Title: PSI 2019/2020 - Quantum Matter Part 1 - Lecture 16

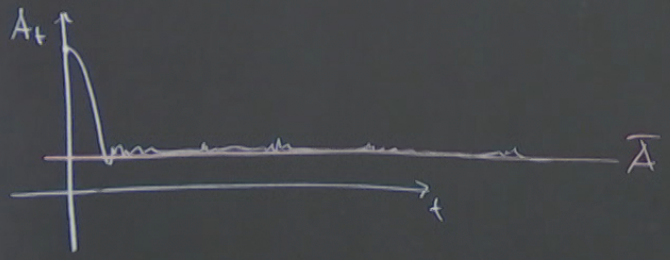
Speakers: Alioscia Hamma

Collection: PSI 2019/2020 - Quantum Matter Part 1

Date: January 23, 2020 - 9:00 AM

URL: http://pirsa.org/20010042

Equilibration in Probability



THEOREM : Time fluctuations are $\leq$?

$|\phi\rangle$ Initial state (pure) $\in \mathcal{H}$; $\phi = |\phi\rangle\langle\phi|$

$H = \sum_n E_n \Pi_n \equiv \sum_n E_n |E_n\rangle\langle E_n|$

non-degenerate

$$\phi_t = e^{-iHt}\,\phi\, e^{iHt} = \sum_{nn'} \phi_{nn'}\, e^{-i(E_n - E_{n'})t}\, |E_n\rangle\langle E_{n'}|$$

$A_t = \mathrm{tr}[A\,\phi_t]$

$$\bar{X} = \lim_{T\to\infty} \frac{1}{T} \int_0^T X_t\, dt$$

$\bar{A} = \overline{tr[A\phi_t]} = tr[A\bar{\phi}] = \sum_n A_{nn}\phi_{nn}$

$\bar{\phi} = \sum_{nn'} \phi_{nn'} |E_n\rangle\langle E_{n'}| \overline{e^{-i(E_n-E_{n'})t}} = \sum_n \phi_{nn} |E_n\rangle\langle E_n| = \begin{pmatrix} \phi_{11} & & 0 \\ & \ddots & \\ 0 & & \phi_{nn} \end{pmatrix}$ ($|\alpha_{11}|^2$)

$\delta_{nn'}$

$= \sum_n \Pi_n \phi \Pi_n \equiv D_H \phi$

$|\phi\rangle = \sum_n \alpha_n |E_n\rangle$

$\phi = |\phi\rangle\langle\phi| = \sum_{nn'} \alpha_n \alpha_{n'}^* |E_n\rangle\langle E_{n'}| \longrightarrow \phi_{nn} = |\alpha_n|^2$ population of $|\phi\rangle$ in $|E_n\rangle$

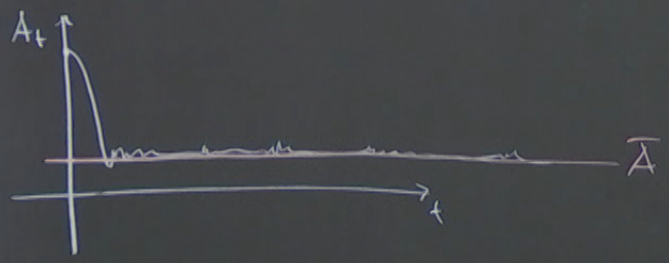
$\phi_{nn'}$

$|E_n\rangle\langle E_n|\phi|E_n\rangle\langle E_n|$

ϕ_{nn}

REMARK Is D_H an effect of time evolution? NO!

Equilibration in Probability



THEOREM: Time fluctuations are $\leq$? $\Delta A^2 = \overline{(A_t - \bar{A})^2}$

$|\phi\rangle$ Initial state (pure) $\in \mathcal{H}$; $\phi = |\phi\rangle\langle\phi|$

$H = \sum_n E_n \Pi_n \equiv \sum_n E_n |E_n\rangle\langle E_n|$

non-degenerate

$$\phi_t = e^{-iHt}\,\phi\, e^{iHt} = \sum_{nn'} \phi_{nn'}\, e^{-i(E_n - E_{n'})t}\, |E_n\rangle\langle E_{n'}|$$

$A_t = \mathrm{tr}(A\,\phi_t)$

$$\bar{X} = \lim_{T\to\infty} \frac{1}{T}\int_0^T X_t\, dt$$

$\bar{A} = \overline{tr[A\phi_t]} = tr[A\bar{\phi}] = \sum_n A_{nn}\phi_{nn}$

$\bar{\phi} = \sum_{nn'} \phi_{nn'} |E_n\rangle\langle E_{n'}| \; \overline{e^{-i(E_n - E_{n'})t}} = \sum_n \phi_{nn} |E_n\rangle\langle E_n| = \begin{pmatrix} \phi_{11} & & 0 \\ & \ddots & \\ 0 & & \phi_{nn} \end{pmatrix}$ $|\alpha_{11}|^2$

$\overline{e^{-i(E_n - E_{n'})t}} = \delta_{nn'}$

$= \sum_n \Pi_n \phi \Pi_n \equiv D_H \phi$

$|\phi\rangle = \sum_n \alpha_n |E_n\rangle$

$\phi = |\phi\rangle\langle\phi| = \sum_{nn'} \alpha_n \alpha_{n'}^* |E_n\rangle\langle E_{n'}| \longrightarrow \phi_{nn} = |\alpha_n|^2$ population of $|\phi\rangle$ in $|E_n\rangle$

$\phi_{nn'}$

$|E_n\rangle\langle E_n|\phi|E_n\rangle\langle E_n| = \phi_{nn}$

$B = \{|\beta\rangle\langle\beta|\}$

$[D_B^X, B] = 0$

REMARK 1 is D_H an effect of time evolution? NO!

