Title: Quantum Field Theory for Cosmology - Lecture 6

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (Kempf)

Date: January 24, 2020 - 4:00 PM

URL: http://pirsa.org/20010005

Note Title

QFT for Cosmology, Achim Kempf, Lecture 6

Recall:

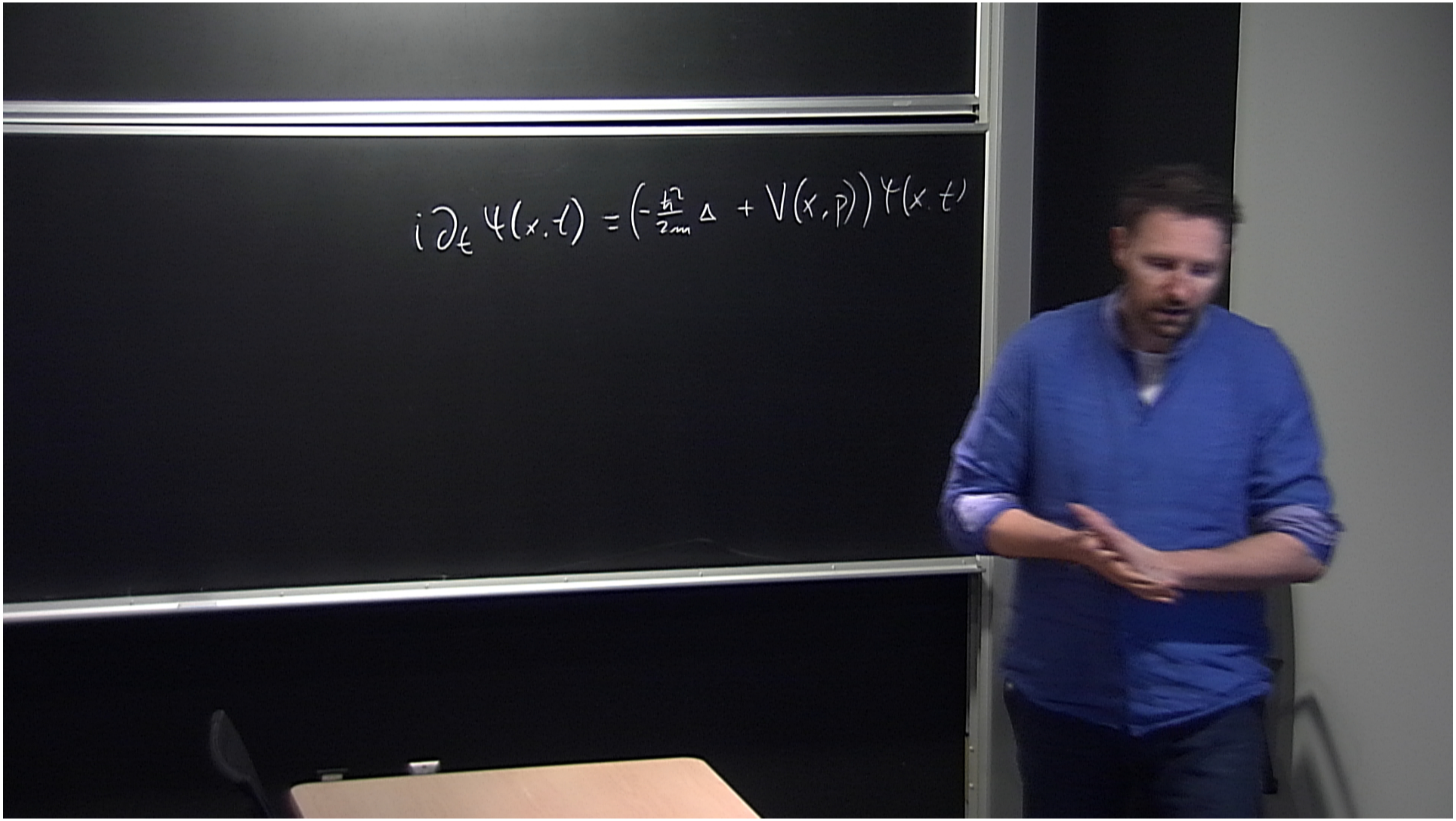
- There are two basic mechanisms to increase the amplitudes of oscillators, i.e., also to excite a field's mode oscillators, i.e. to create particles:

  a.) A time-varying driving force $J(t)$

  b.) A time-varying spring "constant" $\omega(t)$

- We are presently considering case a) :

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$


$i\partial_t \psi(x,t) = \left(-\frac{\hbar^2}{2m}\Delta + V(x,p)\right)\psi(x,t)$

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with a temporary force: $J(t) = 0$ for all $t \notin [0,T]$

Examples: 1. Temporary emission from antenna, 2. Brief interaction (scattering) of particles.

- We defined a convenient variable $a(t)$,

$$a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

so that: $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{1}{\ldots}\, J(t)\left(a^+(t) + a(t)\right)$

b.) A time-varying spring constant $\omega(t)$

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- By using $a(t)$, the problem reduced to solving:

$$* \quad i\dot{a}(t) = \omega\,a(t) - \frac{1}{\sqrt{2\omega}}\,J(t) \qquad \text{(EOM)}$$

$$* \quad [a(t), a^+(t)] = 1 \quad \text{for all } t \qquad \text{(CCR)}$$

We gave a convenient name to $a(t=0)$:

$$a_{in} := a(t=0)$$

(an operator on Hilbert space that we still have to choose.)

$$a(t) = a_{in} e^{-i\omega t} + 1 \frac{i}{\sqrt{2\omega}} e^{-i\omega t} \int_0^t J(t') e^{i\omega t'} dt'$$

$$= \left(a_{in} + 1 \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega t'} dt' \right) e^{-i\omega t}$$

And so, with the definition: $J_0 := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

$$a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } 0 \leq t \leq T \\ \underbrace{(a_{in} + J_0)}_{!!} e^{-i\omega t} & \text{for } T < t \end{cases}$$

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Define: a_{out}

Before and after the force action, we have an undriven harmonic oscillator, solved as always, by $a(t) = a_0\, e^{-i\omega t}$

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$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ a_{out}\, e^{-i\omega t} & \text{for } t>T \end{cases}$$

with $a_{out} = a_{in} + J_0$:

Conservation of the CCRs?

Notice that $[a_{in}, a_{in}^+] = 1$ implies $[a_{out}, a_{out}^+] = 1$.

Here:

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<u>Conservation of the CCRs?</u>

Notice that $[a_{in}, a_{in}^{+}] = 1$ implies $[a_{out}, a_{out}^{+}] = 1$.

