Title: Quantum Field Theory for Cosmology - Lecture 7

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Collection: Quantum Field Theory for Cosmology (Kempf)

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QFT for Cosmology, Achim Kempf, Lecture 7

Note Title

## The driven harmonic oscillator cont'd:

### D. Energy eigenstates

* Recall $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right) J(t)$

$$= \begin{cases} \omega\left(a^+_{in}\, a_{in} + \frac{1}{2}\right) & \text{for } t < 0 \\ \text{something} & \text{for } 0 \leq t \leq T \\ \omega\left(a^+_{out}\, a_{out} + \frac{1}{2}\right) & \text{for } T < t \end{cases}$$

(Q: How come that $\hat{H}(t<0) \neq \hat{H}(t>T)$ ?
A: We use the Heisenberg picture!)

Here, $a_{in} := a(0)$, $a_{out} := a(T)$ and $a_{out} = a_{in} + J_0$

with: $J_0 := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$

* For $t<0$, we diagonalized the Hamiltonian

$$\hat{H}(t) = \omega\left(a^+_{in} a_{in} + \tfrac{1}{2}\right) = \hat{H}_{t<0} = \text{const.}$$

by using $[a_{in}, a^+_{in}] = 1$ to construct its eigenbasis:

$$\hat{H}_{t<0} \, | n_{in} \rangle = E^{(in)}_n \, | n_{in} \rangle$$

Namely:

$$E^{(in)}_n = \omega\left(n + \tfrac{1}{2}\right) \quad , n = 0, 1, 2, 3 \ldots$$

$$| n_{in} \rangle := \frac{1}{\sqrt{n!}} \left(a^+_{in}\right)^n | 0_{in} \rangle$$

Note: The set $\{ | n_{in} \rangle \}$ is a Hilbert basis of the Hilbert space $\mathcal{H}$.

* By $t > T$, the Hamiltonian has become a different operator:

$$\hat{H}(t) = \omega\left(a^{+}_{out} a_{out} + \frac{1}{2}\right) = \hat{H}_{t>T} = \text{const.}$$

What are its eigenvectors $|n_{out}\rangle$ and eigenvalues $E_n^{out}$?

Observation:

We have:

$$[a_{out}, a^{+}_{out}] = 1$$

$\Rightarrow$ we can construct the eigenbasis of $H_{t>T}$ with the same method as the eigenbasis of $H_{t<0}$:

up to normalization and phase

* There is a unique vector $|0_{out}\rangle \in \mathcal{H}$ obeying:

$$a_{out}\,|0_{out}\rangle = 0$$

* We define the set of vectors $\{|n_{out}\rangle\}$:

$$|n_{out}\rangle := \frac{1}{\sqrt{n!}}(a^+_{out})^n\,|0_{out}\rangle$$

The operators $\hat{H}_{t<0}$ and $\hat{H}_{t>T}$ are different and have different eigenvectors: $|n_{in}\rangle$ and $|n_{out}\rangle$. Why are the eigenvalues the same? They both describe a free oscillator of frequency $\omega$.

* Proposition:

$$\hat{H}_{t>T}\,|n_{out}\rangle = E_n^{(out)}\,|n_{out}\rangle \quad \text{with} \quad E_n^{(out)} = \omega\left(n+\frac{1}{2}\right) = E_n^{(in)}$$

* Proposition:

The set $\{|n_{out}\rangle\}$ is a ON Hilbert basis of the Hilbert space $\mathcal{H}$.

## How are the two bases related?

* Recall: Both, $\{|n_{in}\rangle\}$ and $\{|n_{out}\rangle\}$ are ON bases of $\mathcal{H}$.

  $\Rightarrow$ Each basis vector $|n_{in}\rangle$ is a linear combination of the basis vectors $\{|m_{out}\rangle\}$ and vice versa.

* Therefore, in particular:

  There must exist coefficients $\Lambda_n \in \mathbb{C}$ so that:

$$|0_{in}\rangle = \sum_n \Lambda_n |n_{out}\rangle$$

"Bogolubov Transformation"

* Meaning of the $\Lambda_n$?