REMARK 2 is $D_H\phi$ a steady state? YES because $[D_H\phi, H] = 0$

Hypothesis: H obeys the non-resonance condition

$E_j = E_k$ iff $j = k$

$E_j - E_k = E_n - E_m$ iff $\begin{cases} j = n \\ k = m \end{cases}$

$H = \begin{pmatrix} & \\ & \ddots \end{pmatrix} \longrightarrow H + \epsilon V$



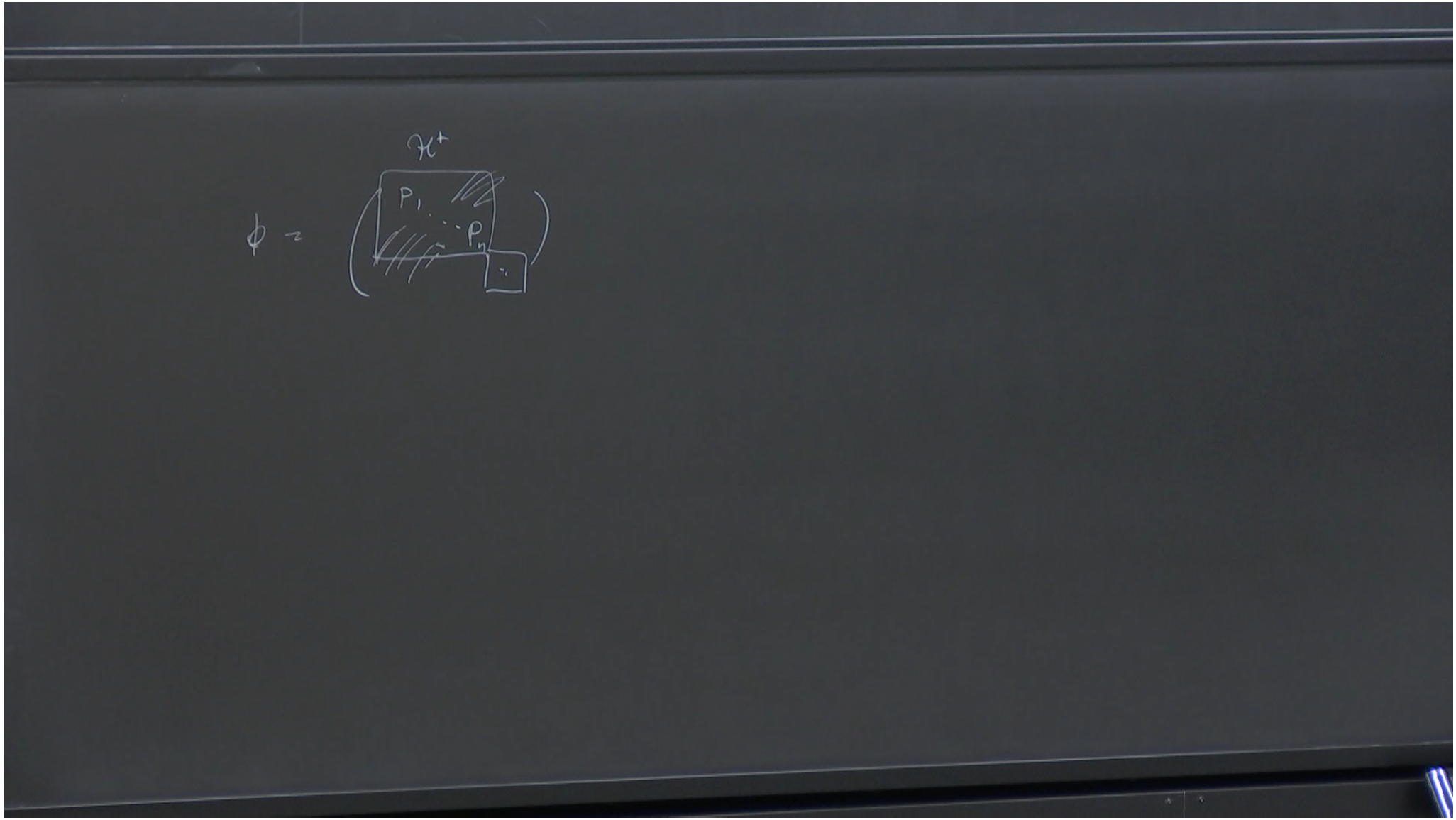
Theorem $\Delta A^2 \leq \|A_p\|^2 \operatorname{tr} \Phi^2$

Theorem $\Delta A^2 \leq \|A_p\|^2 \, (\mathrm{tr}\,\tilde{\Phi}^2)$ → Pur of Φ

Def. $\tilde{\Phi}_t = \Phi_t - \bar{\Phi} = \sum_{h \neq m} \Phi_{nm} e^{-i(E_n - E_m)t}$ $|E_n - E_m|$

$\mathrm{diag}\,\tilde{\Phi}_t = 0 \Rightarrow \mathrm{tr}\,\tilde{\Phi}_t = 0$




$\mathcal{H}^+$
$\phi \sim$
P_1
P_n

Hypothesis : H obeys the non-resonance condition

$E_j = E_k$ iff $j = k$

$E_j - E_k = E_n - E_m$ iff $\begin{cases} j = n \\ k = m \end{cases}$

$H = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix} \rightarrow H(\epsilon V)$



Theorem $\Delta A^2 \leq \|A\|_\rho^2 \left(\mathrm{tr}\, \tilde{\phi}^2\right) \rightarrow$ Purity of ϕ

Def. $\tilde{\phi}_t = \phi_t - \bar{\phi} = \sum_{n \neq m} \phi_{nm} e^{-i(E_n - E_m)t}$ $|E_n \times E_m|$

$\mathrm{diag}\, \tilde{\phi}_t = 0 \Rightarrow \mathrm{tr}\, \tilde{\phi}_t = 0$

$\mathcal{H}^+ = \mathrm{supp}\, \phi \subseteq \mathcal{H}$

obeys the non-resonance condition

$E_j = E_k$ iff $j = k$

$E_j - E_k = E_n - E_m$ iff $\begin{cases} j = n \\ k = m \end{cases}$

$\dots) \longrightarrow \mathcal{H}_t \subseteq \mathcal{V}$

Theorem $\Delta A^2 \leq \|A\|^2 \left(\operatorname{tr} \bar{\Phi}^2\right)$ → Pur of Φ

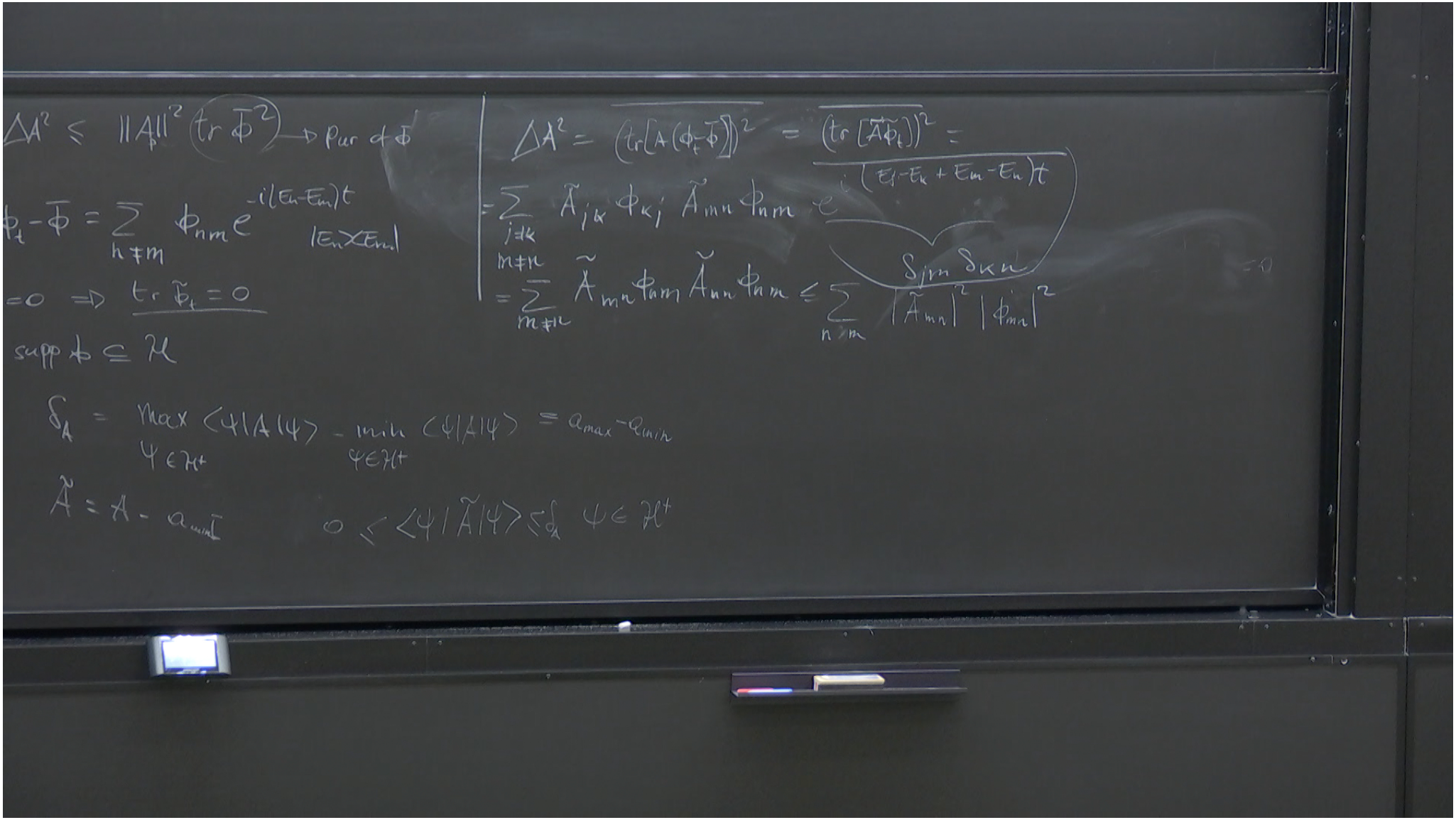
Def. $\tilde{\Phi}_t = \Phi_t - \bar{\Phi} = \sum_{n \neq m} \Phi_{nm} e^{-i(E_n - E_m)t}$ $|E_n \times E_m|$