In fact:

Proof:

Assume $[a_{in}, a^+_{in}] = 1$. Then:

$$[a(t), a^+(t)] = \Big[a_{in}e^{-i\omega t} + \underbrace{1\frac{i}{\sqrt{2\omega}}\int_0^t \ldots dt'}_{\text{number}},\ a^+_{in}e^{+i\omega t} - \underbrace{1\frac{i}{\sqrt{2\omega}}\int_0^t \ldots dt'}_{\text{number}}\Big]$$

$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad \checkmark$$

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$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad \checkmark$$

The initial period, $t<0$:

The dynamical variables:

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

$$\left.\begin{aligned} * \quad & \hat{q}(t) = \frac{1}{\sqrt{2\omega}}\left(a_{in}^{+} e^{i\omega t} + a_{in} e^{-i\omega t}\right) \\ * \quad & \hat{p}(t) = i\sqrt{\frac{\omega}{2}}\left(a_{in}^{+} e^{i\omega t} - a_{in} e^{-i\omega t}\right) \end{aligned}\right\}$$

Exercise: verify

$$* \quad \hat{H}(t) = \omega\left(a^{+}(t)\, a(t) + \frac{1}{2}\right)$$

$$= \omega\left(a_{in}^{+} e^{i\omega t}\, a_{in} e^{-i\omega t} + \frac{1}{2}\right)$$

□ The Hilbert space of states:

* As always, we can write arbitrary Hilbert space vectors as linear combinations of an arbitrary set of basis vectors.

* We could use, for example, the eigenbasis of $\hat{q}(t)$ (or the eigenbasis of $\hat{p}(t)$).

  But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigenbasis for each t.

* However, $\hat{H}$ is time independent (for $t<0$).

The eigenbasis of $\hat{H}$ for $t<0$:

* We have

$$\hat{H}_{t<0} = \omega\left(a^+_{in}\, a_{in} + \tfrac{1}{2}\right)$$

with:

$$[a_{in}, a^+_{in}] = 1 \qquad \text{(CCR)}$$

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in}\,|0_{in}\rangle = 0$$

the Hilbert space vector with zero length

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$ i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar = 1$).

* Then it is eigenvector of $H_{t<0}$:

$$\hat{H}_{t<0}|0_{in}\rangle = \omega\left(a^+_{in} a_{in} + \tfrac{1}{2}\right)|0_{in}\rangle = \tfrac{1}{2}\omega\,|0_{in}\rangle$$

$$H_{t<0} = \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right)$$

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$$\hat{H}_{t<0} |0_{in}\rangle = \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right) |0_{in}\rangle = \frac{1}{2}\omega\, |0_{in}\rangle$$

$\Rightarrow$ We recognize $|0\rangle$: it is the lowest energy eigenvector of $\hat{H}$ (and thus it indeed exists)

$$= \omega \frac{3}{2} a^\dagger |0_{in}\rangle$$

$$= \frac{3}{2}\omega\, |1_{in}\rangle$$

$\Rightarrow$ The state $|1_{in}\rangle$ is eigenstate of $\hat{H}$ with eigenvalue $\frac{3}{2}\omega$. So it must be the 1^{st} excited state.

* Is the vector $|1_{in}\rangle$ normalized?

$$\langle 1_{in}|1_{in}\rangle = \langle 0_{in}| a_{in}\, a^\dagger_{in} |0_{in}\rangle = \langle 0_{in}| a^\dagger_{in}\, a_{in} + 1 |0_{in}\rangle = \langle 0_{in}|0_{in}\rangle = 1$$

✓

* **Proposition:**

The set of vectors $\{|n_{in}\rangle\}_{n=0}^{\infty}$ defined through

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}} (a_{in}^{+})^n |0_{in}\rangle$$

is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

* **Proposition:**

The vectors $|n_{in}\rangle$ are eigenvectors of $\hat{H}_{t<0}$:

