Title: A deformation invariant of 1+1D SQFTs

Speakers: Theo Johnson-Freyd

Series: Quantum Fields and Strings

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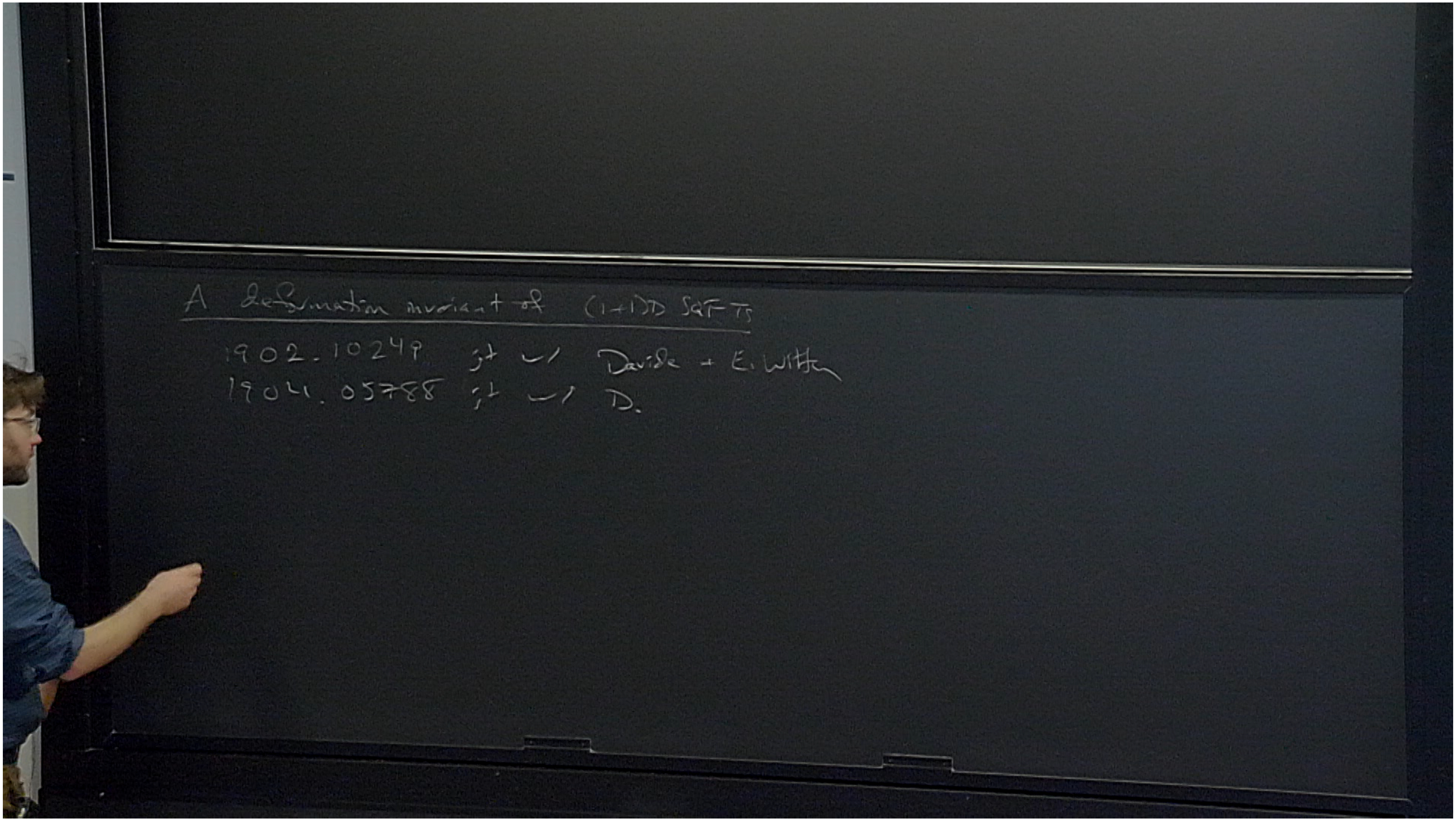
Abstract: The elliptic genus is a powerful deformation invariant of 1+1D SQFTs: if it is nonzero, then it protects the SQFT from admitting a deformation to one with spontaneous supersymmetry breaking. I will describe a "secondary" invariant, defined in terms of mock modularity, that goes beyond the elliptic genus, protecting SQFTs with vanishing elliptic genus. The existence of this invariant supports the hypothesis that the space of minimally supersymmetric 1+1D SQFTs provides a geometric model for universal elliptic cohomology. Based on joint works with D. Gaiotto and E. Witten.


A deformation invariant of (1+1)D SQFTs
1902.10249 jt w/ Davide + E. Witten
1904.05788 jt w/ D.

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General question: What is the topology of your favorite space of QFTs?

$\underbrace{\text{topology}}_{\text{the homotopy type.}}$

A deformation invariant of $\boxed{(1+1)D \text{ SQFTs}}$

For this talk

minimal ($N=(0,1)$) susy

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most basic question: when can two QFTs be related by "deformation"?

A deformation invariant of $\boxed{(1+1)\text{D SQFTs}}$

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For this talk

minimal $(N=(0,1))$ susy

General question: What is the topology of your favorite space of QFTs?

↑ the homotopy type.

most basic question: when can two qfts be related by "deformation"?

Example: if two qfts are "deformation equiv", then certainly all their "invariants" match.

Example: if two qfts are "deformation equiv", then certainly all their "anomalies" match.

E.g. For (1+1)d QFTs, there is a $\underbrace{\text{grav}}$ anomaly

$$c_R - c_L \in \tfrac{1}{2}\mathbb{Z}.$$

e.g. CFT

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E.g. For (1+1)d QFTs, there is a grav anomaly

$$\underbrace{\text{grav}}_{\text{anomaly}}$$

$\approx c_R - c_L \in \frac{1}{2}\mathbb{Z}$.

?if CFT

$SQFT_n :=$ Space of (1+1)d $N=(0,1)$ qfts w/ $c_R - c_L = \frac{n}{2}$.

Given $\mathcal{F} \in SQFT_n$, some natural questions

(1) Is SUSY spontaneously broken in $\mathcal{F}$?

(2) Does $\mathcal{F}$ admit a small deformation, after which SUSY is spont. broken?

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deformation equiv

Some natural questions:

...neously broken in F?

