 - Quantum Matter Part 1 - Lecture 17

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Collection: PSI 2019/2020 - Quantum Matter Part 1

Date: January 23, 2020 - 2:00 PM

URL: <http://pirsa.org/20010043>

$U \mapsto \Gamma_U$ unitary rep. on some space $\mathcal{V}$

$$\mathbb{E}(X) = \int dU \Gamma_U^+ X \Gamma_U$$

$$X \in \mathcal{B}(\mathcal{V})$$

Schur's Lemma

$$[\Gamma_U, A] = 0 \quad \forall U$$

$$A = \sum_{i \in \text{irreps of } \Gamma(U)} \lambda_i \Pi_i \quad \text{where } \lambda_i = \frac{\text{tr}[A \cdot \Pi_i]}{\text{tr } \Pi_i}$$

Example 0 : Γ is irreducible $X = \lambda \mathbb{1}$

$X \Phi(U^+)$

$U \mapsto \Gamma_U$ unitary rep. on some space $\mathcal{V}$

$$\mathbb{E}(X) = \int dU \Gamma_U^+ X \Gamma_U$$

$$X \in \mathcal{B}(\mathcal{D})$$

$$[\Gamma_U, \mathbb{E}(X)] = ?$$

Example 0: Γ is irreducible

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$$X = \lambda \mathbb{1}$$

$$\Gamma_U \int dU \Gamma_U^+ X \Gamma_U = \int dU \Gamma_{UU^+} X \Gamma_U$$

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Example 0: Γ is irreducible

$$X = \lambda \mathbb{1}$$

$$V U^+ = W$$

$$U^+ = V^+ W$$

$$U = W^+ V$$

$X \Phi(U^+)$

$U \mapsto \Gamma_U$ unitary rep. on some space

$$\mathbb{E}(X) = \int dU \Gamma_U^+ X \Gamma_U$$

$$X \in \mathcal{B}(\mathcal{D})$$

$$[\Gamma_V, \mathbb{E}(X)] = ?$$

Example 0: Γ is irreducible

$$X = \lambda \mathbb{1}$$

$$\Gamma_V \int dU \Gamma_U^+ X \Gamma_U = \int dU \Gamma_{VU^+} X \Gamma_U$$

$$= \left(\int dW \Gamma_W^+ X \Gamma_{W^+} \right) \Gamma_V = \mathbb{E}(X) \Gamma_V$$

Schur's Lemma

$$[\Gamma_U, A] = 0 \quad \forall U$$

$$A = \sum_{i \in \text{irrops of } \Gamma(U)} \lambda_i \Pi_i \quad \text{where } \lambda_i = \frac{\text{tr}[A \Pi_i]}{\text{tr } \Pi_i}$$

$$VU^+ = W$$

$$U^+ = V^+ W$$

$$U = W^+ V$$

$$\mathbb{E}(X) = \int dU \, \Gamma_U^+ \times \Gamma_U = \sum_{i \text{ irreps}} \lambda_i \pi_i$$

$$\mathbb{E}_\psi(\text{tr}[A\psi]) \stackrel{?}{=} \text{tr}[A \mathbb{E}_\psi(\psi)]$$

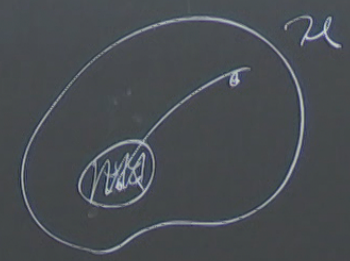
$\lambda_i \pi_i$
 ops

$\psi(\psi)$

$$E_\psi(\psi) = \int dV \, \psi^\dagger \psi = \lambda \mathbb{1} = \frac{\mathbb{1}}{d}$$

$$\lambda = \frac{\text{tr}[\psi^\dagger \psi]}{\text{tr} \mathbb{1}} = \frac{1}{d}$$