$$\hat{H}_{t<0} |n_{in}\rangle = E_n |n_{in}\rangle$$

with

$$E_n = \omega\left(n + \frac{1}{2}\right)$$

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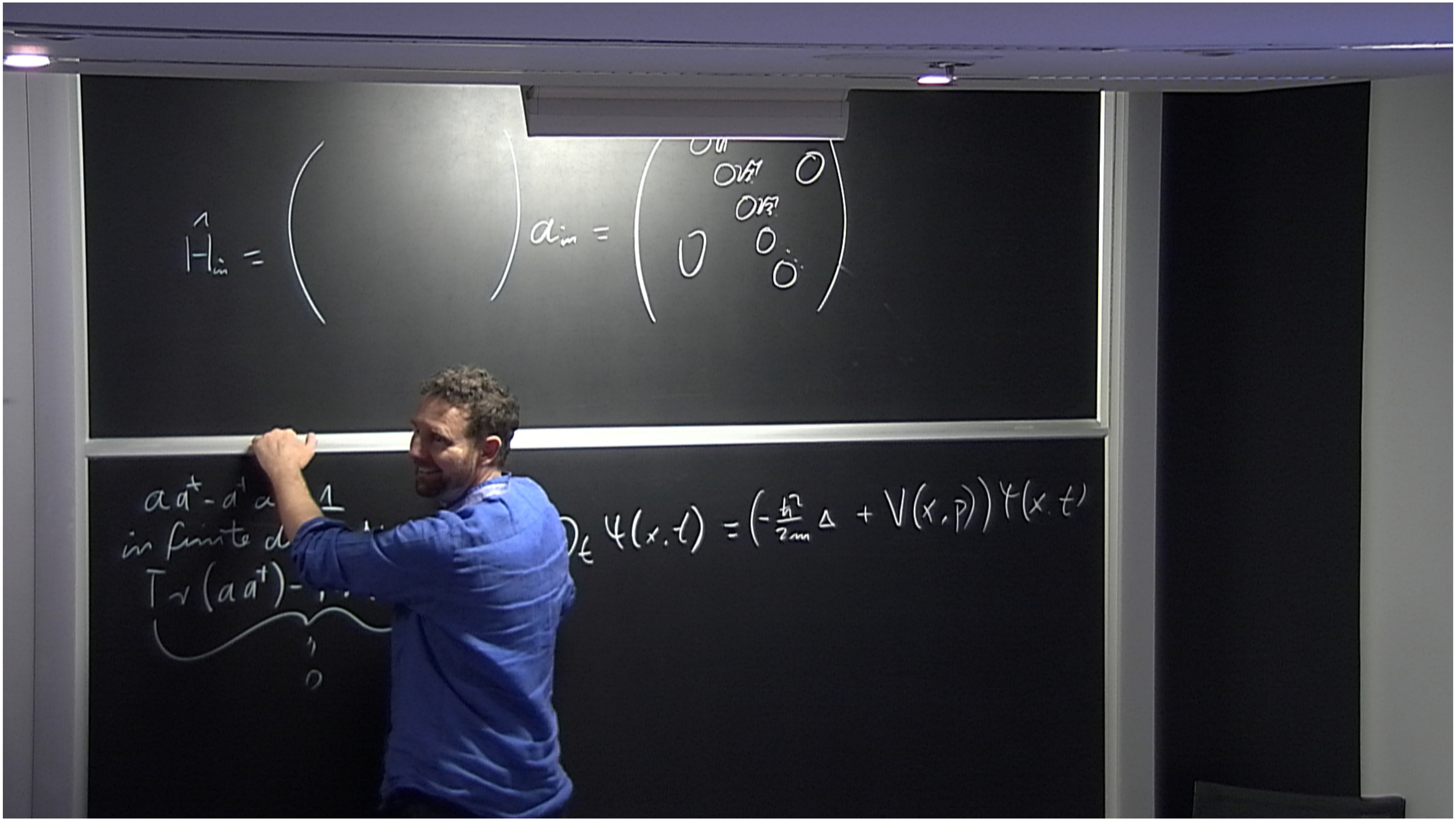
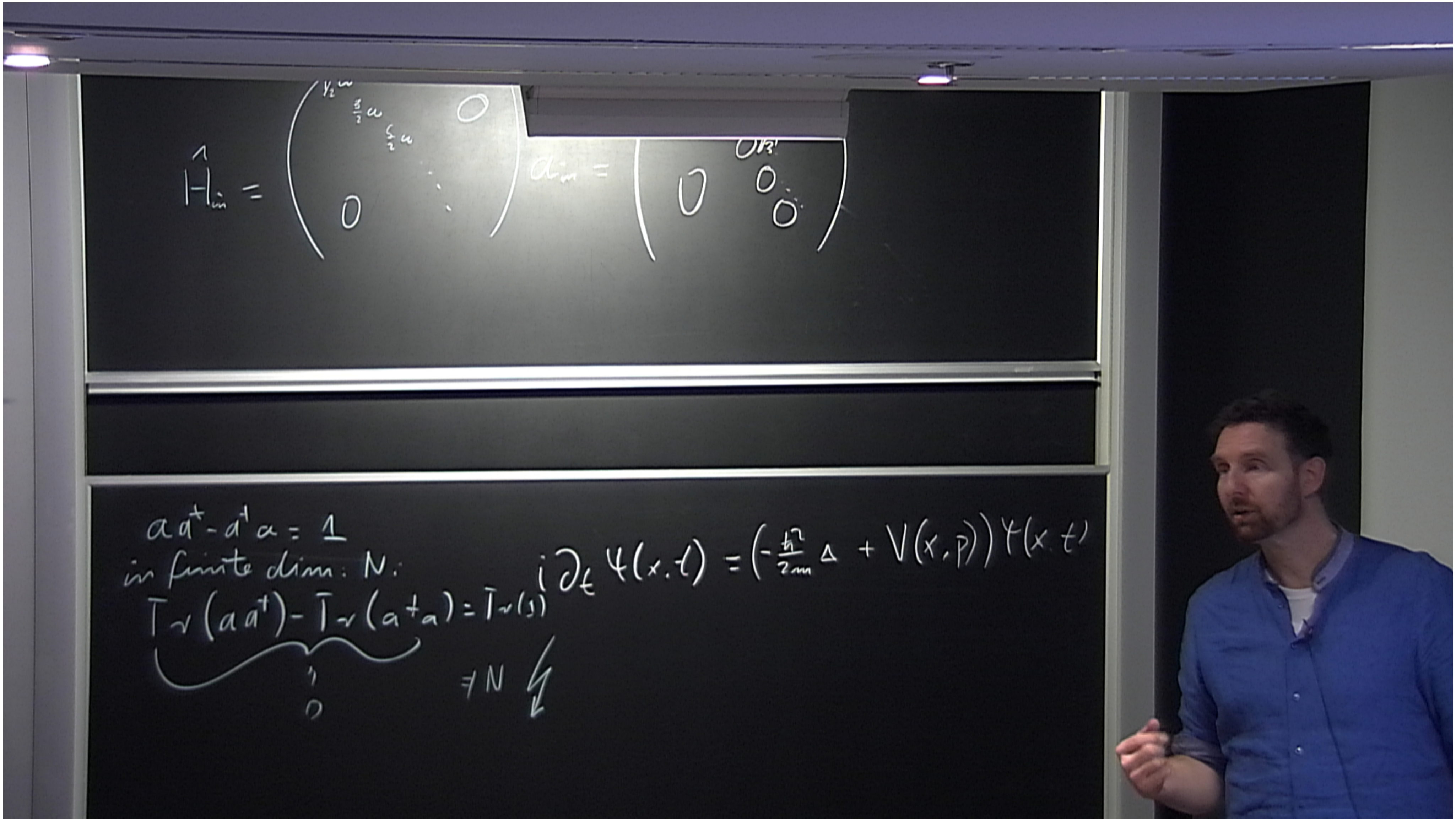
Exercise: verify

* Proposition: $\{|n_{in}\rangle\}$ is complete eigenbasis of $\hat{H}$.




$\hat{H}_{in} =$
$a_{in} =$
$a a^+ - a^+ a = 1$
in finite dim: N:
$Tr(a a^+) - Tr(a^+ a) = Tr(1)$
0
$\neq N$
$i \partial_t \Psi(x,t) = \left(-\frac{\hbar^2}{2m} \Delta + V(x,p)\right) \Psi(x,t)$

Summary re choice of basis for $t<0$:

- The Hamiltonian $\hat{H}(t)$ is constant for $t<0$.
- Thus it has one eigenbasis for all $t<0$, namely $\{|n_{in}\rangle\}$.
- We may expand every arbitrary vector $|\psi\rangle$ of the Hilbert space, $\mathcal{H}$, in this basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n_{in}\rangle$$

- E.g., the state of our quantum system could be:

$$|\psi\rangle = |5_{in}\rangle$$

- The system always stays in state $|\psi\rangle = |5_{in}\rangle$.

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Recall: - But $|\psi\rangle = |5_{in}\rangle$ generally ceases to be eigenvector of $\hat{H}(t)$ for $t > 0$!