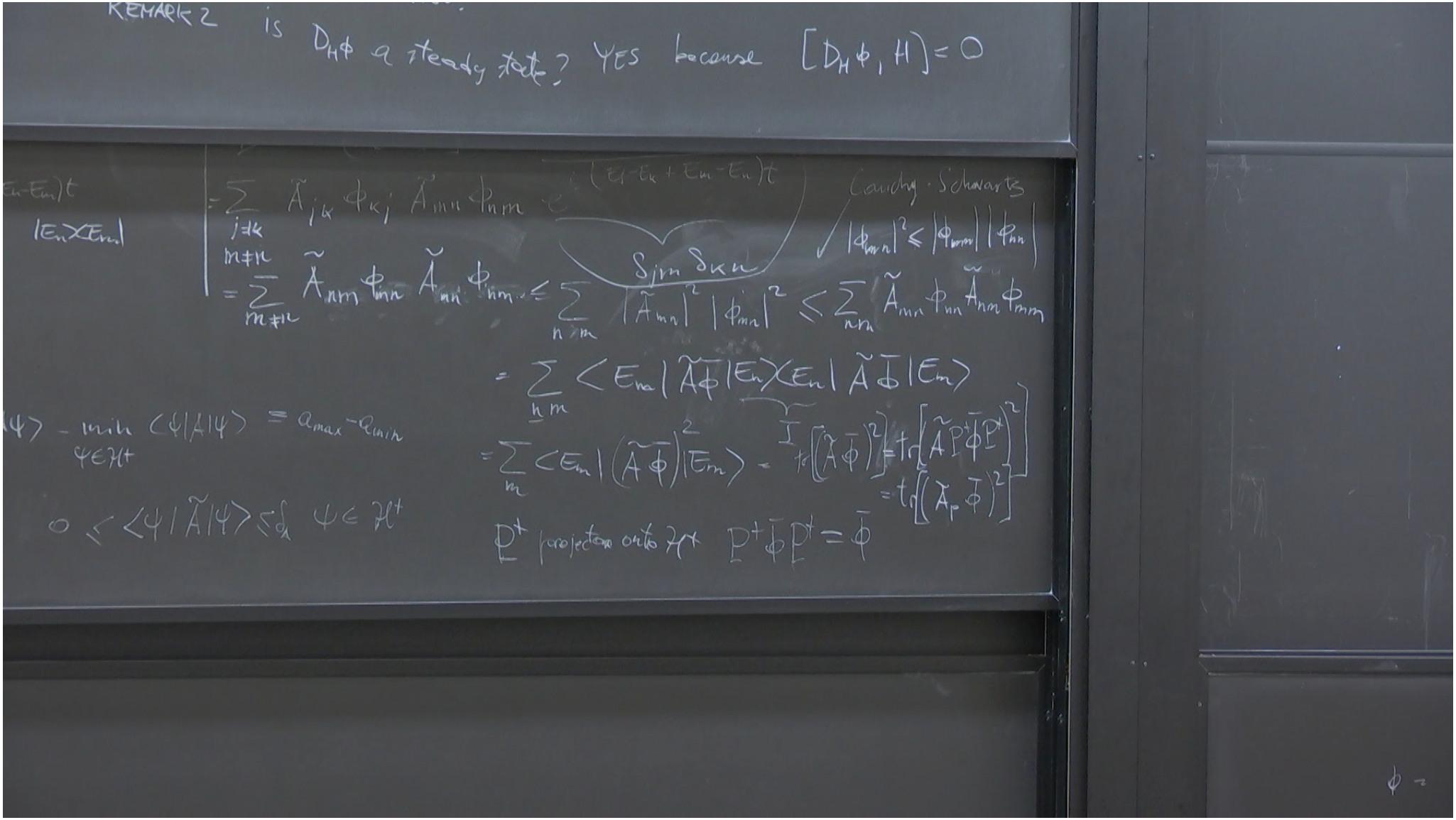
$\operatorname{diag} \tilde{\Phi}_t = 0 \Rightarrow \operatorname{tr} \tilde{\Phi}_t = 0$

$\mathcal{H}^+ = \operatorname{supp} \Phi \subseteq \mathcal{H}$

norm of $\hat{A}$ in $\mathcal{H}_t$ $\quad \delta_A = \max_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle - \min_{\psi \in \mathcal{H}^+} \langle \psi | A | \psi \rangle = a_{max} - a_{min}$

$\tilde{A} = A - a_{min} \mathbb{1}$ $\quad 0 \leq \langle \psi | \tilde{A} | \psi \rangle \leq \delta_A \quad \psi \in \mathcal{H}^+$


$\Delta A^2 \leq \|A_\rho\|^2 \, (\mathrm{tr}\,\bar{\Phi}^2)$ → Pur of Φ
$\Phi_t - \bar{\Phi} = \sum_{n \neq m} \Phi_{nm} e^{-i(E_n - E_m)t}$
$|E_n \times E_m|$
$= 0 \Rightarrow \mathrm{tr}\,\tilde{\Phi}_t = 0$
$\mathrm{supp}\,\tilde{\Phi} \subseteq \mathcal{H}$
$\delta_A = \max_{\psi \in \mathcal{H}^+} \langle\psi|A|\psi\rangle - \min_{\psi \in \mathcal{H}^+} \langle\psi|A|\psi\rangle = a_{max} - a_{min}$
$\tilde{A} = A - a_{min}\mathbb{1}$
$0 \leq \langle\psi|\tilde{A}|\psi\rangle \leq \delta_A \quad \psi \in \mathcal{H}^+$
$\Delta A^2 = \overline{(\mathrm{tr}[A(\Phi_t - \bar{\Phi})])^2} = \overline{(\mathrm{tr}[\tilde{A}\tilde{\Phi}_t])^2} =$
$= \sum_{\substack{j \neq k \\ m \neq n}} \tilde{A}_{jk} \Phi_{kj} \tilde{A}_{mn} \Phi_{nm} e^{i(E_j - E_k + E_m - E_n)t}$
$\delta_{jm}\delta_{kn}$
$= \sum_{m \neq n} \tilde{A}_{mn} \Phi_{nm} \tilde{A}_{nm} \Phi_{nm} \leq \sum_{n \neq m} |\tilde{A}_{mn}|^2 |\Phi_{nm}|^2$