$$\Gamma : U \rightarrow \Gamma = U$$

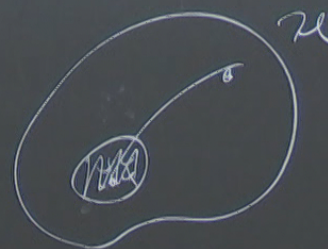


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$$\mathbb{E}_\psi(\psi) = \int dV \, \rho_\psi \psi \rho_\psi^\dagger = \lambda \mathbb{1} = \frac{\mathbb{1}}{d}$$

$$\lambda = \frac{\text{tr}[\rho_\psi^\dagger \rho_\psi]}{\text{tr} \mathbb{1}} = \frac{1}{d}$$



$$\text{tr}(\mathbb{1} \int U \psi U^\dagger) = \int dV \, \text{tr}[\psi U U^\dagger] = \int dV \cdot 1 = 1$$

$$\mathbb{E}(X) = \int dU \Pi_U^+ X \Pi_U = \sum_{i \text{ irreps}} \lambda_i \pi_i$$

$$\mathbb{E}_\psi(\psi) = \int dU \Pi_U \psi \Pi_U^+ = \lambda \mathbb{1} = \frac{\mathbb{1}}{d}$$

$$\begin{aligned} \mathbb{E}_\psi(\text{tr}[A\psi]) &\stackrel{?}{=} \text{tr}[A \mathbb{E}_\psi(\psi)] = \frac{1}{d} \text{tr}[A \mathbb{1}] \\ &= \frac{1}{d} \sum_n a_n \end{aligned}$$

$$\lambda = \frac{\text{tr}[\Pi_U^{\psi\psi}]}{\text{tr} \mathbb{1}} = \frac{1}{d} \quad \Pi_U : \psi \rightarrow \Pi_U \psi = \psi$$

$$\text{tr}[\mathbb{1} \int dU \psi \psi^\dagger] = \int dU \text{tr}[\psi \psi^\dagger] = \int dU \cdot 1 = 1$$

$$\mathbb{E}(X) = \int dU \Pi_U^\dagger X \Pi_U = \sum_{i \text{ irreps}} \lambda_i \pi_i$$

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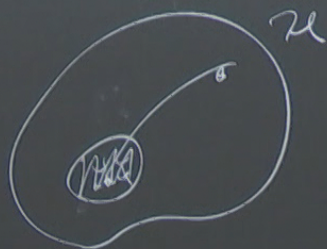
$$\lambda = \frac{\text{tr}[\Pi_U^\psi \psi]}{\text{tr} \mathbb{1}} = \frac{1}{d}$$

$\Pi_U : U \rightarrow \Pi_U = U$

$$\text{tr}[\mathbb{1} \int dU \psi U^\dagger] = \int dU \text{tr}[\psi U U^\dagger] = \int dU \cdot 1 = 1$$

$$\lambda = \lambda \mathbb{1} = \frac{\mathbb{1}}{\alpha}$$

$$\Gamma_U = U$$



$$\omega = \overline{\phi}$$

$$\Delta A_{\phi}^2 = \|A\|^2 \text{tr} \omega^2$$

$$\int_U \omega = \int dV \cdot 1 = 1$$

$$\mathbb{E}_{\phi} [\text{tr} (D_H \phi)^2]$$

$$\text{tr } X^2 = \text{tr} [T^{(2)} X^{\otimes 2}]$$

$$\text{tr} \left[\sum_{ij} |ij\rangle \langle ij| \sum_{mn, m'n'} X_{mn} X_{m'n'} |m\rangle \langle n| \otimes |m'\rangle \langle n'| \right]$$

$$X^2 \in \mathcal{H}$$

$$X \otimes X \in \mathcal{H} \otimes \mathcal{H}$$

$$|i\rangle \otimes |j\rangle$$

$$T^{(2)} = \sum_{ij} |ij\rangle \langle ij|$$

$$T^{(2)} |ij\rangle = |ij\rangle$$

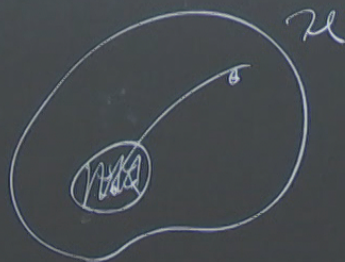
$$X = \sum_{mn} X_{mn} |m\rangle \langle n|$$