... small deformation,

...sy is spnt. broken?

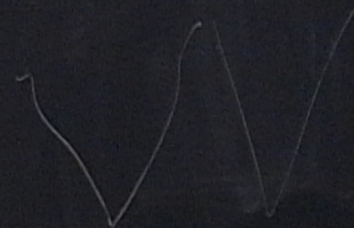
...eformation equiv

...d by some path in $SQFT_n$

...t SUSY breaking?

"deformation equivalence":

- related by "usual" (hep-th) deformation
- RG flow
- thys w/ the same IR.



E.g. $\mathcal{N}=(0,1)$ sigma model
with target round S^3.

E.g. $N=(0,1)$ sigma model with target round S^3 $\Big] \leftarrow$ to give a sigma model, I need to choose (quantum) B-field

E.g. $N=(0,1)$ sigma model with target round S^3 ← to give a sigma model, I need to choose (quantum) B-field

$S^3_k :=$ sigma model where the B-field has k units of H-flux.

$k \in \mathbb{Z}$.

Aside. For a general M, there might be an obstruction to existence of a $(0,1)$-sigma model. In general, "B-field" $\in$ torsor for $H^3(M)$

In S^3 case, this torsor is canonically trivialized

E.g. ... with target round S^3) ← to give a sigma model, I need to choose (quantum) B-field

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What is the IR limit of S^3_k?

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What is the IR limit of S^3_k?

Answer: if $k \neq 0$, then

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What is the IR limit of S^3_k?

Answer: if $k \neq 0$, then $S^3_k \rightsquigarrow$ $(0,1)$ WZW model $G = SU(2)$
levels $(|k|-1, |k|+1)$

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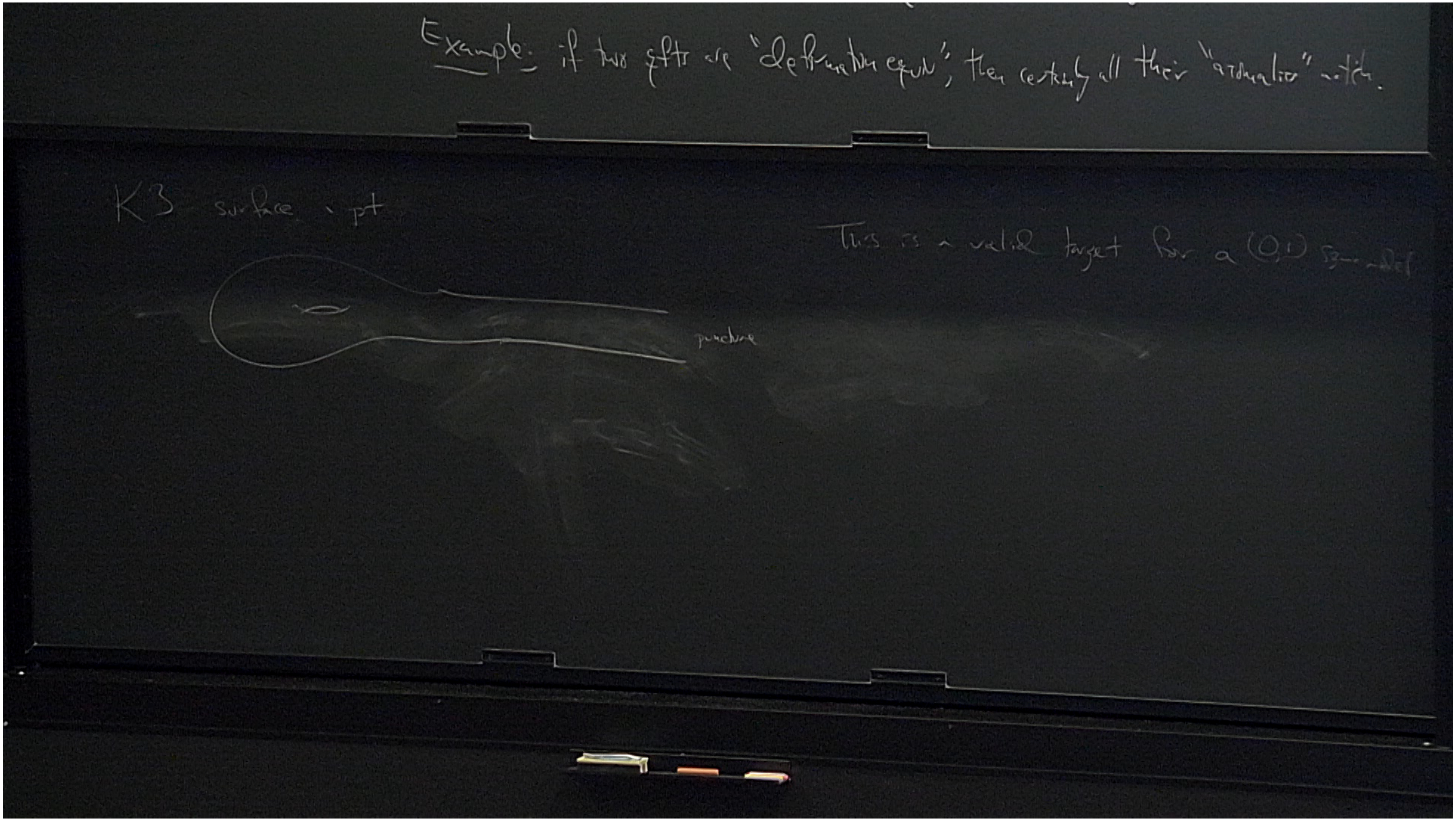
Answer: • if $k \neq 0$, then $S^3_k \rightsquigarrow$ (0,1) WZW model $G = SU(2)$ levels $(|k|-1, |k|+1)$

• if $k=0$, S^3_0 has spont susy breaking.