The period $t > T$:

The period $t > T$: (after the force ceased to act)

- Once the driving force acts, $\hat{H}(t)$ starts to change.
- **But:** After the force finished, $t > T$, the Hamiltonian simply reads

$$\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) - \frac{a^+(t) + a(t)}{\sqrt{2\omega}}\,\cancel{J(t)} \quad \text{for } t > T$$

$$J(t) = 0$$

and from above, therefore:

$$\hat{H}(t) = \omega\left(a^+_{out}\,\cancel{e^{i\omega t}}\,a_{out}\,\cancel{e^{-i\omega t}} + \tfrac{1}{2}\right) \quad \text{with} \quad a_{out} = a_{in} + J_0$$

$$\Rightarrow \quad \hat{H}_{t>T} = \omega\left(a^+_{out}\,a_{out} + \tfrac{1}{2}\right) \Rightarrow \hat{H} \text{ is then constant again!}$$

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Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

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□ Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

Compare $t<0$ to $t>T$:

QFT:

A. Motion: $\bar{q}(t)$ (large $\bar{q}$ means large $\bar{\phi}_k$ means large waves)

B. Resonance: best $J(t)$? (consider e.g. antenna)

C. Energy expectation: $\bar{E}(t)$ (large $\bar{E}$ means large $\bar{E}_k$ means energy in mode k)

D. Energy eigenstates: $\{|E_n(t)\rangle\}$ (particle creation)

We will consider the example where the system starts out in the lowest energy state (the vacuum):

A. Motion $\bar{q}(t)$:

$$\bar{q}(t) = \langle \psi | \hat{q}(t) | \psi \rangle$$

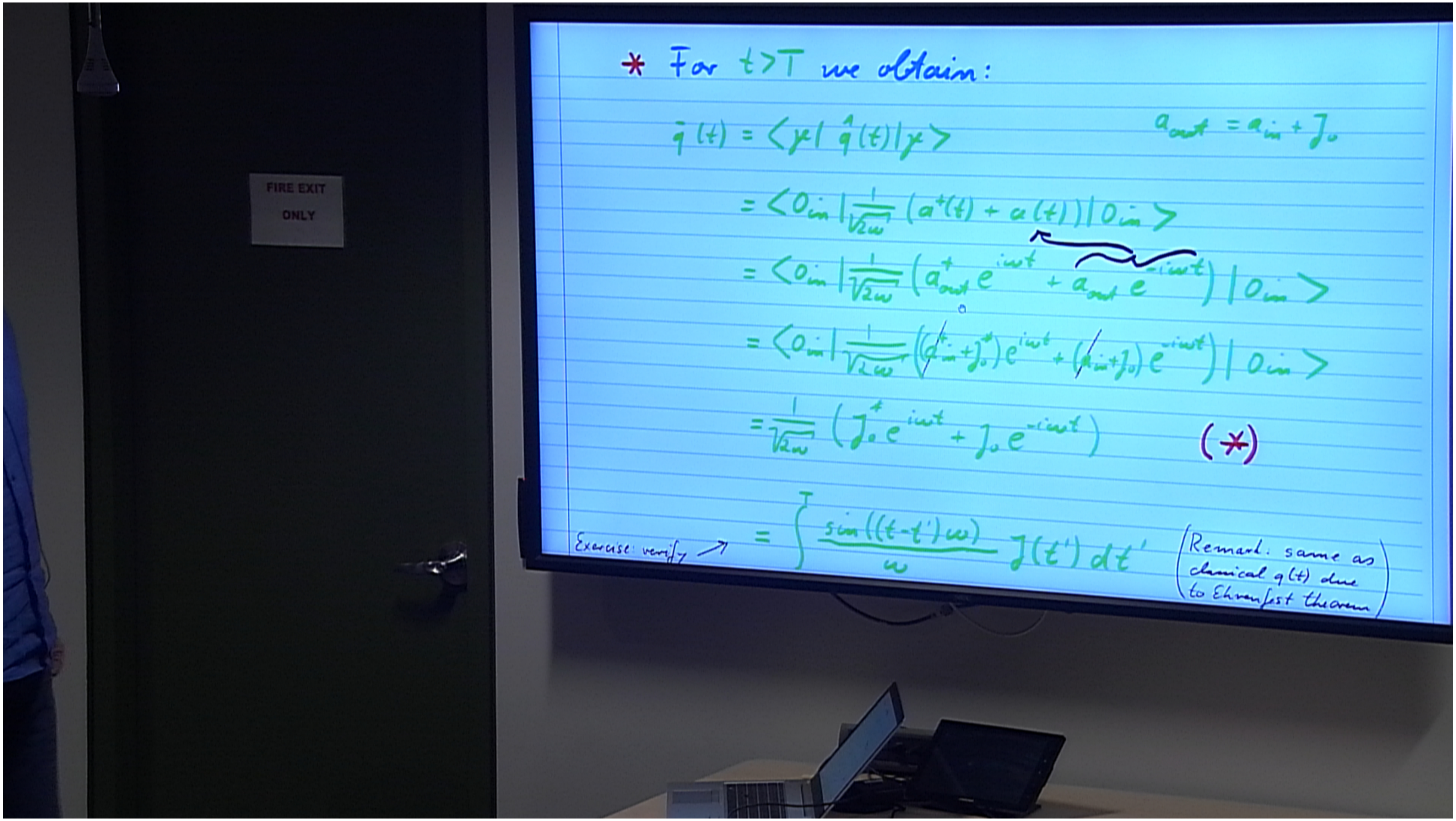
$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} (a^+(t) + a(t)) | 0_{in} \rangle$$

* For $t<0$ we obtain:

$$\bar{q}(t) = \frac{1}{\sqrt{2\omega}} \langle 0_{in} | a^+_{in} e^{i\omega t} + a_{in} e^{-i\omega t} | 0_{in} \rangle$$

$$= 0$$

This was expected since for $t<0$ the system's state $|0_{in}\rangle$


FIRE EXIT
ONLY
* For $t>T$ we obtain:
$\bar{q}(t) = \langle \chi | \hat{q}(t) | \chi \rangle$
$a_{out} = a_{in} + J_0$
$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+(t) + a(t)\right) | 0_{in} \rangle$
$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+_{out} e^{i\omega t} + a_{out} e^{-i\omega t}\right) | 0_{in} \rangle$
$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left((\cancel{a^+_{in}} + J_0^*) e^{i\omega t} + (\cancel{a_{in}} + J_0) e^{-i\omega t}\right) | 0_{in} \rangle$
$= \frac{1}{\sqrt{2\omega}} \left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t}\right)$ (*)
Exercise: verify $\nearrow$
$= \int^T \frac{\sin((t-t')\omega)}{\omega} J(t')\, dt'$
(Remark: same as classical $q(t)$ due to Ehrenfest theorem)

B. Resonance:

* The amplitude of the excited motion of the oscillator is determined by J_0, as equation (*) shows.

* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$$

* For $t > T$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \qquad \text{(always)}$$

$$= \langle 0_{in} | \omega \left(a^{+}_{out} a_{out} + \tfrac{1}{2} \right) | 0_{in} \rangle \qquad (\text{for } t > T)$$

$$= \omega \langle 0_{in} | (\cancel{a}^{+}_{in} + J_0^{*})(\cancel{a}_{in} + J_0) + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \langle 0_{in} | J_0^{*} J_0 + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \left(\tfrac{1}{2} + |J_0|^2 \right)$$

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Remark: We notice that the oscillator's energy increases the more the larger $|J_0|$, i.e., from B, the closer the driving force is to the oscillator's natural frequency ω.

Remark: In QFT, say when electrical current drives electromagnetic field modes, the closer a mode's ω_k is to the frequency of the current, the more this mode gets excited.

Implication: $|0_{out}\rangle \neq |0_{in}\rangle = |\psi\rangle$

- Ground state $|0_{out}\rangle$ of

$$H_{t>T} = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) = \omega\left(a^+_{out}\,a_{out} + \tfrac{1}{2}\right)$$

has eigenvalue $\omega/2$, i.e.:

$$a_{out}|0_{out}\rangle = 0.$$

- Therefore: $a_{out}|\psi\rangle = a_{out}|0_{in}\rangle$

$$= (a_{in} + J_0)|0_{in}\rangle$$

$$= J_0|\psi\rangle \neq 0$$

Vacuum state $|0_{out}\rangle$ of

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$$= J_0|\psi\rangle \neq 0$$

$\Rightarrow$ At late times: $|\psi\rangle \neq |0_{out}\rangle$

Q: So what kind of excited state is $|\psi\rangle$ at late times?

Vacuum state $|0_{out}\rangle$ of

$$H_{t>T} = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) = \omega\left(a_{out}^+\, a_{out} + \tfrac{1}{2}\right)$$

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Therefore: $a_{out}\,|\psi\rangle = a_{out}\,|0_{in}\rangle$

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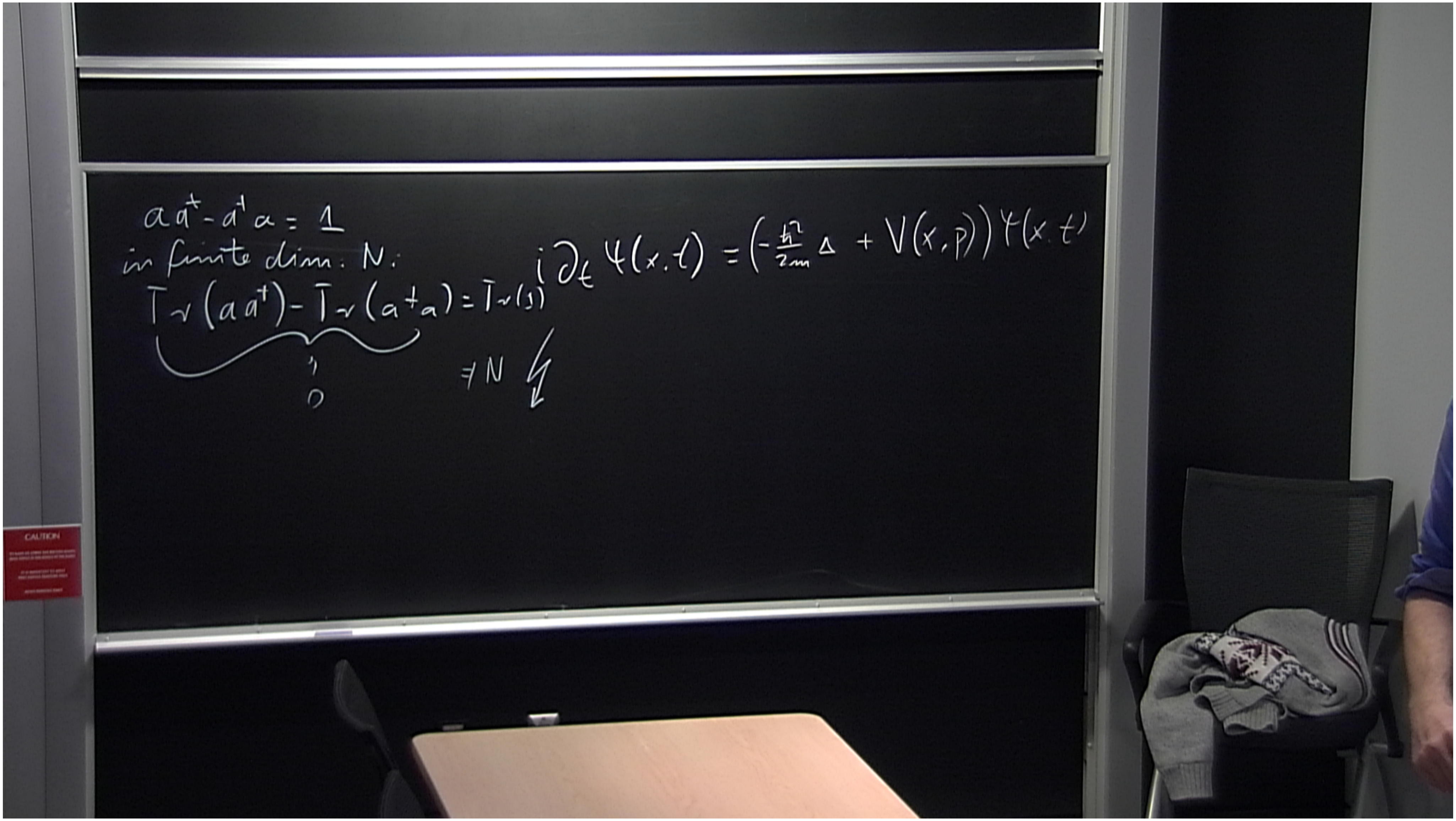

$aa^{\dagger} - a^{\dagger}a = 1$
in finite dim: N:
$\underbrace{\mathrm{Tr}(aa^{\dagger}) - \mathrm{Tr}(a^{\dagger}a)}_{= 0} = \mathrm{Tr}(1)$
$= N$
$i\partial_t \psi(x,t) = \left(-\frac{\hbar^2}{2m}\Delta + V(x,p)\right)\psi(x,t)$

Q: So what kind of excited state is $|\chi\rangle$ at late times?