$$= \left(\int dW \dots \Gamma_W \times \Gamma_W \right) \Gamma_V = \mathbb{E}(x) \Gamma_V U^T = V^T W \quad U = W^T V$$

$$dV \Gamma_V \Psi \Gamma_V^T = \lambda \mathbb{1} = \frac{\mathbb{1}}{d}$$

$$\Gamma : U \rightarrow \Gamma_U = U$$

$$= \frac{1}{d}$$



$$\omega = \overline{\phi}$$

$$\Delta A_\phi^2 = \|A\|^2 (\text{tr } \omega^2)$$

$$\int dV \text{tr} [\Psi \Psi^T] = \int dV \cdot 1 = 1$$

$$\mathbb{E}_\phi [\text{tr} (D_H \phi)^2] =$$

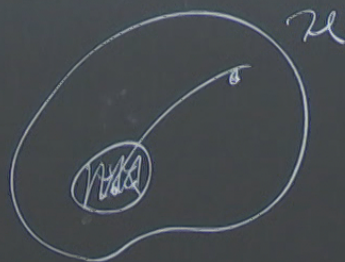
$$\mathbb{E}_\phi [\text{tr} (T^{(1)} (D_H \phi)^2)] = \text{tr} [T^{(2)} D_H^2 (\mathbb{E}(\phi^2))]$$

$$= \left(\int dW \dots \Gamma_W \times \Gamma_W \right) \Gamma_V = \mathbb{E}(x) \Gamma_V U^T = V^T W \quad U = W^T V$$

$$dV \Gamma_U \Psi \Gamma_U^T = \lambda \mathbb{1} = \frac{\mathbb{1}}{d}$$

$$\Gamma : U \rightarrow \Gamma_U = U$$

$$= \frac{1}{d}$$



$$\omega = \overline{\phi}$$

$$\Delta A_\phi^2 \rightarrow \|A\|^2 (\text{tr } \omega^2)$$

$$\int dV \text{tr} [\Psi \Psi U^T] = \int dV \cdot 1 = 1$$

$$\mathbb{E}_\phi [\text{tr} (D_H \phi)^2] =$$

$$\mathbb{E}_\phi [\text{tr} [T^{(1)} (D_H \phi)^2]] = \text{tr} [T^{(2)} D_H^2 (\mathbb{E}(\phi^2))]$$

$$\mathbb{E}(X) = \int dU \Pi_U^\dagger X \Pi_U = \sum_{i \text{ irreps}} \lambda_i \pi_i$$

$$\mathbb{E}_\psi(\psi) = \int dU \Pi_U \psi$$

$$\mathbb{E}_\psi(\text{tr}[A\psi]) \stackrel{?}{=} \text{tr}[A \mathbb{E}_\psi(\psi)] = \frac{1}{d} \text{tr}[A \mathbb{1}]$$

$$= \frac{1}{d} \sum_n a_n$$

$$\lambda = \frac{\text{tr}[\Pi_U^\dagger \Pi_U]}{\text{tr} \mathbb{1}} = \frac{1}{d}$$

$$D_H(x) \stackrel{\otimes 2}{=} \left(\sum_n \pi_n X \pi_n \right)^{\otimes 2}$$

$$\text{tr}[\mathbb{1} \int dU \psi U^\dagger] = \int dU \text{tr}[\psi U^\dagger]$$

$$\mathbb{E}(\phi^{\otimes 2}) = \int dU \quad U^{\otimes 2} \phi_0^{\otimes 2} U^{\dagger \otimes 2}$$

$$\begin{array}{c} \phi \otimes \phi \\ (U \phi_0 U^\dagger)^{\otimes 2} \end{array}$$

$$\begin{array}{ccc} T: U & \longrightarrow & U^{\otimes 2} \\ \cap & & \searrow \\ \mathcal{U}(n) & & \text{Lin}(\mathcal{H}^{\otimes 2}) \end{array}$$

$$\mathbb{E}(\phi^{\otimes 2}) = \int dU \ U^{\otimes 2} \phi_0^{\otimes 2} U^{\dagger \otimes 2} = \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_+]}{\text{tr} \Pi_+} \Pi_+ + \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_-]}{\text{tr} \Pi_-} \Pi_-$$