Example: if two qfts are "deformation equiv", then certainly all their "anomalies" match.
K3 surface \ pt
This is a valid target for a (0,1) sigma model
puncture

A deformation invariant of $\boxed{(1+1)D\ SQFTs}$

1902.10249 jt w/ Davide + E. Witten

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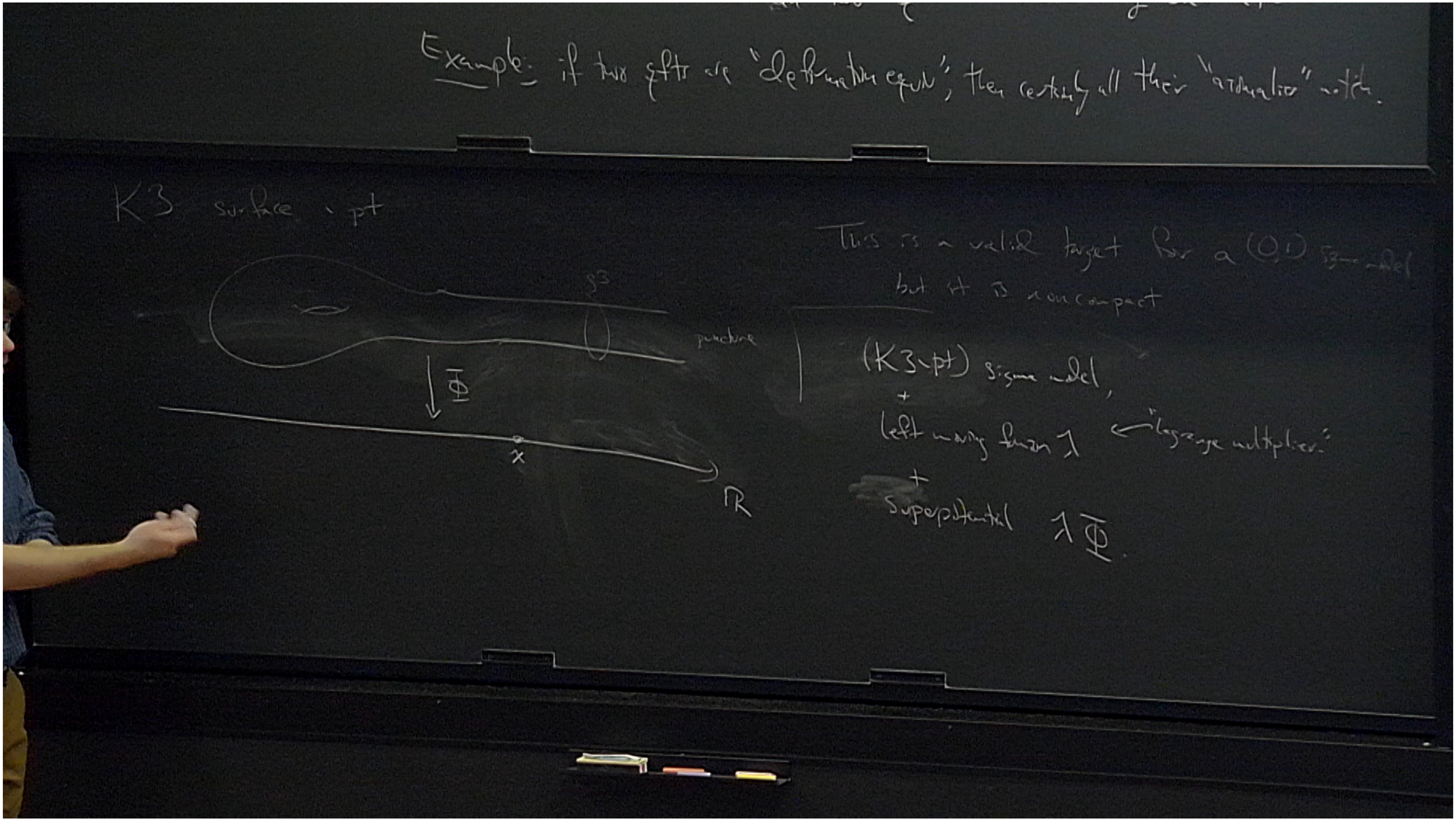
For this talk: minimal $(N=(0,1))$ susy.

General question: What is the topology of your favorite compact space of QFTs?

$\uparrow$ the homotopy type.

most basic question: when can two qfts be related by "deformation"?

Example: if two qfts are "deformation equiv", then certainly all their "anomalies" match.



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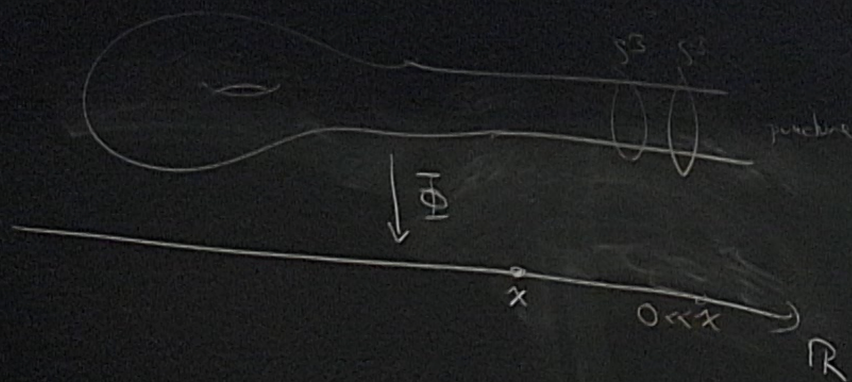
K3 surface ∖ pt
S3
puncture
Φ
x
R
This is a valid target for a (0,1) sigma model
but it is noncompact
(K3∖pt) sigma model,
+
left moving fermion λ
"Lagrange multiplier"
+
Superpotential λΦ.

Example: if two qfts are "deformation equiv", then certainly all their "anomalies" match.

$K3 \setminus pt$ surface $\setminus$ pt



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$(K3 \setminus pt)$ sigma model,

$+$

left moving fermion λ ← "lagrange multiplier"

$+$

superpotential $\lambda(\Phi - x)$

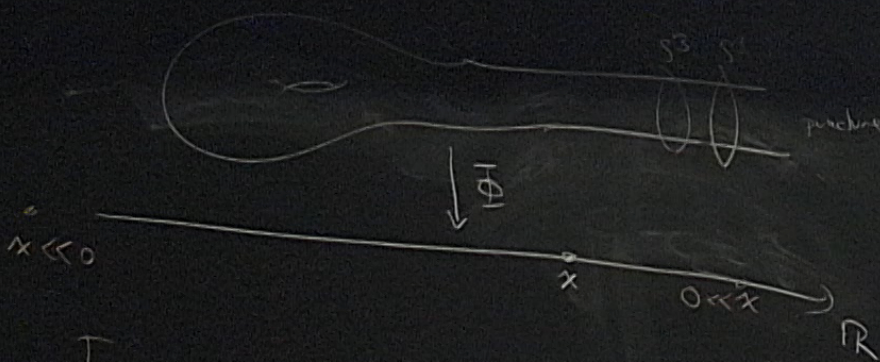
family of SQFTs depending on $x \in \mathbb{R}$. $\mathcal{F}_x$

$x \gg 0$, $\mathcal{F}_{+\infty} = S^3_{k=24}$.