A: Since $|\chi\rangle$ is eigenstate of a lowering operator,

$$a_{out}|\chi\rangle = J_0 |\chi\rangle$$

$|\chi\rangle$ is what is called a Coherent State.

Recall:

Coherent states saturate the uncertainty relation:

If $|\psi\rangle$ is a coherent state, then

A: Since $|\chi\rangle$ is eigenstate of a lowering operator,

$$a_{out}|\chi\rangle = J_0 |\chi\rangle$$

$|\chi\rangle$ is what is called a Coherent State.

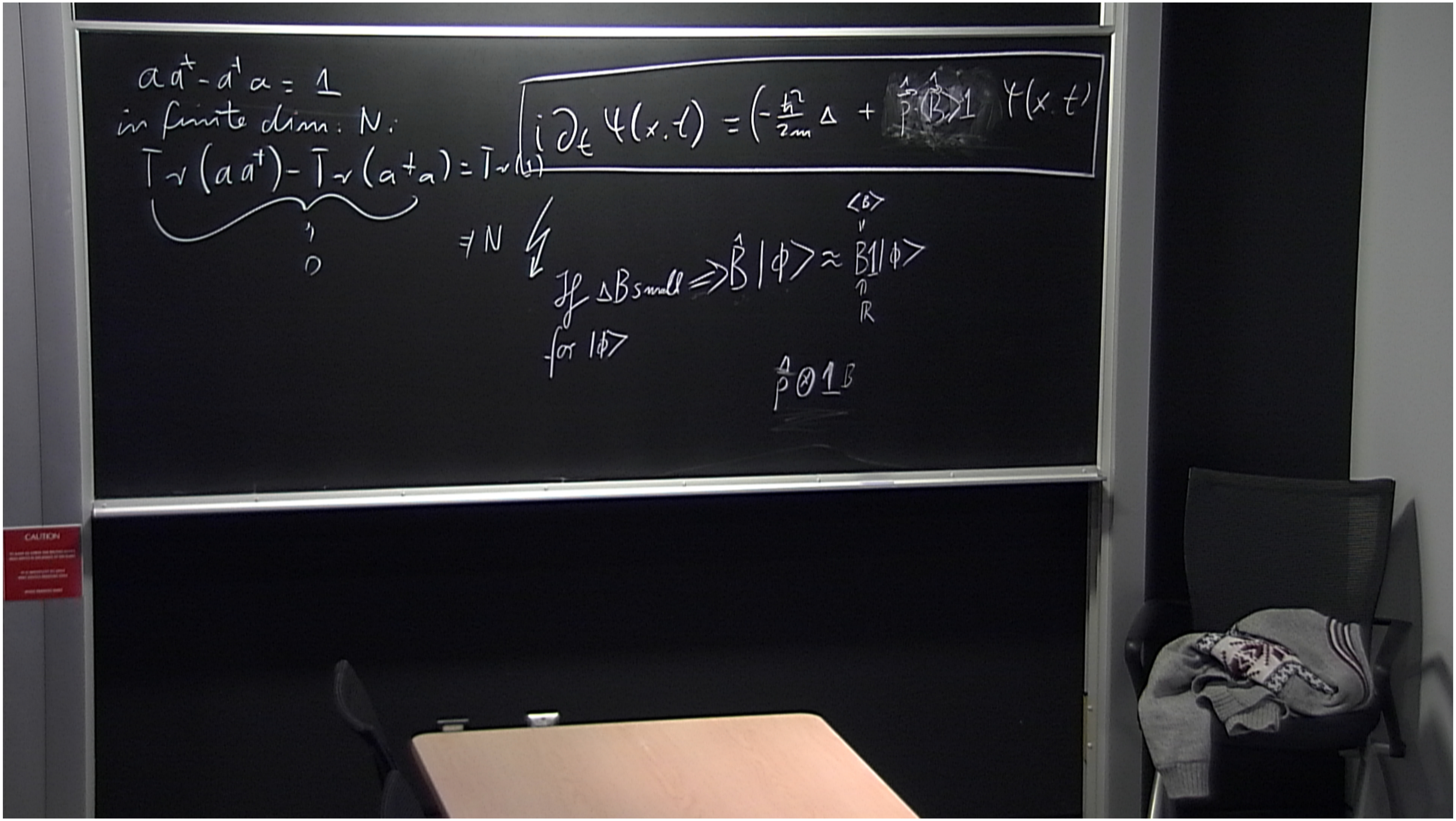
Recall:

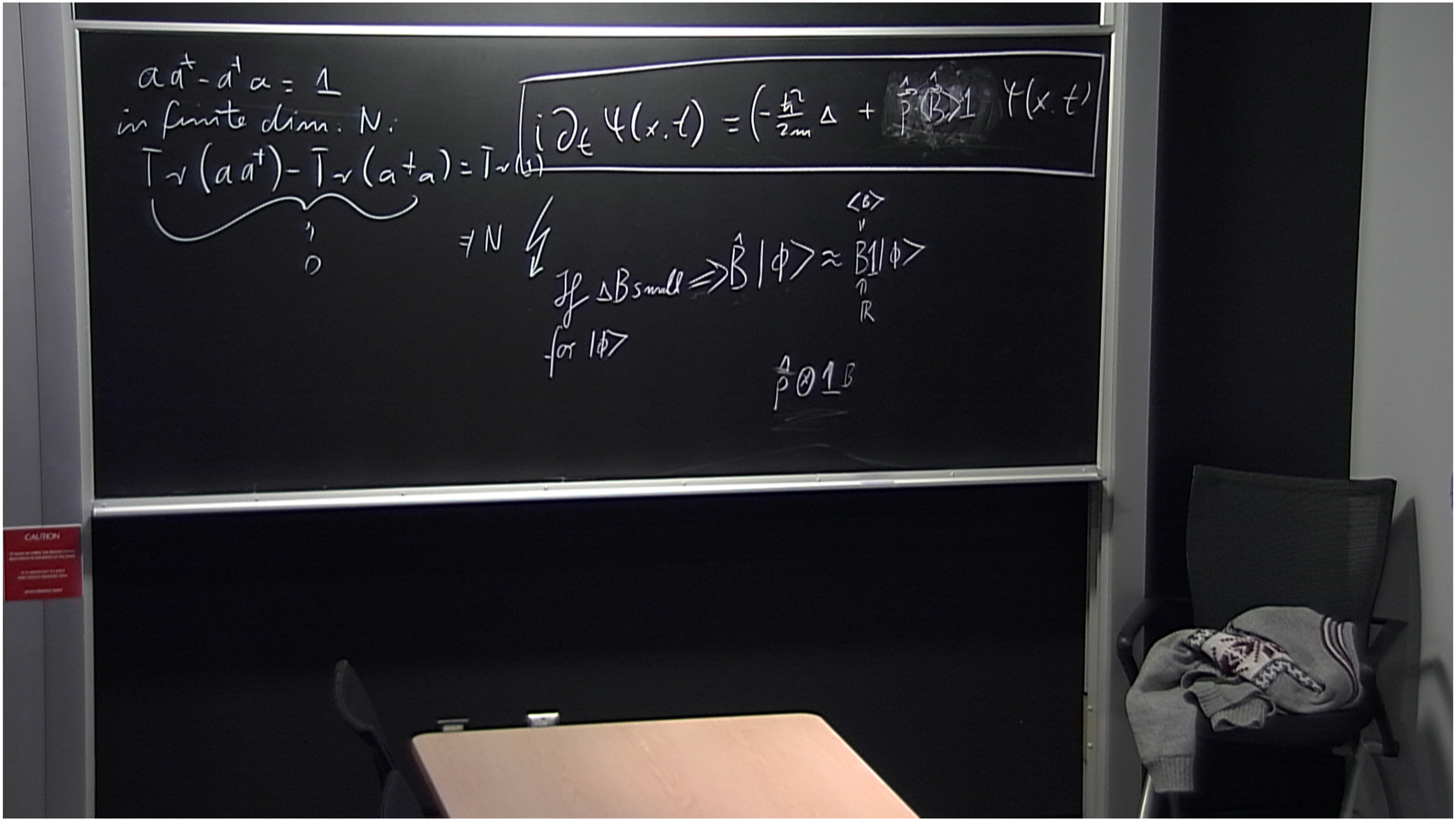
Coherent states saturate the uncertainty relation:

If $|\psi\rangle$ is a coherent state, then

$$\Delta q_{|\psi\rangle} \, \Delta p_{|\psi\rangle} = \frac{\hbar}{2}$$

→ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:


aa† - a†a = 1
in finite dim: N:
Tr(aa†) - Tr(a†a) = Tr(1)
0
≠ N
i∂t Ψ(x,t) = (- ħ²/2m Δ + p̂·⟨B⟩1) Ψ(x,t)
⟨B⟩
If ΔB small ⟹ B̂|φ⟩ ≈ B1|φ⟩
for |φ⟩
R
p̂ ⊗ 1B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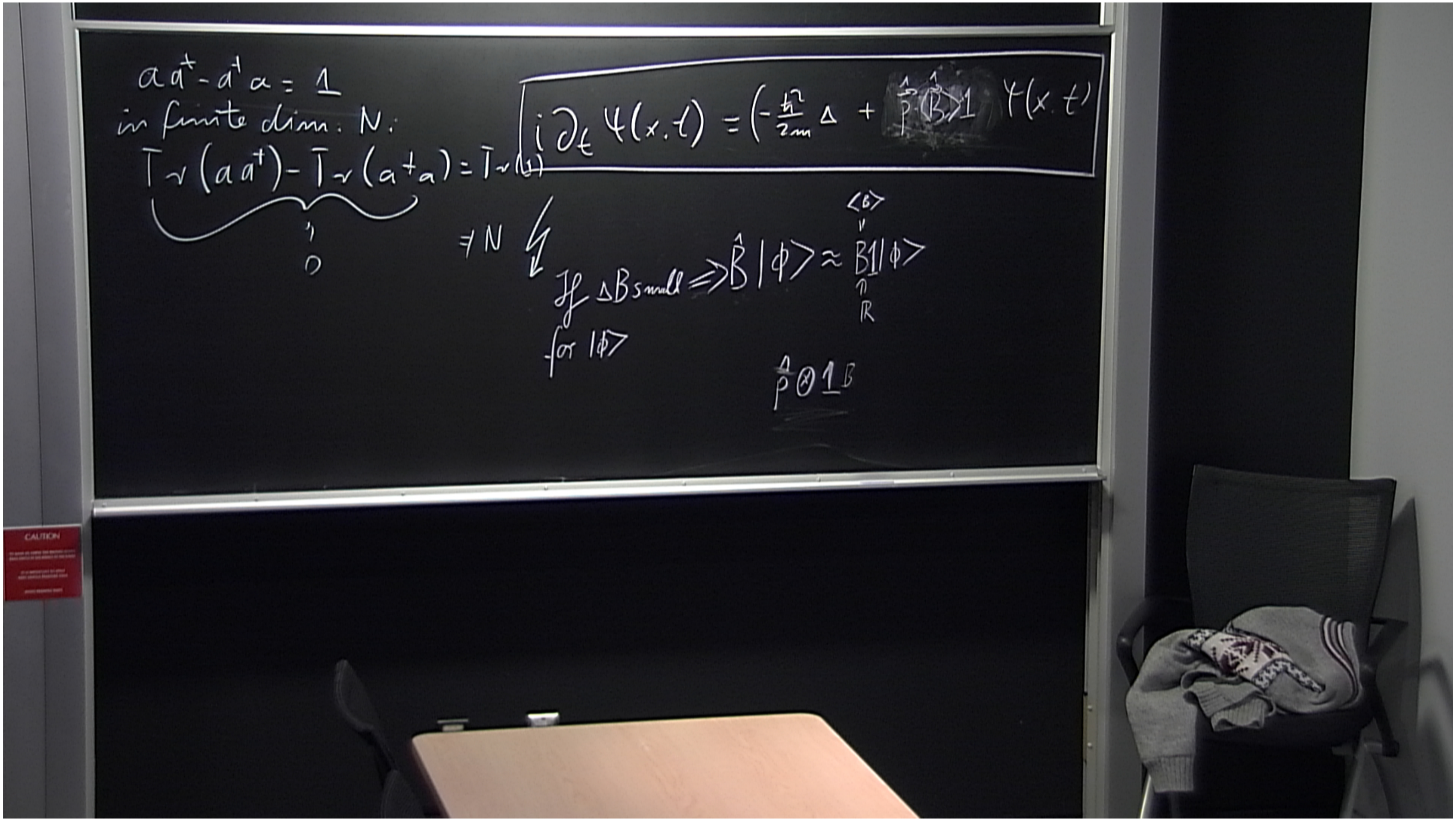
i∂t Ψ(x.t) = (-ħ²/2m Δ + p̂ ⟨B̂⟩1) Ψ(x.t)
⟨B⟩
If ΔB small ⟹ B̂ |φ⟩ ≈ B1|φ⟩
for |φ⟩
ℝ
p̂ ⊗ 1B
CAUTION



aa† - a†a = 1
in finite dim. N:
Tr(aa†) - Tr(a†a) = Tr(1)
0
≠ N
i∂t Ψ(x,t) = (- ħ²/2m Δ + p̂·⟨B̂⟩1) Ψ(x,t)
⟨B⟩
If ΔB small ⇒ B̂ |φ> ≈ B1 |φ>
R
for |φ>
p̂ ⊗ 1B