- Recall: The system's state is frozen in state $|\psi\rangle = |0_{in}\rangle$.

- Assume we measure at a time $t > T$ the energy, i.e., we measure

$$\hat{H}(t) = \omega\left(a_{out}^{+} a_{out} + \tfrac{1}{2}\right)$$

- What is the probability amplitude for finding the energy eigenvalue $E_n$?

- Clearly:

$$\text{prob.amp.}(|n_{out}\rangle \text{ at } t > T) = \langle n_{out}|\psi\rangle$$

i.e.:

$$\text{prob.}(|n_{out}\rangle \text{ at } t > T) = |\langle n_{out}|\psi\rangle|^2$$

Calculate:

$$\langle n_{out} | \psi \rangle = \langle n_{out} | 0_{in} \rangle$$

$$= \langle n_{out} | \sum_m \Lambda_m | m_{out} \rangle$$

$$= \Lambda_n$$

$\Rightarrow$ If the oscillator started in its ground state, then at time $t > T$ the probability for finding the oscillator in its n'th excited state is given by:

$$\text{prob.}(|n_{out}\rangle \text{ at } t > T) = |\Lambda_n|^2$$

Remark: In QFT, this will be the prob. for finding n particles after charges and currents $J(x,t)$ excited the vacuum.

## Calculation of $\Lambda_n$:

**Proposition:** $\Lambda_n = e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n$

**Proof:** The claim is that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$.

We need to check that indeed: $a_{in}|0_{in}\rangle = 0$

Using $a_{out} = a_{in} + J_0$, we need to check: $(a_{out} - J_0)\,|0_{in}\rangle = 0$

Indeed:

$$(a_{out} - J_0) \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2} (a_{out} - J_0) \sum_n J_0^n \overbrace{\frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n!}}}^{\frac{1}{n!}} (a_{out}^+)^n |0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2}(a_{out} - J_0)\, e^{J_0 a_{out}^+} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}|J_0|^2}\Big(\underbrace{a_{out} e^{J_0 a^+_{out}}} - J_0 e^{J_0 a^+_{out}}\Big)|0_{out}\rangle$$

$\|$ using $AB = [A,B] + BA$

$$= e^{-\frac{1}{2}|J_0|^2}\Big(\overbrace{\underbrace{[a_{out}, e^{J_0 a^+_{out}}]} + e^{J_0 a^+} a_{out}} - J_0 e^{J_0 a^+_{out}}\Big)|0_{out}\rangle$$

$\|$

$$\overset{(*)}{=} e^{-\frac{1}{2}|J_0|^2}\Big(\overbrace{(J_0 - J_0) e^{J_0 a^+_{out}}} + e^{J_0 a^+_{out}} a_{out}\Big)|0_{out}\rangle = 0 \quad ✓$$

**Note:** In the last step, (*), we used that: $[a_{out}, e^{J_0 a^+_{out}}] = J_0 e^{J_0 a^+_{out}}$.

**Exercise:** Show that, more generally, $[a,a^+]=1$ implies $[a, f(a^+)] = f'(a^+)$ by induction.

Hint: Show that: $[a,a^+]=1$, $[a,a^{+2}] = 2a^+$, $[a,a^3] = 3a^2$, $\ldots$, $[a, a^{+n}] = n(a^+)^{n-1}$

**Exercise:** Verify that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$ obeys $\langle 0_{in}|0_{in}\rangle = 1$.