"when can two qfts be related by deformation"?

Example: if two qfts are "deformation equiv", then certainly all their "anomalies" match.

K3 surface ∖ pt



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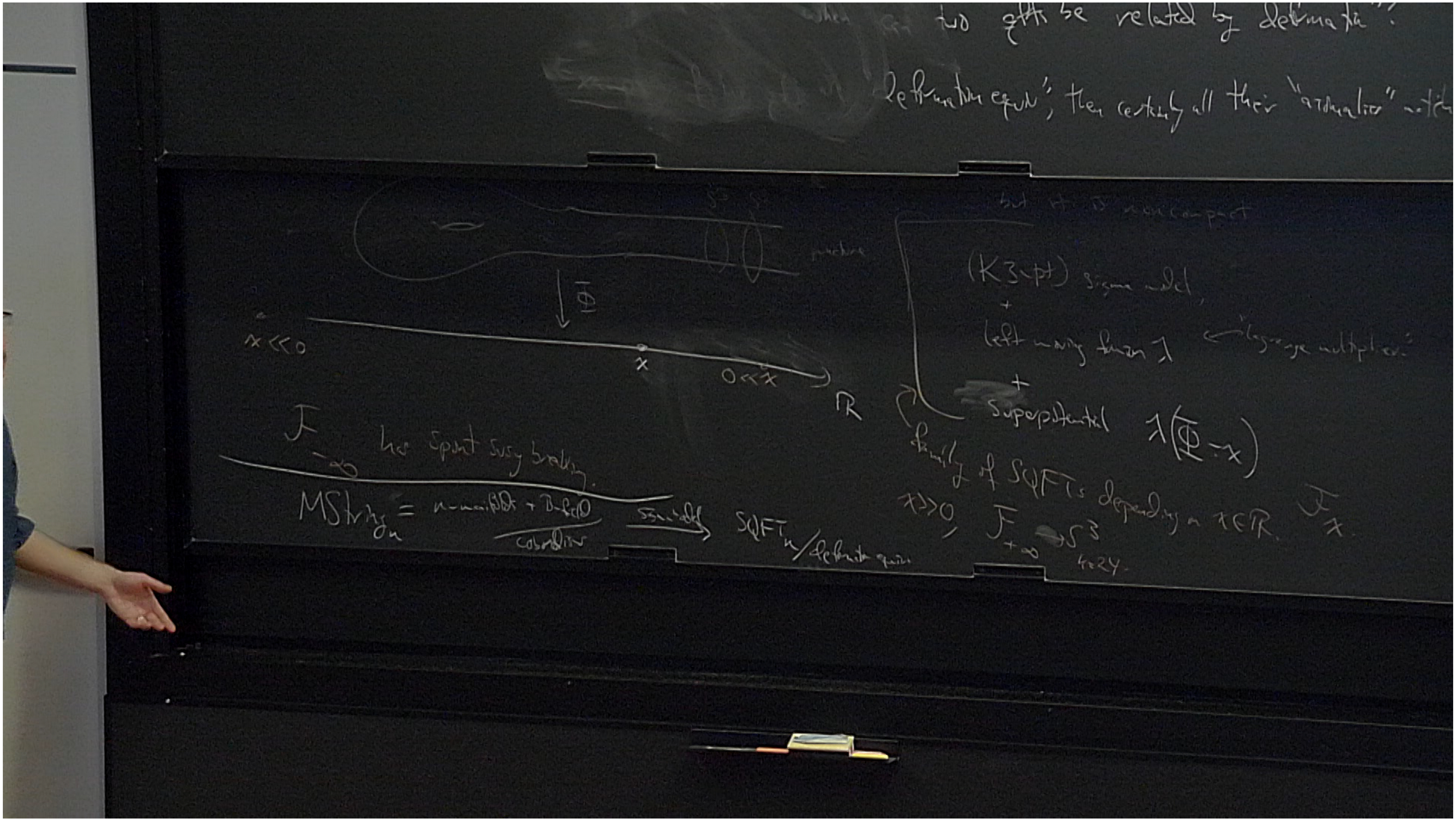
$\mathcal{F}_{-\infty}$ has spont susy breaking.

$x \gg 0$, $\mathcal{F}_{+\infty} \simeq S^3$ $k=24$.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Φ
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(K3×pt) sigma model,
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left moving fermion λ
"lagrange multiplier"
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superpotential λ(Φ − x)
family of SQFTs depending on x∈ℝ.
F_x
F_{-∞} has spont susy breaking.
MString_n = n-manifolds + B-field / cobordism
sigma model
SQFT_n / deformation
x>>0, F_{+∞} ≃ S^3

$\mathcal{F}_{-\infty}$ has spont susy breaking.

$MString_n = \dfrac{n\text{-manifolds} + B\text{-field}}{\text{cobordism}} \xrightarrow{\text{susy }\sigma\text{-model}} SQFT_n$ / deformation

family of SQFTs depending on $x \in \mathbb{R}$. $\mathcal{F}_x$.

$x \gg 0$, $\mathcal{F}_{+\infty} \to S^3_{k=24}$.

A deformation invariant of $\boxed{(1+1)D \text{ SQFTs}}$

Q: How to protect an SQFT from property (3)?

A. Find a deformation invariant which is 0 if susy is spont broken
$\neq 0$ in your SQFT.

Most famous example: Elliptic / Witten genus.

"Compact"
Spectral condition on Hamiltonian.

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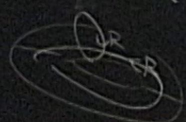
given $\mathcal{F} \in SQFT_n$, partition fn of $\mathcal{F}$ on tori w/ non-bounding spin str

$$Z_{RR}(\mathcal{F})$$



$\mathcal{F}_{-\infty}$ has spont susy breaking.

$MStrings_{\Omega_n}$ = nonminimal + Ball? cobordisms $\xrightarrow{\text{something}}$ $SQFT_n$/deformation

family of SQFTs depending on $\lambda \in \mathbb{R}$. $\mathcal{F}_\lambda$

$\lambda(\Phi - x)$

$\lambda \gg 0$, $\mathcal{F}_{+\infty} \to S^3$ $4 \cdot 24$.