Exercise: Show that if $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\alpha \in \mathbb{C}$

and $\hat{q} = \frac{1}{\sqrt{2\omega}}(a^+ + a)$, $\hat{p} = i\sqrt{\frac{\omega}{2}}(a^+ - a)$ $\quad (*)$

Then, $\langle\alpha|\hat{q}|\alpha\rangle = \frac{1}{\sqrt{2\omega}}(\alpha^* + \alpha)$

$$\langle\alpha|\hat{p}|\alpha\rangle = i\sqrt{\frac{\omega}{2}}(\alpha^* - \alpha)$$

and: $\Delta q(t)\,\Delta p(t) = \frac{1}{2}$

Remarks:

- Notice that because $a|\alpha\rangle = \alpha|\alpha\rangle$, the operator a does **not** reduce the excitation (or particle) number of $|\alpha\rangle$.
- If the ω in $(*)$ is chosen to be not the frequency of the

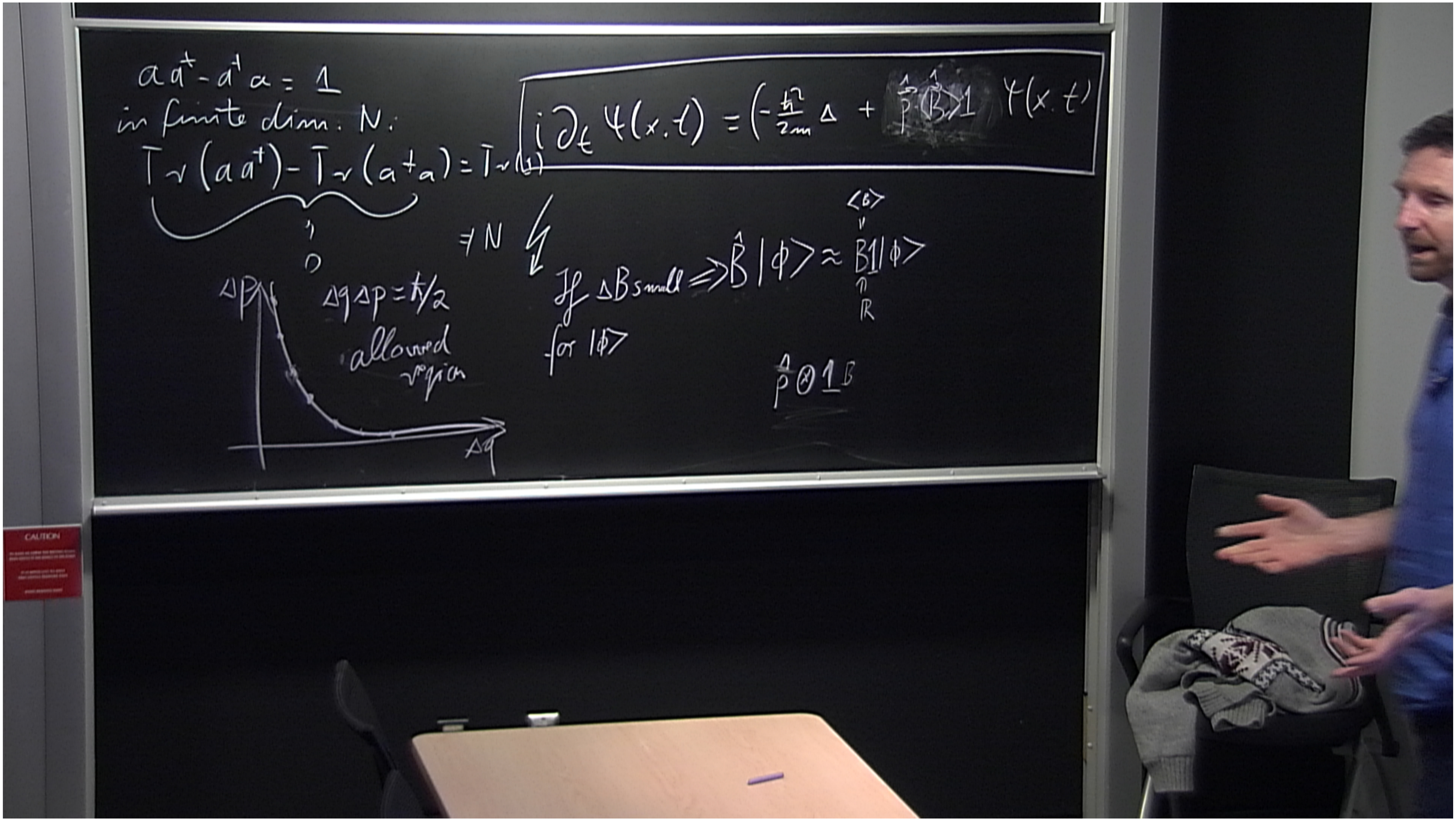
Then, $\langle \alpha | \hat{q} | \alpha \rangle = \frac{1}{\sqrt{2\omega}} (\alpha^* + \alpha)$

$$\langle \alpha | \hat{p} | \alpha \rangle = i\sqrt{\frac{\omega}{2}} (\alpha^* - \alpha)$$

and: $\Delta q(t) \, \Delta p(t) = \frac{1}{2}$

Remarks:

- Notice that because $a|\alpha\rangle = \alpha|\alpha\rangle$, the operator a does **not** reduce the excitation (or particle) number of $|\alpha\rangle$.
- If the ω in (*) is chosen to be not the frequency of the harmonic oscillator of the Hamiltonian, then $|\alpha\rangle$ is called a Squeezed State.


aa† - a†a = 1
in finite dim. N:
Tr(aa†) - Tr(a†a) = Tr(1)
≠ N
Δq Δp = ħ/2
allowed region
Δp
Δq
iħ ∂t Ψ(x,t) = (- ħ²/2m Δ + p̂ ⟨B̂⟩ 1) Ψ(x,t)
If ΔB small ⟹ B̂|φ⟩ ≈ ⟨B⟩1|φ⟩
for |φ⟩
p̂ ⊗ 1B