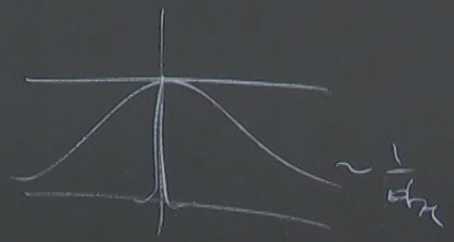

REMARK 2
is $D_H\phi$ a steady state? YES because $[D_H\phi, H] = 0$
Cauchy-Schwartz
$|\phi_{mn}|^2 \le |\phi_{mm}|\,|\phi_{nn}|$
$\delta_{jm}\delta_{kn}$
$\|A\phi\|^2 - \min_{\psi\in\mathcal{H}^+} \langle\psi|A|\psi\rangle = a_{max} - a_{min}$
$0 \le \langle\psi|\tilde{A}|\psi\rangle \le \delta_A \quad \psi\in\mathcal{H}^+$
P^+ projection onto $\mathcal{H}^+$ $P^+\bar{\phi}P^+ = \bar{\phi}$

$$\Delta A^2 \leq \mathrm{tr}\left[(\tilde{A}_P \Phi)^2\right] \leq \delta_A^2 \left(\mathrm{tr}\, \Phi^2\right)$$

exercise

$O(1)$ in $\dim \mathcal{H}$

$$\frac{1}{d_{\mathcal{H}}} \leq \sum_n p_n^2 \leq 1$$



$A_t = \mathrm{tr}[\hat{A}\phi_t]$ may have typical value

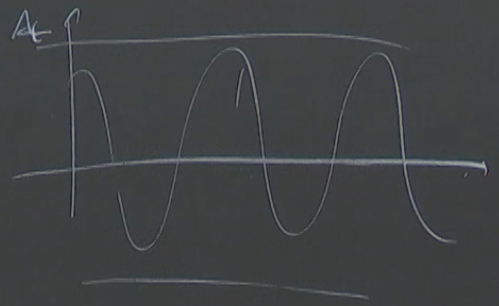
$\Delta A^2 \leq \|A\|^2 (\mathrm{tr}\,\bar{\phi}^2)$ ← purity

ϕ initial state ; $\bar{\phi} = \mathcal{D}_H \phi$ completely dephased state

ϕ $\quad \hat{A} = \phi^{-1}_{mn} |E_m\rangle\langle E_n| + h.c.$

$$\begin{pmatrix} 0 & \phi^{-1}_{mn} \\ \phi^{-1}_{nm} & 0 \end{pmatrix}$$

$$tr[\hat{A}\phi_t] = 2\cos\omega t$$



ity

ϕ completely dephased state

ϕ $\quad \hat{A} = \phi^{-1}_{mn} |E_m\rangle\langle E_n| + h.c.$

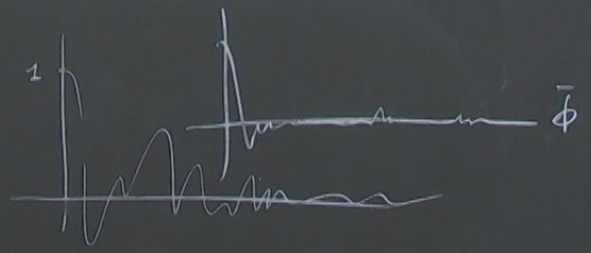
Quantum Quench

$H(\lambda)$ $\quad |\psi_0(\lambda_1)\rangle$

$t=0 \quad H(\lambda_1) \longrightarrow H(\lambda_2)$

$|\psi(t)\rangle = e^{-iH(\lambda_2)t} |\psi_0(\lambda_1)\rangle = e^{-iH(\lambda_2)t} \sum_n \alpha_n |E_n(\lambda_2)\rangle$

$\mathrm{tr}\,\bar{F}^2 = \overline{|\langle \psi_0(\lambda_1)|\psi(t)\rangle|^2} = \bar{L}_t$ Loschmidt Echo



$H(\lambda_2)$

$\sum_n \alpha_n |E_n(\lambda_2)\rangle$

$\hat{A} = \phi_{mn}^{-1} |E_m\rangle\langle E_n| + h.c.$

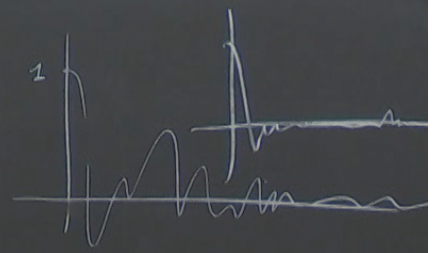
$\mathrm{tr}\,\bar{\phi}^2 = \overline{|\langle\psi_0(\lambda_1)|\psi(t)\rangle|^2} = \bar{L}_t$ Loschmidt Echo