A deformation invariant of $\boxed{(1+1)D \text{ SQFTs}}$

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$\hookrightarrow$ to define Z_{RR}.

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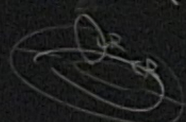

partition fn of $\mathcal{F}$ on flat tori w/ non-bounding spin str

$Z_{RR}(\mathcal{F})$

space of these is 3-real-dim: $\tau, \bar{\tau}$, volume

Some facts about $Z_{RR}(\mathcal{F})$ for $\mathcal{F}$ a $(0,1)$ SQFT:

(0) it is a def. invariant (uses that the string w/in {compact SQFTs})

A priori, $Z_{RR}(\mathcal{F})(\tau, \bar{\tau}, \text{volume})$ real analytic

$SL(2,\mathbb{Z})$-modular

(1) $\dfrac{\partial Z}{\partial \bar{\tau}} = \dfrac{\partial Z}{\partial \text{vol}} = 0$

Why does (1) hold?

$$\frac{\partial Z}{\partial \Sigma} = \langle T_{\Sigma,\Sigma} \rangle = \langle \hat{G}[G_\Sigma] \rangle$$

Why does (1) hold? (sketch)

$$\frac{\partial Z}{\partial \bar\tau} \propto \langle T_{\bar z \bar z} \rangle \propto \langle \hat{G}[G_{\bar z}] \rangle$$

for compact Σ,

in the nonbounding spin str.,

$$\langle \hat{G}[\mathcal{O}] \rangle = 0$$

Some facts about $Z_{RR}(\mathcal{F})$ for $\mathcal{F}$ a $(0,1)$ SQFT:

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(1) $\dfrac{\partial Z}{\partial \bar{\tau}} = \dfrac{\partial Z}{\partial \text{vol}} = 0$

(2) q-expansion of $Z_{RR}(\tau)$ is in $\mathbb{Z}((q))$

as rotating the S^1_R.

Why does (1) hold? (sketch)

$\dfrac{\partial Z}{\partial \bar{z}_{\text{vol}}} \propto \langle T_{\bar{z}\bar{z}} \rangle \propto \langle \hat{G}[G_{\bar{z}}] \rangle$

$T_{z\bar{z}}$

for (compact) SQFTs in the nonboundary spin str,

$\langle \hat{G}[\mathcal{O}] \rangle = 0$

Why (2)?

(2) q-expansion of $Z_{RR}(\tau)$ is in $\mathbb{Z}((q))$.

q, rotating the S^1_R.

why does (1) hold? (sketch)

$$\frac{\partial Z}{\partial \bar\tau} \propto \langle T_{\bar z \bar z} \rangle \propto \langle \hat{G}[G_{\bar z}] \rangle$$

vol $T_{z\bar z}$

for compact SQFTs in the nonbounding spin str, $\langle \hat{G}[\mathcal{O}] \rangle = 0$.

why (2)?

Z_{RR} can be computed by first compactifying on S^1_R and recognizing

Z_{RR} = index in the SQM sense.

Some facts about $Z_{RR}(\mathcal{F})$ for $\mathcal{F}$ a (0,1) SQFT:

(0) it is a def. invariant (uses that I'm staying w/in {compact SQFTs})

A priori, $Z_{RR}(\mathcal{F})(\tau, \bar{\tau}, \text{volume})$ real analytic

$SL(2,\mathbb{Z})$-modular

allowed a pole at $\tau = \infty$

(weakly) holo modular fn of wt $\frac{-c}{2}$

(up to some normalizing factor of $\eta(\tau)$)

(1) $\frac{\partial Z}{\partial \bar{\tau}} = \frac{\partial Z}{\partial \text{vol}} = 0$

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as rotating the S^1_R

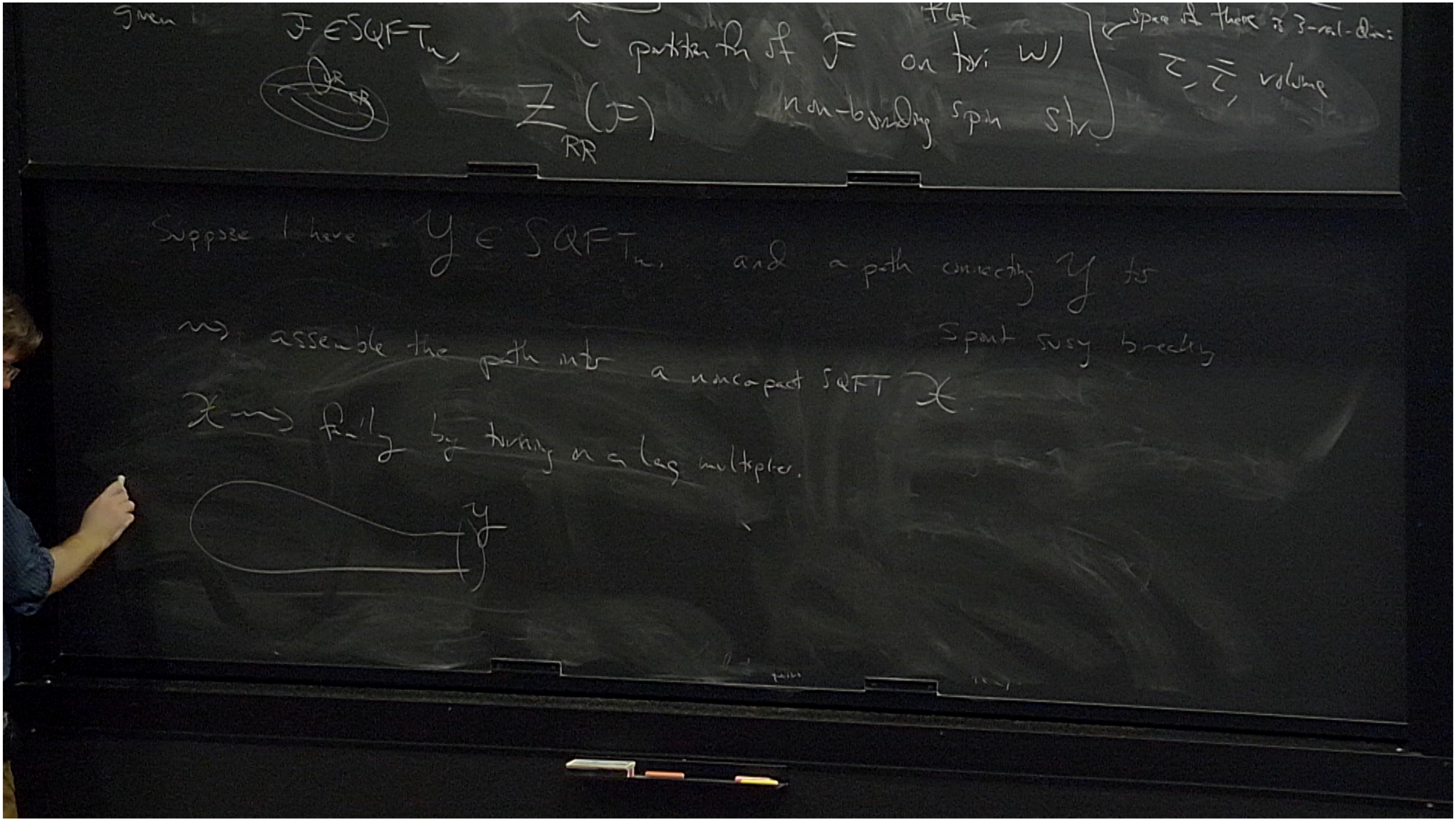
Z_{RR} can be computed by ... R ...