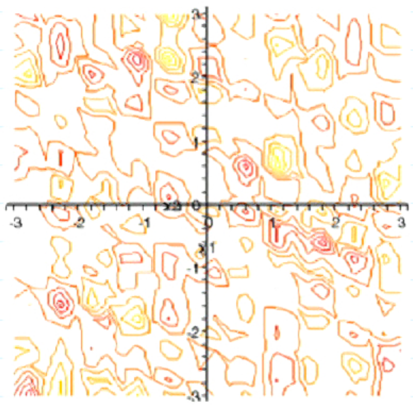
$$\partial_x x - x\partial_x = 1 \qquad [\partial_x, f(x)] = f'(x)$$

$$\underbrace{\partial_x x \psi(x)}_{=} - x\partial_x \psi(x)$$

$$\psi(x) + x\cancel{\partial_x \psi(x)} - x\cancel{\partial_x \psi(x)} = \psi(x)$$

# Apply this strategy to the mode oscillators in QFT:

Making waves...



Making EM waves...

$e^-$

**Recall:**

$$\hat{H}(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) \quad + J(x,t)\,\hat{\phi}(x,t)\; d^3x$$

**Example interpretation:**

* $\hat{\phi}(x,t)$ may be viewed as a slightly simplified version of the quantum electromagnetic field.

* $J(x,t)$ may be viewed as a simplified version of a given classical electric charge and current density function.

Example:

A (Klein-Gordon) charge traveling a path $\tilde{x}^i(t)$:

Then: $J(x,t) = q\,\delta(x - \tilde{x}(t))$

## In- and out periods

Let us consider the case where

$$J(x,t)=0 \quad \text{for all} \quad t \notin [0,T]$$

$\Rightarrow$ It suffices to consider the periods $t<0$ and $t>T$ in both of which $J(x,t)=0$ (and then to relate the bases).

## The free (i.e., undriven) QFT: $(t<0 \text{ or } t>T)$

$$\hat{H}(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t)\ d^3x$$

* We need to solve:

$$\dot{\hat{\pi}}(x,t) - (\Delta - m^2)\,\hat{\phi}(x,t) = 0$$

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\,\delta^3(x-x')$$

* Fourier transformed,

$$\hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\phi}(x,t)\, e^{-ikx}\, d^3x$$

we need to solve:

$$\ddot{\hat{\phi}}_k(t) + (\underbrace{k^2 + m^2}_{=:\,\omega_k^2})\,\hat{\phi}_k(t) = 0 \qquad (EoM)$$

Definition: $\omega_k^2 := k^2 + m^2$, where $k^2 = \sum_{i=1}^{3} k_i^2$

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i\,\delta^3(k+k') \qquad (CCRs)$$

* Recall: $\hat{\phi}^+(x,t) = \hat{\phi}(x,t)$ means $\hat{\phi}_k^+(t) = \hat{\phi}_{-k}(t)$.

## Solution strategy due to Fock:

* Proceed analogously to the driven oscillator, e.g., during $t<0$ :

  - Introduce new variables:

    QM: $a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$

    QFT: $a_k(t) := \sqrt{\frac{\omega_k}{2}}\,\hat{\phi}_k(t) + i\frac{1}{\sqrt{2\omega_k}}\,\hat{\pi}_k(t)$

  - Equation of motion and CCRs:

    QM: $\dot{a}(t) = -i\,\omega\, a(t) \qquad [a(t), a^+(t)] = 1$

    Exercise: verify

    QFT: $\dot{a}_k(t) = -i\,\omega_k\, a_k(t) \qquad [a_k(t), a^+_{k'}(t)] = \delta^3(k-k')$

  - Remark: Valid only while no force and while $\omega$ is constant.

- Solution, using an initial condition:

QM: $a(t) = e^{-i\omega t} a_{in}$ , $[a_{in}, a^+_{in}] = 1$

QFT: $a_k(t) = e^{-i\omega_k t} a_{in_k}$ , $[a_{in_k}, a^+_{in_{k'}}] = \delta^3(k-k')$

- Explicitly ⇒

QM: $$\hat{q}(t) = \frac{1}{\sqrt{2}}\left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a^+_{in}\right)$$

QFT: $$\hat{\phi}_k(t) = \frac{1}{\sqrt{2}}\left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a^+_{in_{-k}}\right) \qquad (S)$$