Quantum Quench

$H(\lambda)$ $|\psi_0(\lambda_1)\rangle$

Q. Ising

$\Pi_k \left(1 - \frac{\sin^2 \chi_k}{2}\right) \sim e^{-N} \sim \frac{1}{d}$

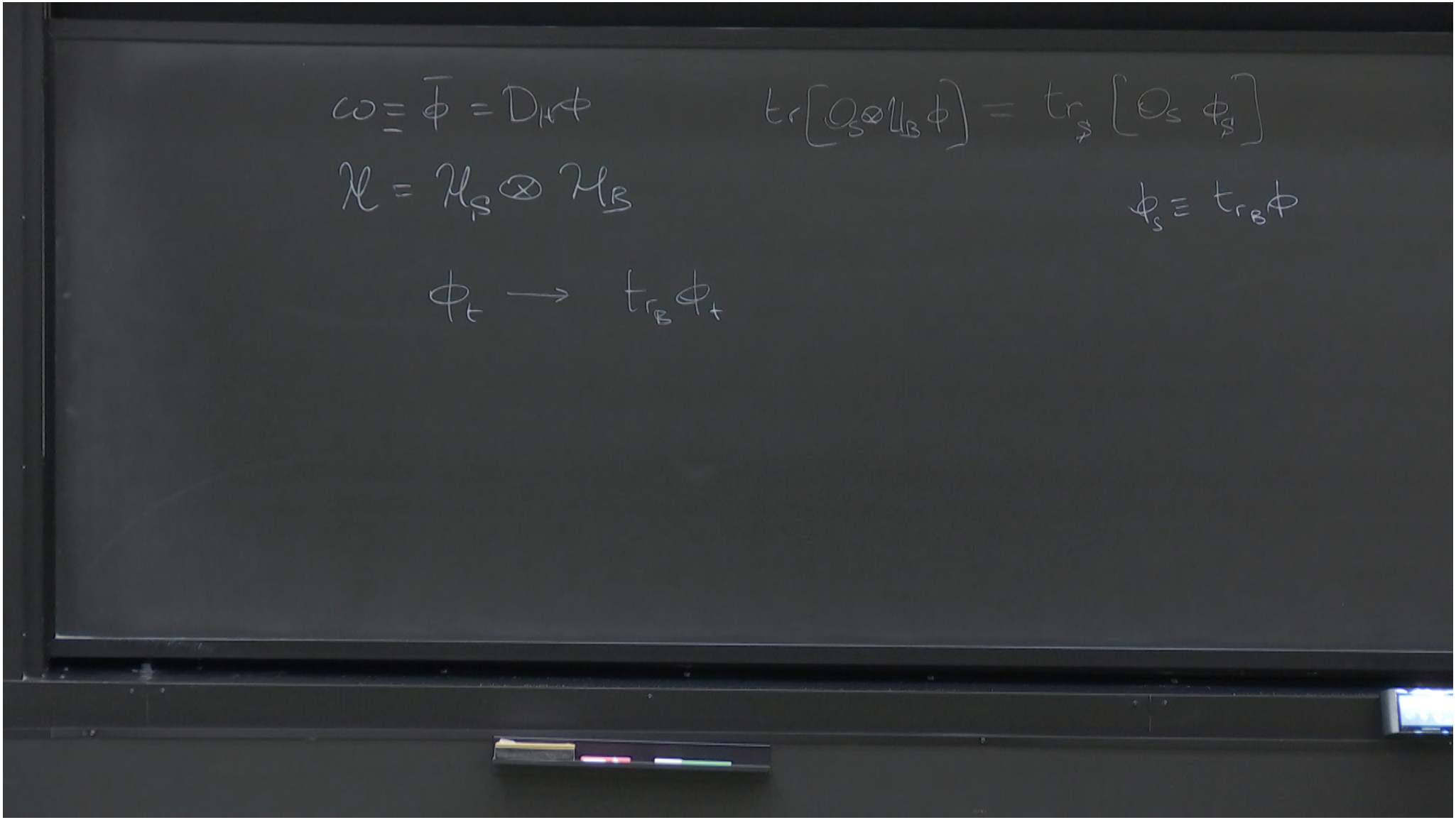


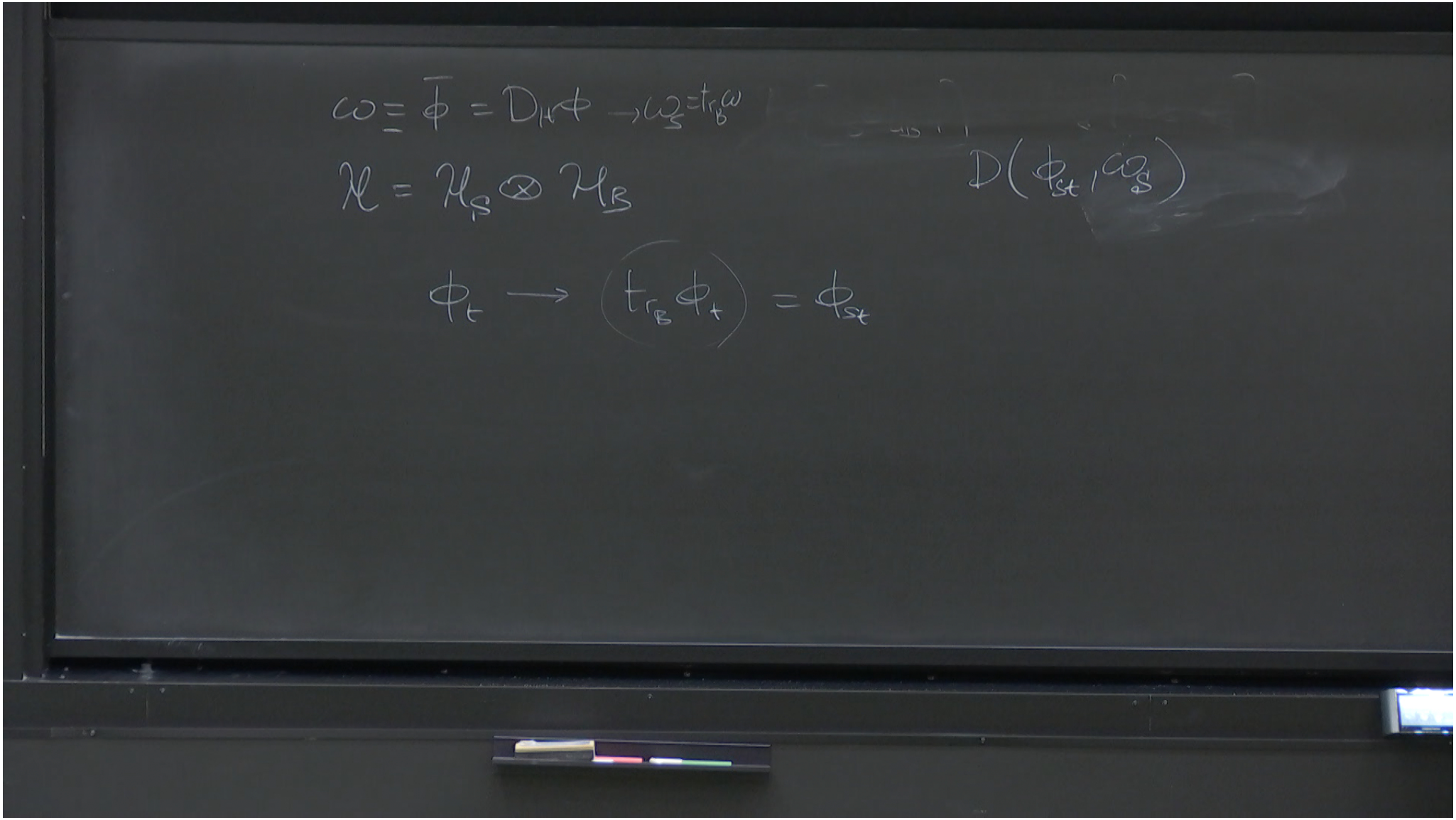
$t=0$ $H(\lambda_1) \longrightarrow H(\lambda_2)$