Z_{RR} = index in the SQM sense.

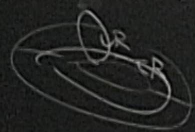
given: $\mathcal{F} \in SQFT_n$, $Z_{RR}(\mathcal{F})$ partition fn of $\mathcal{F}$ on tori w/ non-bounding spin str. space of these is 3-real-dim: $\tau, \bar{\tau}$, volume

Suppose I have $\mathcal{Y} \in SQFT_n$, and a path connecting $\mathcal{Y}$ to spont susy breaking

$\rightsquigarrow$ assemble the path into a noncompact SQFT $\mathcal{X}$.


given $\mathcal{F} \in SQFT_n$,
$Z_{RR}(\mathcal{F})$
partition fn of $\mathcal{F}$ on tori w/ flat non-bounding spin str
space of these is 3-real-dim: $\tau, \bar{\tau}$, volume
Suppose I have $\mathcal{Y} \in SQFT_n$, and a path connecting $\mathcal{Y}$ to spont susy breaking
⇝ assemble the path into a noncompact SQFT $\mathcal{X}$.
$\mathcal{X}$ ⇝ finally by turning on a Lag multiplier.
$\mathcal{Y}$

given $\mathcal{F} \in SQFT_n$,



$Z_{RR}(\mathcal{F})$ ← partition fn of $\mathcal{F}$ on tori w/ non-bounding spin str, flat

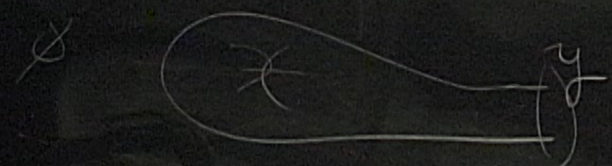
← space of these is 3-real-dim: $\tau, \bar{\tau}$, volume

Suppose I have $\mathcal{Y} \in SQFT_n$, and a path connecting $\mathcal{Y}$ to

⇝ assemble the path into a noncompact SQFT $\mathcal{X}$.

Spont susy breaking

$\mathcal{X}$ ⇝ finally by turning on a Lag multiplier.



A deformation invariant of $\boxed{\text{(1+1)D SQFTs}}$

Look at $Z_{RR}(\mathcal{X})$

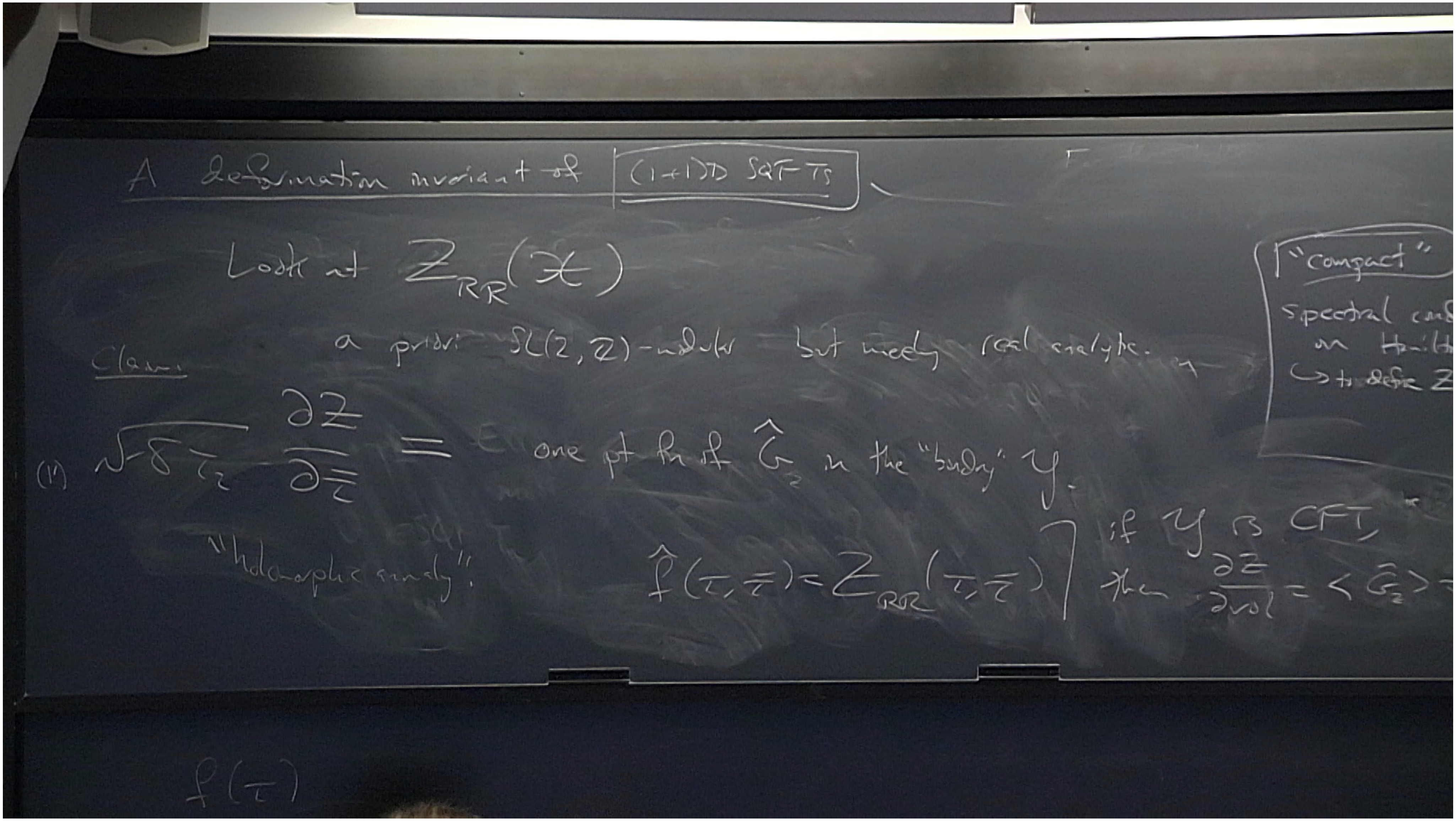
Claim. a priori $SL(2,\mathbb{Z})$-modular but merely real analytic.