Exercise: verify

$$\left(\text{i.e.: } \hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in_k} e^{-i\omega_k t + ikx} + a^+_{in_k} e^{i\omega_k t - ikx}\right) d^3k\right)$$

## The Hilbert space of states:

* Analogous to the case of QM, there is a vector, $|0_{in}\rangle \in \mathcal{H}$, which obeys:

$$a_{in_k}|0_{in}\rangle = 0, \text{ now for all vectors } k.$$

$$\left(|0_{in}\rangle = \bigotimes_k |0_{in_k}\rangle\right)$$

* The Hamiltonian reads (for $t<0$):

Exercise: verify

$$\hat{H} = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}_k^+ \hat{\pi}_k + \omega^2 \hat{\phi}_k^+ \hat{\phi}_k \, d^3k$$

This $\infty$ is called a "Infrared divergence"

$$= \int_{\mathbb{R}^3} \omega_k \left(a_{in_k}^+ a_{in_k} + \frac{1}{2}\delta^3(0)\right) d^3k$$

In a box: $\hat{H} = L^{-3/2}\sum_k \omega_k \left(a_{in_k}^+ a_{in_k} + \frac{1}{2}\right)$ (because $\delta(k,k')$ is now $\delta_{k,k'}$)

Notice: The divergence $\sum_k L^{-3/2}\omega_k \frac{1}{2} = \infty$ is an "ultraviolet divergence".

## After the driving ends, $t > T$:

* One obtains: $a_k(t) = e^{-i\omega_k t} a_{out_k}$ with $a_{out_k} = a_{in_k} + J_{0_k}$

$$J_{0_k} := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J_k(t') e^{i\omega t'} dt'$$

Here: $J_k(t)$ is the Fourier transform of $J(x,t)$.

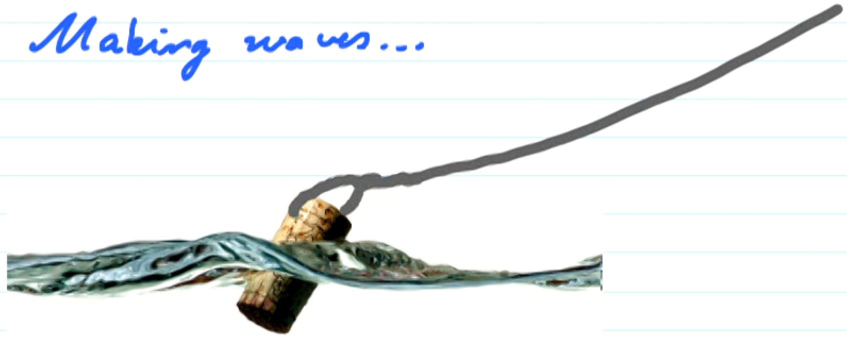
* Construct the out-basis $\{|n_{out_k}\rangle\}$ from:

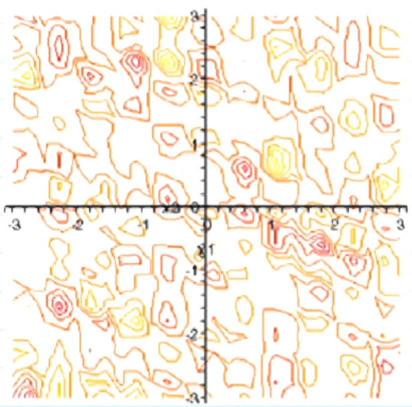
$$a_{out_k} |0_{out}\rangle = 0$$

⇝ Can calculate, e.g., $|\langle n_{out_k} | 0_{in}\rangle|^2$ i.e., the probability for $J(x,t)$ to have created $n$ particles of momentum $k$.

# Recall:

Making waves...