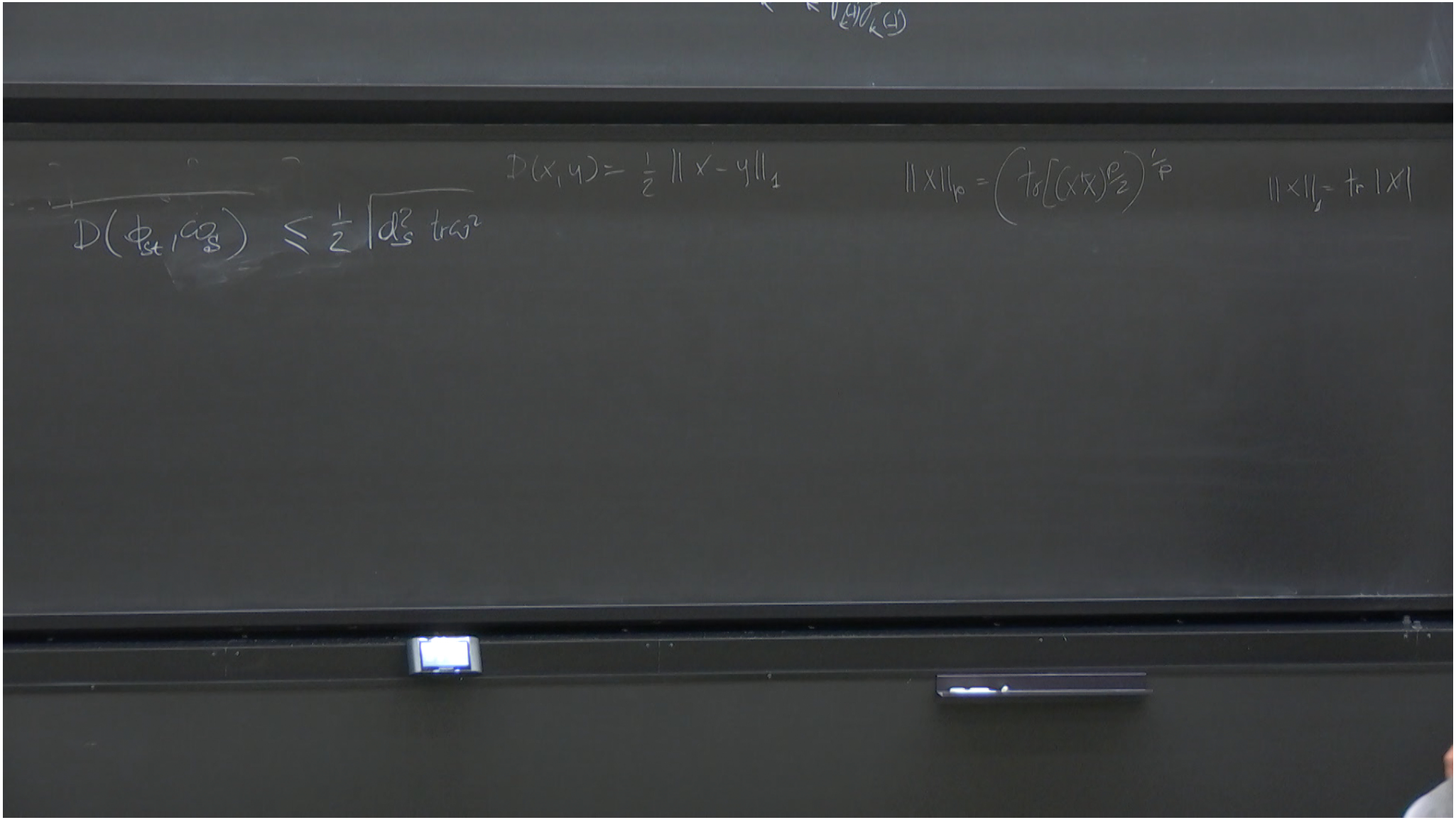
$|\psi(t)\rangle = e^{-iH(\lambda_2)t} |\psi_0(\lambda_1)\rangle = e^{-iH(\lambda_2)t} \sum_n \alpha_n |E_n(\lambda_2)\rangle$

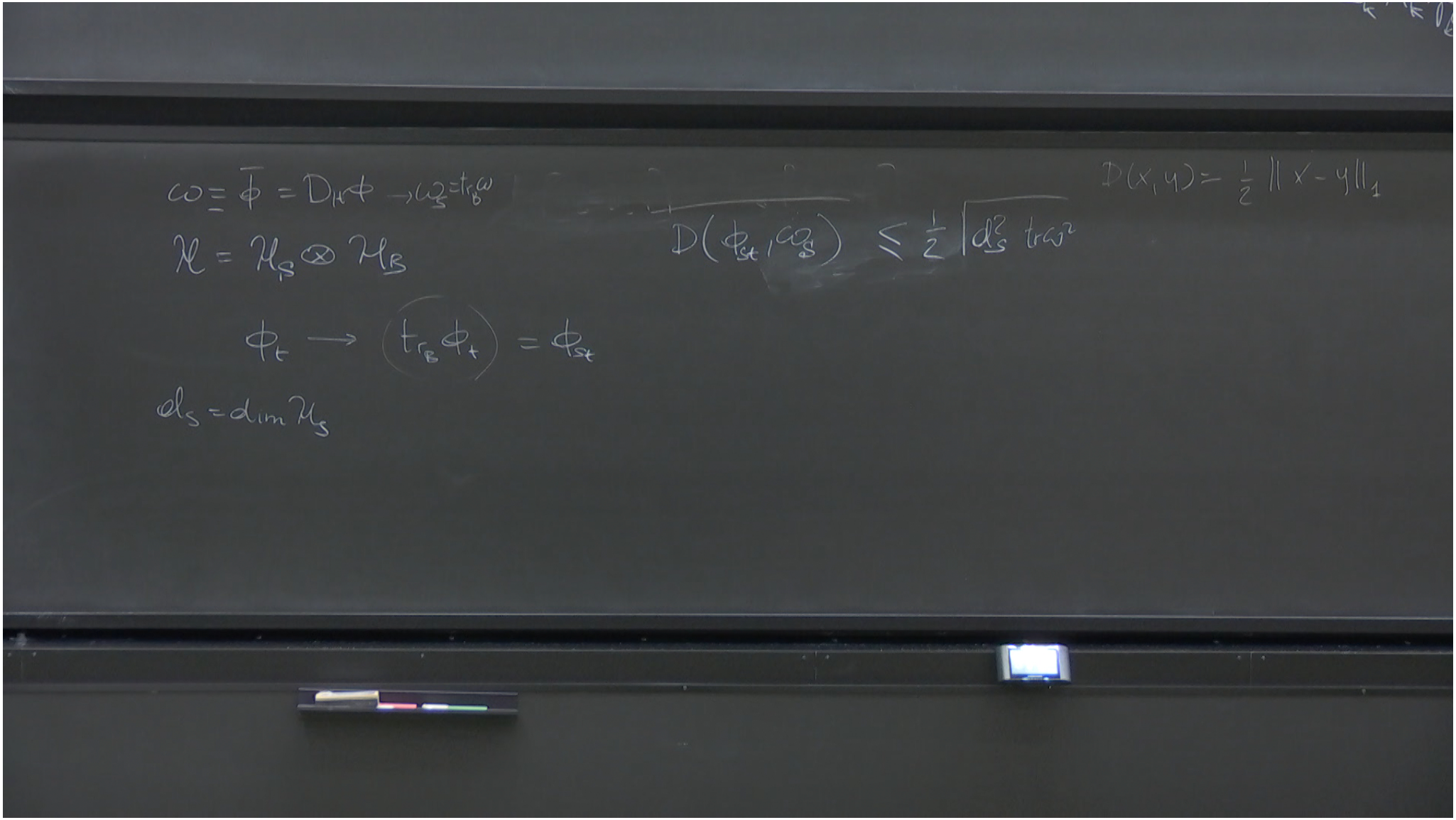
$\chi_k \equiv \theta_k^{(\lambda_2)} - \theta_k^{(\lambda_1)}$