$$\sqrt{-8}\,\tau_2 \frac{\partial Z}{\partial \bar{\tau}} = \text{one pt fn of } \hat{G} \text{ in the "bndry" } \mathcal{Y}.$$

"compact"

spectral condition on Hamiltonian

↳ to define Z_{RR}.



A deformation invariant of (1+1)D SQFTs
Look at Z_RR(𝒳)
Claim
a priori SL(2,Z)-modular but merely real analytic.
(1') √-8 τ_2 ∂Z/∂τ̄ = one pt fn of Ĝ_2 in the "bndry" Y.
"holomorphic anomaly"
f̂(τ,τ̄) = Z_RR(τ,τ̄)
if Y is CFT, then ∂Z/∂vol = < Ĝ_2 >
"compact"
spectral cond
f(τ)

A deformation invariant of $\boxed{(1+1)D \text{ SQFTs}}$

Look at $Z_{RR}(\mathcal{T})$

a priori $SL(2,\mathbb{Z})$-modular but merely real analytic.

Claim:

$$\sqrt{-g}\,\tau_2 \frac{\partial Z}{\partial \bar{\tau}} = \text{one pt fn of } \hat{G}_2 \text{ in the "bndry" } \mathcal{T}.$$

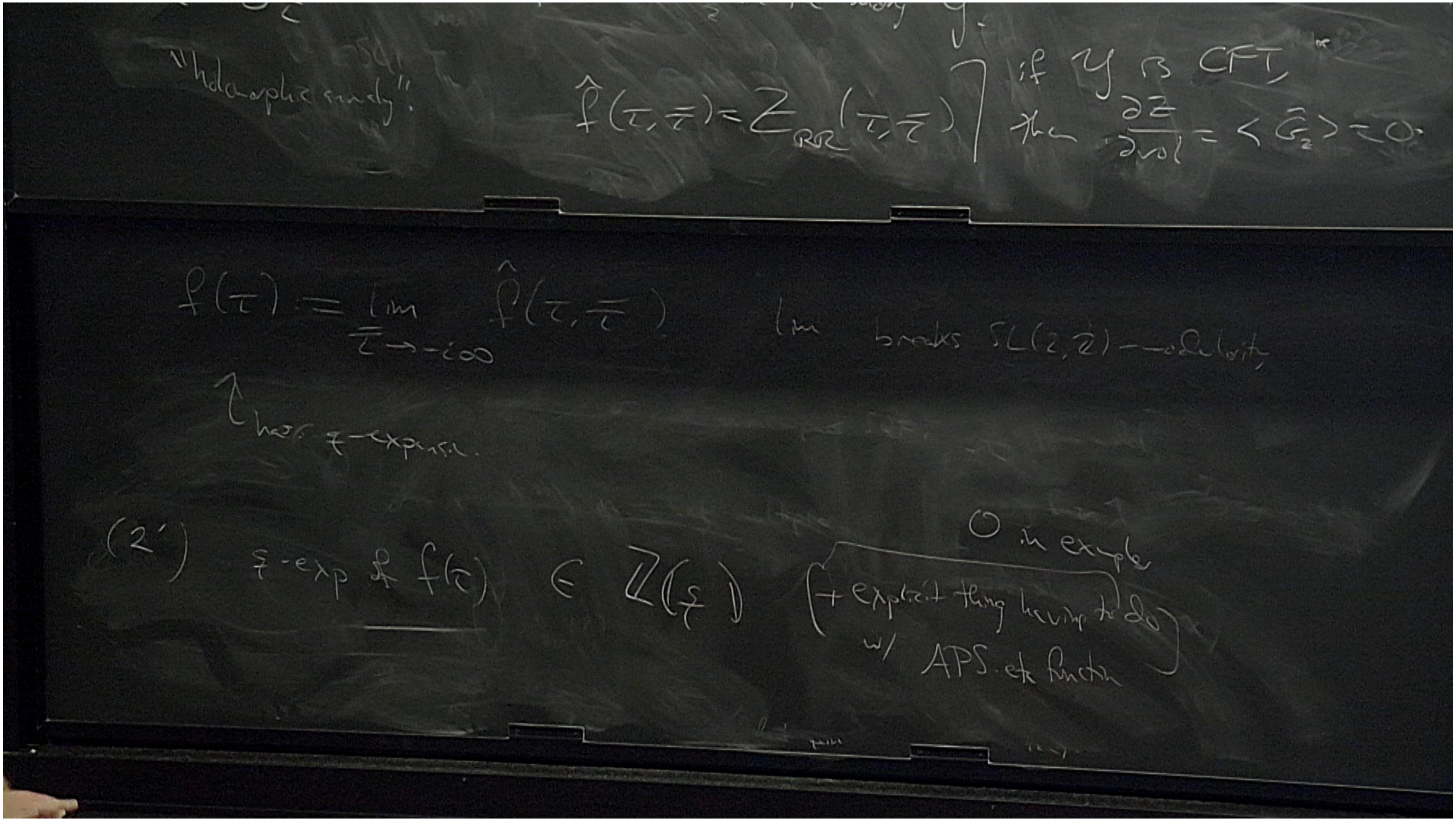
"holomorphic anomaly"!

$$\hat{f}(\tau,\bar{\tau}) = Z_{RR}(\tau,\bar{\tau})$$

if $\mathcal{T}$ is CFT, then $\frac{\partial Z}{\partial \mathrm{vol}} = \langle \hat{G}_2 \rangle = 0$.