$$\mathcal{H}_{\pm} = \{|\psi\rangle\}$$

$$\phi \otimes \phi$$

$$(U \phi U^\dagger)^{\otimes 2}$$

$$\Gamma: U \longrightarrow U^{\otimes 2}$$

$$\cap$$

$$\mathcal{U}(n)$$

$$\mathcal{L}(\mathcal{H}^{\otimes 2})$$

$$\pi_+ + \frac{\text{tr}[\phi^{\otimes 2} \pi_-]}{\text{tr} \pi_-} \pi_-$$

$$\mathcal{H}_{\pm} = \{ |\psi\rangle \in \mathcal{H}^{\otimes 2} \mid T^{(2)}|\psi\rangle = \pm |\psi\rangle \}$$

$$\mathcal{H}^{\otimes 2})$$

$$\pi_+ = \frac{1(\mathcal{H}^{\otimes 2}) + T^{(2)}}{2}$$

$$\pi_+^2 = \pi_+$$

$$[T^{(2)}]^2 = 1$$

$$\frac{\text{tr} [\phi^{\otimes 2} \pi_+]}{\text{tr} \pi_+} \pi_+ + \frac{\text{tr} [\phi^{\otimes 2} \pi_-]}{\text{tr} \pi_-} \pi_-$$

$$\mathcal{H}_{\pm} = \{ |\psi\rangle \in \mathcal{H}^{\otimes 2} : T^{(2)} |\psi\rangle = \pm |\psi\rangle \}$$

$$\dim(\mathcal{H}^{\otimes 2})$$

$$\pi_{\pm} = \frac{1_{\mathcal{H}^{\otimes 2}} \pm T^{(2)}}{2}$$

$$\pi_+^2 = \pi_+$$

$$[T^{(2)}]^2 = 1$$

$$T^{(2)} \pi_{\pm} \psi = \pm \pi_{\pm} \psi$$

$$\frac{\text{tr} [\phi^{\otimes 2} \pi_+]}{\text{tr} \pi_+} \pi_+ + \frac{\text{tr} [\phi^{\otimes 2} \pi_-]}{\text{tr} \pi_-} \pi_-$$

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$$\pi_{\pm} = \frac{1_{\mathcal{H}^{\otimes 2}} \pm T^{(2)}}{2}$$

$$T^{(2)}\pi_{\pm}\psi = \pm \pi_{\pm}\psi$$

$$\pi_{\pm}^2 = \pi_{\pm}$$

$$[T^{(2)}]^2 = 1$$

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$$\begin{array}{c} \phi \otimes \phi \\ (U \phi U^\dagger)^{\otimes 2} \end{array}$$

$$\begin{array}{ccc} \Gamma: U & \longrightarrow & U^{\otimes 2} \\ \cap & & \downarrow \\ \mathcal{U}(n) & & \text{Lin}(\mathcal{H}^{\otimes 2}) \end{array}$$

$$\text{tr} \Pi_+ =$$

$$T^{(2)} \Pi_+ \psi =$$

$$\mathbb{E}(\phi^{\otimes 2}) = \int dU \ U^{\otimes 2} \phi_0^{\otimes 2} U^{\dagger \otimes 2} = \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_+]}{\text{tr} \Pi_+} \Pi_+ + \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_-]}{\text{tr} \Pi_-} \Pi_-$$

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$$\mathcal{H}_{\pm} = \{ |\psi\rangle \in \mathcal{H}^{\otimes 2} : T^{(2)} |\psi\rangle = \pm |\psi\rangle \}$$

$$\phi \otimes \phi$$

$$(U \phi_0 U^{\dagger})^{\otimes 2}$$

$$T: U \longrightarrow U^{\otimes 2}$$

$$\cap$$

$$\mathcal{U}(n)$$

$$\dim(\mathcal{H}^{\otimes 2})$$

$$\Pi_{\pm} = \frac{1_{\mathcal{H}^{\otimes 2}} \pm T^{(2)}}{2}$$

$$\text{tr} \Pi_{\pm} = \frac{1}{2} (d^2 \pm d)$$

$$= \frac{1}{2} d(d \pm 1)$$

$$T^{(2)} \Pi_{\pm} \psi = \pm \Pi_{\pm} \psi$$

$$\text{tr} T^{(2)} = \sum_{i,j} \overline{\langle i | i \rangle} \langle j | j \rangle = \sum_i 1 = d$$