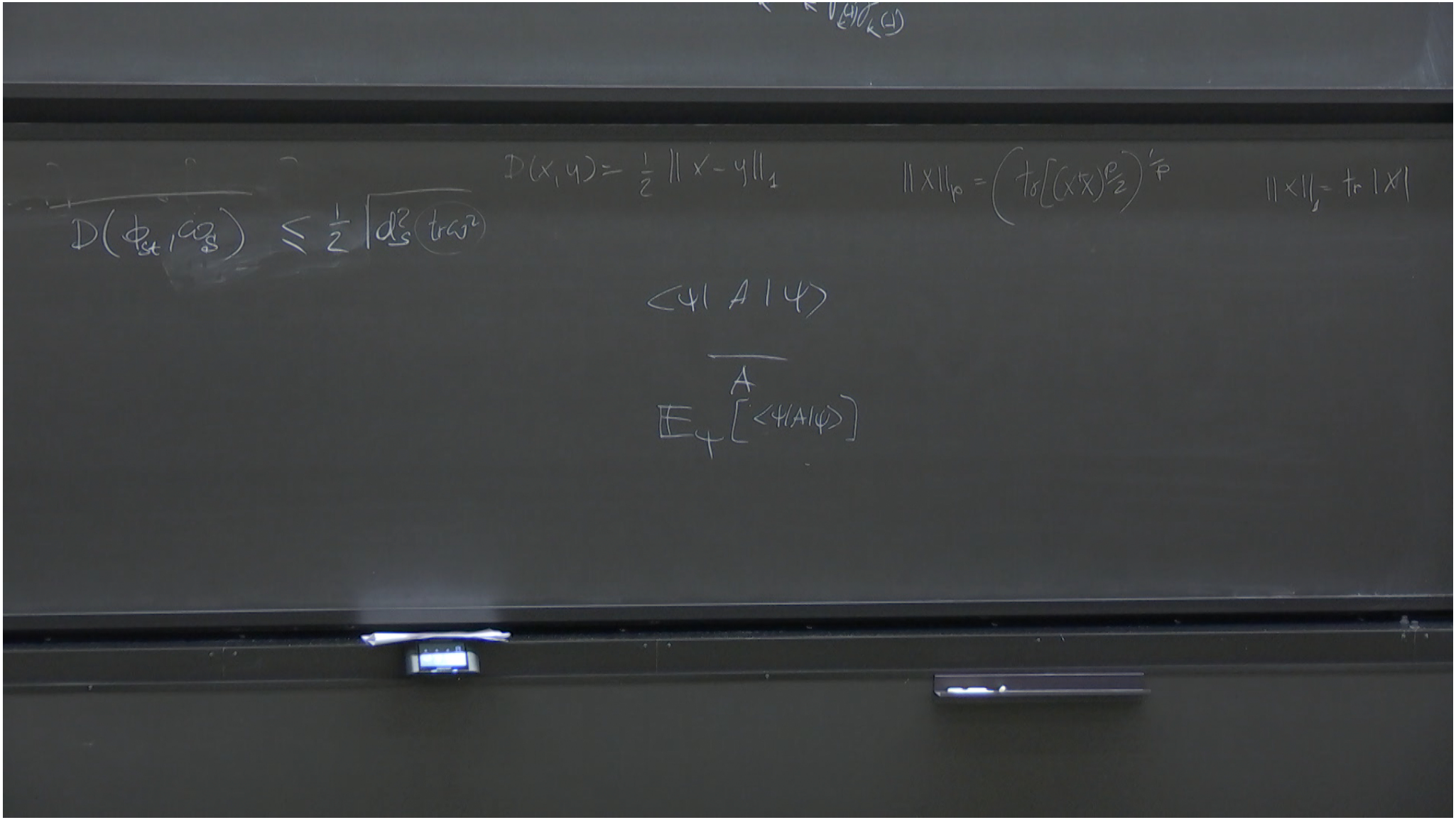
$H = \sum_k \Lambda_k^{(1)} \gamma_k^{\dagger(1)} \gamma_k^{(1)}$

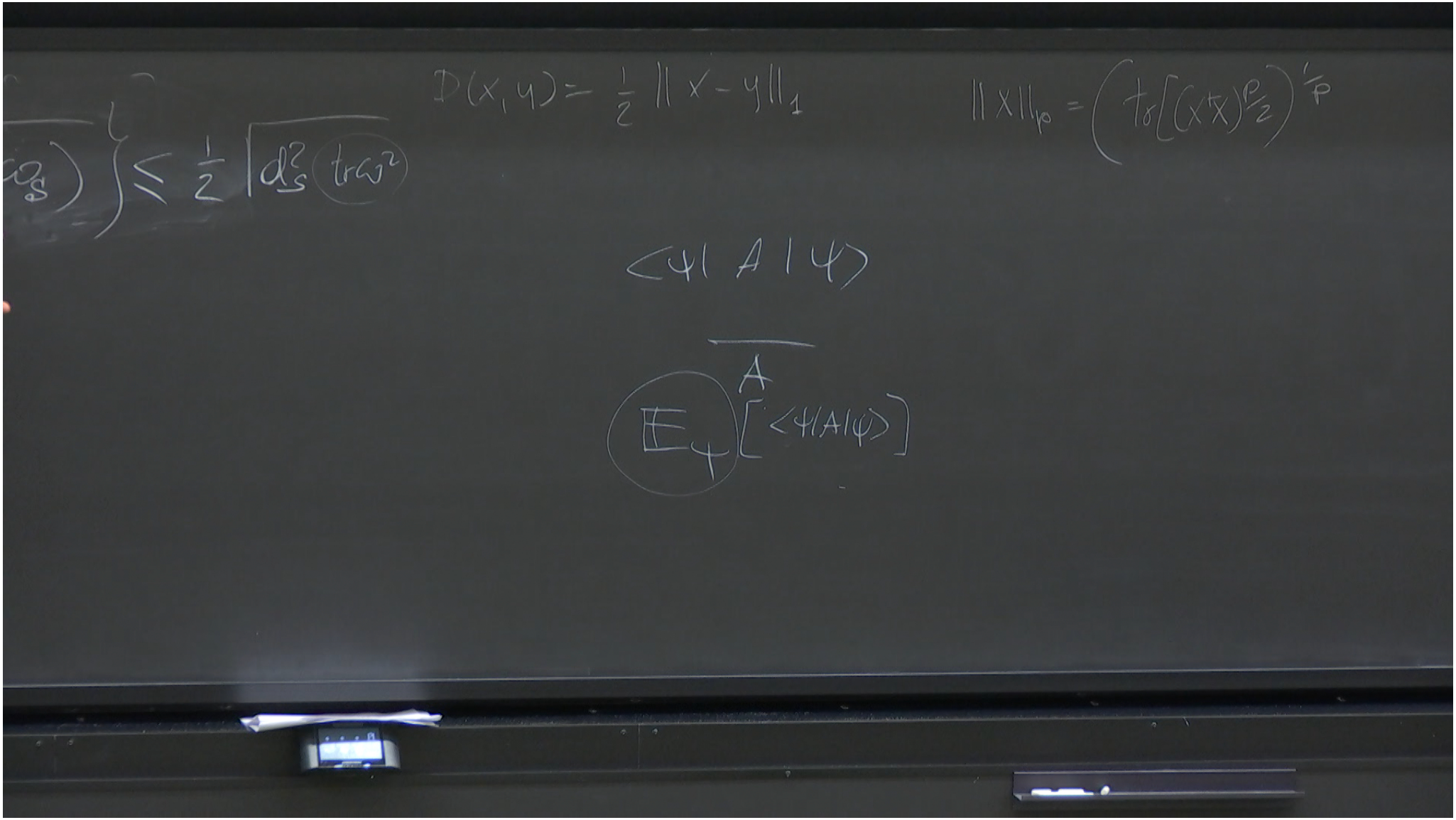

$\omega = \bar{\phi} = D_M \phi$
$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$
$\phi_t \longrightarrow tr_B \phi_t$
$tr[O_S \otimes 1_B \phi] = tr_S [O_S \phi_S]$
$\phi_S \equiv tr_B \phi$