"compact"
spectral condition on Hamiltonian
↪ to define Z_{RR}.

$f(\tau) = \lim \hat{f}(\tau,\bar{\tau})$ lim breaks $SL(2,\mathbb{Z})$-modularity


"holomorphic anomaly":
$\hat{f}(\tau,\bar{\tau}) = Z_{RR}(\tau,\bar{\tau})$
if Y is CFT, then $\frac{\partial Z}{\partial vol} = \langle \bar{G}_2 \rangle = 0$.
$f(\tau) = \lim_{\bar{\tau} \to -i\infty} \hat{f}(\tau,\bar{\tau})$
lim breaks SL(2,Z)-modularity
hol. q-expansion.
(2') q-exp of $f(\tau) \in \mathbb{Z}((q))$
0 in example
(+ explicit thing having to do w/ APS eta function)

(2) q-expansion of $Z_{RR}(\tau)$ is in $\underbrace{\mathbb{Z}[[q]]}_{\text{is, rotating the } S^1_R}$.

<u>Summary:</u>

Given $\mathcal{Y}^{SCFT}$ $\quad g(\tau,\bar{\tau}) = \langle \hat{G} \rangle$

If $\mathcal{Y}$ is null definable, then

$\exists\ \hat{f}(\tau,\bar{\tau})$ solving

(2) ξ-expansion of $Z_{RR}(\tau)$ is in $\mathbb{Z}((\xi))$.

is rotating the S^1_R.

<u>Summary:</u>

Given $\mathcal{Y}$ SCFT: $g(\tau,\bar\tau) = \langle \hat{G} \rangle$

If $\mathcal{Y}$ is null deformable, then $\exists\, \hat{f}(\tau,\bar\tau)$ solving $\boxed{(*)}$ w/ $\mathbb{Z}$-ξ-expan.

$g \rightsquigarrow [\hat{f}] \in \dfrac{\text{real and modular}}{\text{holomorphic}}$

$\rightsquigarrow [f] \in \dfrac{\mathbb{C}((\xi))}{MF}$

<u>Conversely:</u> This class $[f] \in \dfrac{\mathbb{C}((\xi))}{MF + \mathbb{Z}((\xi))}$ is a def inv. of $\mathcal{Y}$.

$0 = [f] \in \dfrac{\mathbb{C}((\xi))}{MF + \mathbb{Z}((\xi))}$

breaks $SL(2,\mathbb{Z})$-modularity

For S^3_k,

this class is

$k\, c_2 + m\kappa + \mathbb{Z}$[illegible]

0 in example

(+ explicit thing having to do w/ APS eta function) breaks $SL(2,\mathbb{Z})$ — modularity

$\mathcal{O}$ in example

(+ explicit thing having to do w/ APS. etc function)

For $S^3_{k,1}$

this class is

$k\,c_2 + m\kappa + \mathbb{Z}(\kappa\text{-}?)$

So $S^3_{k=1}$ not null definable.

$$(1') \quad \boxed{\sqrt{-g}\,\tau_2 \frac{\partial Z}{\partial \bar{\tau}} = g(\tau,\bar{\tau})} \quad \text{one pt fn of } \hat{G}_2 \text{ in the "bndry" } \mathcal{Y}.$$

"holomorphic anomaly".

$$\hat{f}(\tau,\bar{\tau}) = Z_{RR}(\tau,\bar{\tau})$$

if $\mathcal{Y}$ is CFT, then $\frac{\partial Z}{\partial \mathrm{vol}} = \langle \hat{G}_2 \rangle = 0$.

basic fields in WZW

all: bosonic WZW $SU(2)$ level $k-1$. J, $\bar{J}$

and three right moving fermions ψ_1, ψ_2, ψ_3

$$\hat{G} = \sqrt{-1}\sqrt{\frac{2}{k+1}} :\psi_1\psi_2\psi_3: + \# :\bar{J}_i \psi_i:$$

... larity

For $S^3_k \to WZW$

this class is

$k\,G_2 + \ldots$

so S^3

$k=1$ not

null defined.

$$(1') \quad \boxed{\sqrt{-g}\,\tau_2 \frac{\partial Z}{\partial \bar\tau} = g(\tau,\bar\tau)_{\text{one}}}$$ pt fn of $\hat{G}_2$ in the "bndry" Y.

"holomorphic anomaly".

$$\hat{f}(\tau,\bar\tau) = Z_{RR}(\tau,\bar\tau)$$ if Y is CFT, then $\frac{\partial Z}{\partial \text{vol}} = \langle \hat{G}_2 \rangle = 0$.

basic fields in WZW

bosonic WZW $SU(2)$ level $k-1$. $J, \bar{J}$

and three right moving fermions ψ_1, ψ_2, ψ_3

$$\hat{G} = \sqrt{-1}\sqrt{\frac{2}{k+1}} :\psi_1\psi_2\psi_3: + \# \cancel{:\bar{J}_i \psi_i:} \to 0$$

one-pt fn = $\eta(\bar\tau)^3 \, Z^{\text{bosonic}}_{WZW}$

For $S^3_k \to WZW$

this class is

$k\,G_2 + \ldots$

So S^3

k_1 not

null/defm...

(1') $\boxed{\sqrt{-g}\,\tau_2 \frac{\partial Z}{\partial \bar\tau} = \hat g(\tau,\bar\tau)}$ one pt fn of $\hat G_z$ in the "bndry" Y.

"holomorphic anomaly"!

$\hat f(\tau,\bar\tau) = Z_{RR}(\tau,\bar\tau)$] if Y is CFT, then $\frac{\partial Z}{\partial \mathrm{vol}} = \langle \hat G_z \rangle = 0$.

one-pt fn $= \sqrt{\frac{2}{k+1}}\, \eta(\bar\tau)^3 \left(Z^{\text{bosonic}}_{WZW} \right) \leftarrow |\text{vect-valued modular fn}|^2$

So S^3_{k+1} not null-deformable.

Harvey-Murthy-Nazaroglu give an $\hat f(\tau,\bar\tau)$