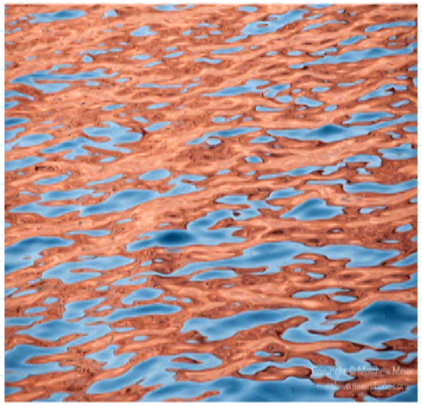


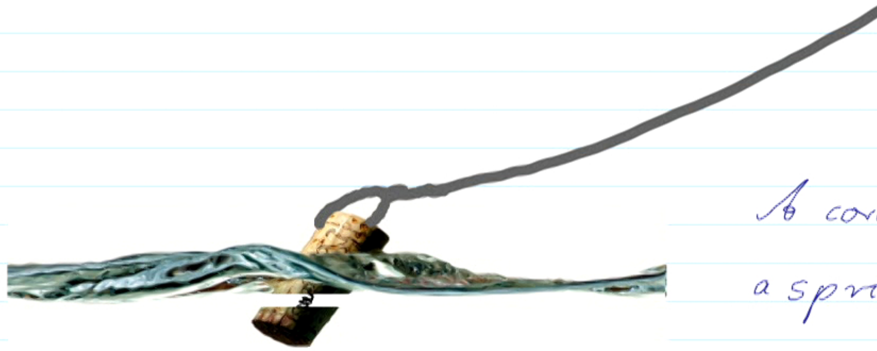
Making EM waves...

$e^-$

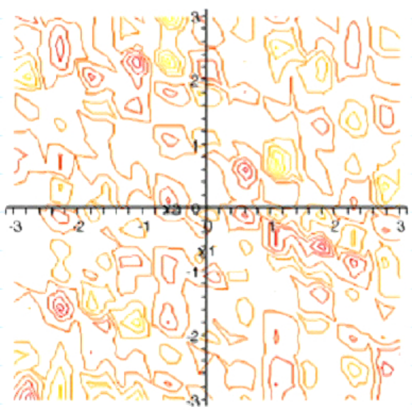
Described by $J(x,t)$.

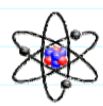
**Upgrade:** Give the charge $J(x,t)$ its own dynamics:





A cork with a spring.





Often, only one atomic transition is of interest. Then, we can model the atom as a 2-level system.

Described by QM, e.g. atom or qubit

number-valued, classical

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)\left(\Delta - m^2\right)\hat{\phi}(x,t) \;+\; J(x,t)\,\hat{\phi}(x,t)\; d^3x$$

and upgrade it to:

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} 1 \otimes \left(\hat{\pi}^2(x,t) - \hat{\phi}(x,t)\left(\Delta - m^2\right)\hat{\phi}(x,t)\right) + \hat{J}(x,t) \otimes \hat{\phi}(x,t)\; d^3x$$

an operator acting on the Hilbert space of the atom.

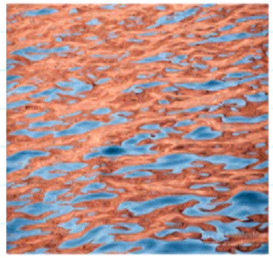
with $\hat{J}(x,t) = \hat{Q}(t)\,\delta(x - \tilde{x}(t))$

The Hilbert space: $\mathcal{H}_{total} = \mathcal{H}_{atom} \otimes \mathcal{H}_{field}$

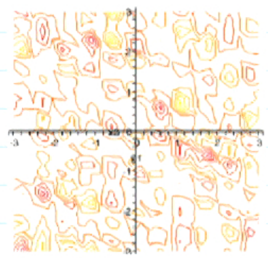
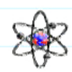
Simplified notation:

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)\left(\Delta - m^2\right)\hat{\phi}(x,t) \;+\; \hat{J}(x,t)\,\hat{\phi}(x,t)\; d^3x$$

The charged systems can act as emitters and as receivers of waves:

And, quantumly, they can act as emitters and receivers of particles!

**Definition:** (Unruh, deWitt):

A "particle", such as a photon is what a "particle detector", such as an atom, can detect, by getting excited.