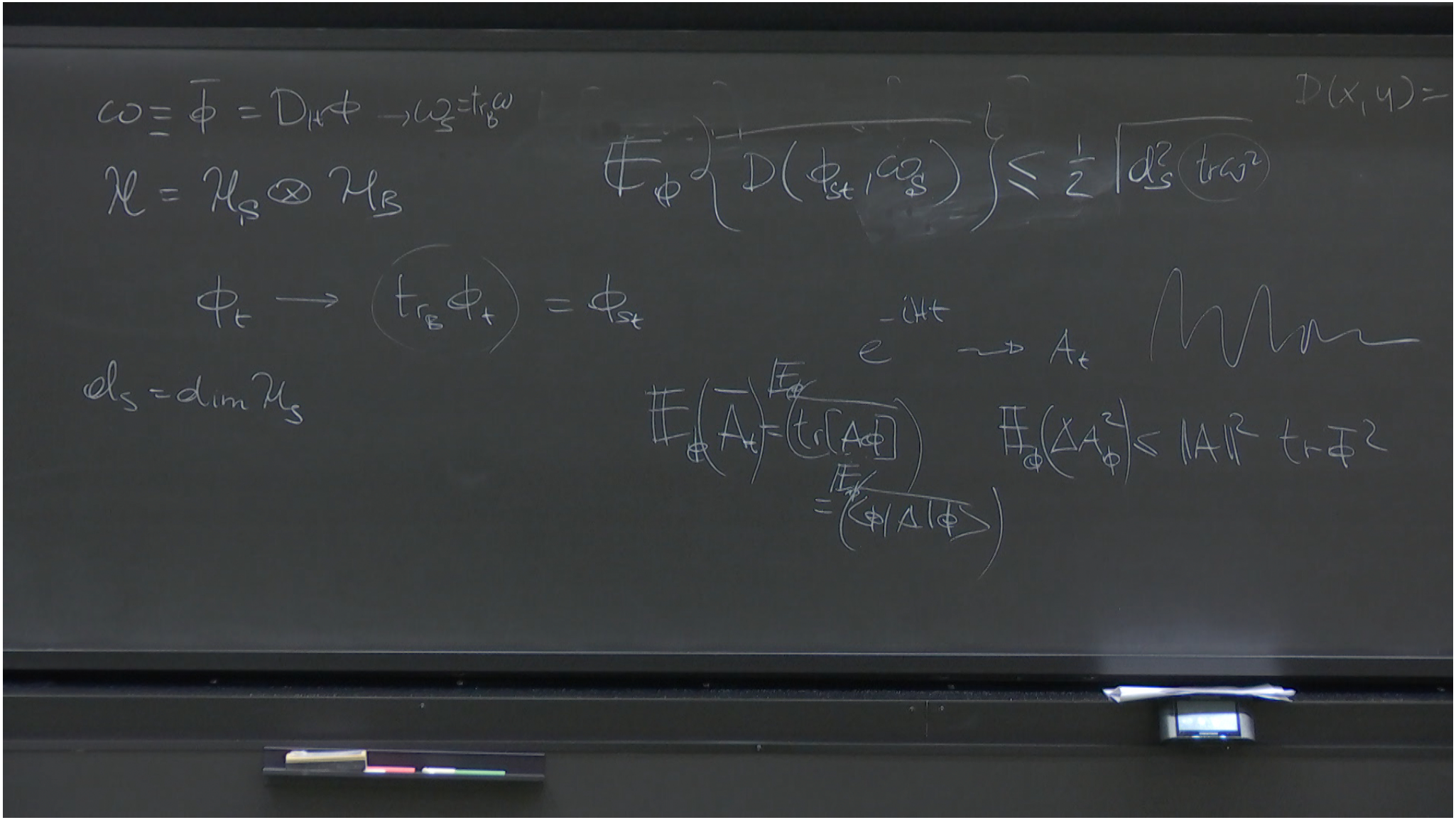

$\omega = \dot{\phi} = D_H \phi \rightarrow \omega_S = t_{r_B} \omega$
$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$
$D(\phi_{st}, \omega_S)$
$\phi_t \longrightarrow t_{r_B} \phi_t = \phi_{st}$


$D(x,y) = \frac{1}{2}\|x-y\|_1$
$\|X\|_p = \left(\mathrm{tr}\left[(X^*X)^{\frac{p}{2}}\right)^{\frac{1}{p}}$
$\|X\|_1 = \mathrm{tr}\,|X|$
$D(\Phi_{st}, \omega_s) \leq \frac{1}{2}\sqrt{d_s^2\,\mathrm{tr}\,\omega^2}$


$\omega = \bar{\phi} = D_{KL}\phi \to \omega_S = \mathrm{tr}_B \omega$
$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$
$D(\phi_{St}, \omega_S) \leq \frac{1}{2}\sqrt{d_S^2\, \mathrm{tr}\omega^2}$
$D(x,y) = \frac{1}{2}\|x-y\|_1$
$\phi_t \longrightarrow (\mathrm{tr}_B \phi_t) = \phi_{St}$
$d_S = \dim \mathcal{H}_S$


$D(x,y) = \frac{1}{2}\|x-y\|_1$
$\|X\|_p = \left(\mathrm{tr}\left[(X^\dagger X)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}}$
$\|X\|_1 = \mathrm{tr}|X|$
$D(\phi_{st}, \omega_S^2) \leq \frac{1}{2}\sqrt{d_S^2\,\mathrm{tr}\omega^2}$
$\langle\psi|A|\psi\rangle$
$\overline{A}$
$\mathbb{E}_\psi[\langle\psi|A|\psi\rangle]$


$D(x,y) = \frac{1}{2}\|x-y\|_1$
$\|X\|_p = \left(\mathrm{tr}\left[(X^*X)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}$
$\leq \frac{1}{2}\sqrt{d_s^2\,\mathrm{tr}\omega^2}$
$\langle\psi|A|\psi\rangle$
$\overline{A}$
$\mathbb{E}_\psi[\langle\psi|A|\psi\rangle]$


$\omega = \bar{\phi} = D_{H+t} \to \omega_S = tr_B \omega$
$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$
$\mathbb{E}_\phi \left\{ D(\phi_{st}, \omega_S) \right\} \leq \frac{1}{2} \sqrt{d_S^2 \, tr\omega^2}$
$D(x,y) =$
$\phi_t \longrightarrow tr_B \phi_t = \phi_{st}$
$e^{-iHt} \rightsquigarrow A_t$
$d_S = \dim \mathcal{H}_S$
$\mathbb{E}_\phi(\bar{A}_t) = tr[A\phi]$
$= \langle \phi | A | \phi \rangle$
$\mathbb{E}_\phi(\Delta A_\phi^2) \leq \|A\|^2 \, tr\phi^2$

$$\int d\phi\, f(\phi) = \int dU\, f(U|\phi_0\rangle\langle\phi_0|U^\dagger)$$

$|\phi\rangle = U|\phi_0\rangle$

$\mathcal{H} \simeq \mathbb{C}^n$

$\int dU = 1$

$V \in U(n)$

$\int d(VU) = \int dU = \int d(UV)$

$\langle \phi | U^\dagger \rangle$

$U \longrightarrow \Gamma_U$ unitary rep. on some space V

$\mathbb{E}(X) = \int dU \, \Gamma_U^\dagger X \Gamma_U$

$X \in \mathcal{B}(V)$

Schur's Lemma

$[\Gamma_U, X] = 0 \quad \forall U$

$X = \sum_{i \in \text{irreps of } \Gamma(U)} \lambda_i \Pi_i$ where $\lambda_i =$

Example 0 : Γ is irreducible

$U \longmapsto \Gamma_U$ unitary rep. on some space V

$$\mathbb{E}(X) = \int dU \, \Gamma_U^+ X \Gamma_U$$

$X \in \mathcal{B}(V)$

Schur's Lemma

$[\Gamma_U, X] = 0 \quad \forall U$

$X = \sum_{i \in \text{irreps of } \Gamma(U)} \lambda_i \Pi_i$ where $\lambda_i =$

Example 0 : Γ is irreducible $X = \lambda \mathbb{1}$