$$\begin{aligned}
 E(\phi^{\otimes 2}) &= \int dU \, U^{\otimes 2} \phi_0^{\otimes 2} U^{\dagger \otimes 2} = \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_+]}{\text{tr} \Pi_+} \Pi_+ + \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_-]}{\text{tr} \Pi_-} \Pi_- \\
 &= \frac{2}{d(d+1)} \Pi_+ = \frac{1_{\mathcal{H}^{\otimes 2}} + T^{(2)}}{d(d+1)} \quad \text{where } \mathcal{H}^{\otimes 2} = \mathcal{H}_+ \oplus \mathcal{H}_- \\
 &\quad \phi_0^{\otimes 2} \quad (U \phi_0 U^\dagger)^{\otimes 2} \quad \Pi : U \longrightarrow \underbrace{\mathcal{H}^{\otimes 2}}_{\dim(\mathcal{H}^{\otimes 2})} \quad \Pi_+ = \frac{1_{\mathcal{H}^{\otimes 2}} + T^{(2)}}{2} \\
 |\phi_0\rangle\langle\phi_0| \otimes |\phi_0\rangle\langle\phi_0| \text{tr} \Pi_\pm &= \frac{1}{2} (d^2 \pm d) \\
 \begin{matrix} || \\ \phi_0^{\otimes 2} \end{matrix} &= \frac{1}{2} d(d \pm 1) \quad \text{tr} T^{(2)} = \frac{1}{2} \sum_{i,j} \langle ii | jj \rangle = \sum_i 1 = d \\
 &\quad T^{(2)} \Pi_\pm \psi = \pm \Pi_\pm \psi
 \end{aligned}$$

$$\begin{aligned}
 E(\phi^{\otimes 2}) &= \int dU \, U^{\otimes 2} \phi_0^{\otimes 2} U^{\dagger \otimes 2} = \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_+]}{\text{tr} \Pi_+} \Pi_+ + \frac{\text{tr}[\phi_0^{\otimes 2} \Pi_-]}{\text{tr} \Pi_-} \Pi_- \\
 &= \frac{2}{d(d+1)} \Pi_+ = \frac{\mathbb{1}_{\mathcal{H}^{\otimes 2}} + T^{(2)}}{d(d+1)} \quad \Pi_+ = \frac{\mathbb{1}_{\mathcal{H}^{\otimes 2}} + T^{(2)}}{2} \\
 &\quad \phi \otimes \phi \quad \quad \quad \Pi_- = \frac{\mathbb{1}_{\mathcal{H}^{\otimes 2}} - T^{(2)}}{2} \\
 &\quad (U \phi_0 U^\dagger)^{\otimes 2} \quad \quad \quad \Pi_+^2 = \Pi_+ \\
 &\quad \quad \quad \Pi : U \longrightarrow \underbrace{U^{\otimes 2}}_{\dim(\mathcal{H}^{\otimes 2})} \\
 &\quad \quad \quad \quad \quad \quad \mathcal{U}(n) \\
 |\phi_0\rangle\langle\phi_0| \otimes |\phi_0\rangle\langle\phi_0| \text{tr} \Pi_\pm &= \frac{1}{2} (d^2 \pm d) \\
 \begin{matrix} |1\rangle \\ \phi_0^{\otimes 2} \end{matrix} &\quad \quad \quad = \frac{1}{2} d(d \pm 1) \\
 \text{tr} T^{(2)} &= \frac{1}{2} \sum_{i,j} \langle 1i | 1j \rangle = \sum_i 1 = d
 \end{aligned}$$

$\mathcal{H}^{\otimes 2} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$\mathcal{H}_\pm = \{ |\psi\rangle \in \mathcal{H}^{\otimes 2} : T^{(2)} |\psi\rangle = \pm |\psi\rangle \}$