Title: Quantum Field Theory for Cosmology - Lecture 5

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Collection: Quantum Field Theory for Cosmology (Kempf)

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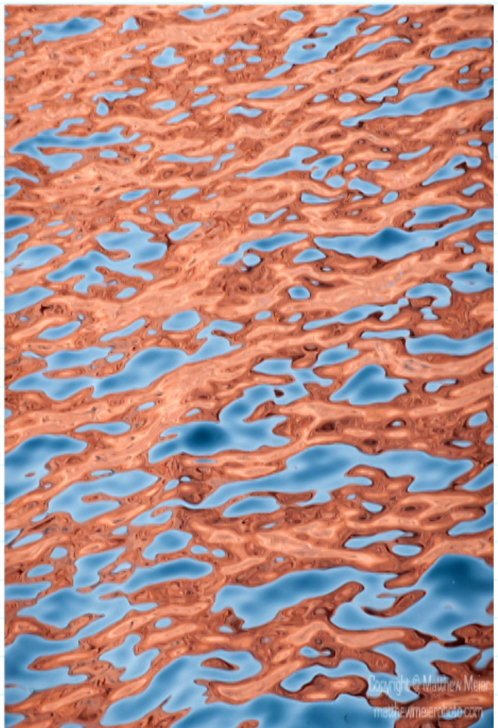
QFT for Cosmology, Achim Kempf, Lecture 5

Note Title

Particles in QFT



Back in the Heisenberg picture, to solve the QFT is to solve:

- The hermiticity conditions:

$$\hat\phi^+(x,t) = \hat\phi(x,t) \quad , \quad \hat\pi^+(x,t) = \hat\pi(x,t)$$

- The canonical commutation relations:

$$[\hat\phi(x,t), \hat\pi(x',t)] = i\,\delta(x-x')$$

- The equations of motion:

$$\dot{\hat\pi}(x,t) - \Delta\hat\phi(x,t) + m^2\hat\phi(x,t) = 0$$

$$\hat\pi(x,t) = \dot{\hat\phi}(x,t)$$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To simplify:

- **Infrared regularization:**

  Box size $L \times L \times L$ with periodic boundary conditions.

  ↑ Project: uses Dirichlet boundary conditions.

- **Then Fourier series expansion:**

$$\hat{\phi}(x,t) = L^{-3/2} \sum_k \hat{\phi}_k(t)\, e^{ikx}$$

↑ $k = \frac{2\pi}{L}(n_1, n_2, n_3),\ n_i \in \mathbb{Z}$

Obtain:

$$\ddot{\hat{\phi}}_k(t) = -\left(k^2 + m^2\right)\hat{\phi}_k(t) \quad \text{and} \quad [\hat{\phi}_k, \dot{\hat{\phi}}_{k'}] = i\,\delta_{k,-k'}$$

$$\hat{H} = \sum_k \hat{H}_k \quad \text{with} \quad \hat{H}_k = \frac{1}{2}\hat{\pi}_k^+ \hat{\pi}_k + \frac{1}{2}\hat{\phi}_k^+ (k^2 + m^2)\hat{\phi}_k$$

i.e.: $$\hat{H} = \sum_k \left(\frac{1}{2}\hat{\pi}_k^+(t)\hat{\pi}_k(t) + \frac{1}{2}\hat{\phi}_k^+(t)\left(k^2 + m^2\right)\hat{\phi}_k(t)\right)$$

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Crucial observations:

* For each wave vector $k = (k_1, k_2, k_3)$ there is an independent harmonic oscillator with frequency $\omega_k = \sqrt{k^2 + m^2}$ and spectrum $\text{spec}(H_k) = \hbar\omega_k \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$.

$\Rightarrow$ The excitation levels of H_k differ by the energy

$$E = \omega_k = \sqrt{k^2 + m^2} \qquad (\hbar = 1)$$

* This is also the energy of a particle of momentum k!

$\rightsquigarrow$ Hypothesis: $\boxed{\text{Mode excitation = particle creation}}$

Does this interpretation work?

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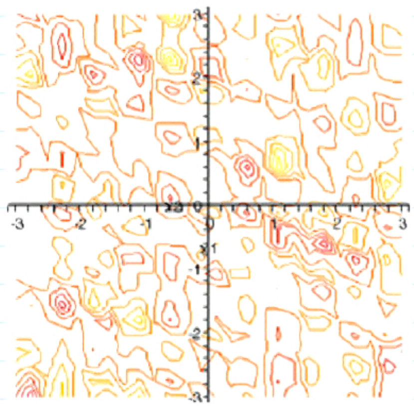
Water:

$\phi(x,t)$

Probe amplitudes,
e.g., with a cork:

Quantum field:

$\hat{\phi}(x,t)$



Probe amplitudes, e.g.,
with atoms.

Use as a
detector for
the field's particles
(e.g. photons for EM field)

One finds:

- ☐ Interpretation works but is acceleration and curvature dependent.
- ☐ Interpretation simple only in Minkowski space for inertial detectors.

Note: Conventional particle physics is based on that special case.

Then: Which is, e.g., the state $|\psi\rangle$ in which we have

3 particles of momentum k_a and 7 particles of momentum k_b?

$$|\psi\rangle = |n_{k_a}=3,\ n_{k_b}=7,\ \text{all other } n_k=0\rangle$$

$$= |n_{k_a}=3\rangle \otimes |n_{k_b}=7\rangle \left(\bigotimes_{\substack{\text{all other} \\ k_c}} |n_{k_c}=0\rangle \right)$$

Energy:
$$\hat{H}_k|\psi\rangle = \begin{pmatrix} \hbar\omega_k\left(\frac{1}{2}+3\right) & \text{if } k=k_a \\ \hbar\omega_k\left(\frac{1}{2}+7\right) & \text{if } k=k_b \\ \hbar\omega_k\,\frac{1}{2} & \text{if } k\neq k_a,k_b \end{pmatrix}|\psi\rangle$$

$$\Rightarrow \qquad \hat{H}|\psi\rangle = \left(3\omega_{k_a} + 7\omega_{k_b} + \sum_{\text{all } k}\frac{1}{2}\omega_k\right)|\psi\rangle$$

And one can have linear combinations:

Which is, e.g., the state $|x\rangle$ in which we have

3 particles of momentum k_a or 7 particles of momentum k_b,

with probability amplitudes α, $\beta = \sqrt{1-|\alpha|^2}$?

$$|x\rangle = \alpha \, |n_{k_a} = 3, \text{other } n_k = 0\rangle + \beta \, |n_{k_b} = 7, \text{other } n_k = 0\rangle$$

Notice: This is not a state of fixed particle number!

(Remark: Some particle species have a number conservation law, e.g., leptons, i.e. $e^-, \mu^-, \tau^-, \nu_{e^-}, \nu_{\mu^-}, \nu_{\tau^-}$, (where the antiparticles count negatively).
"Superselection rule" then says we can't have such linear combinations: only number eigenstates allowed.)

Mechanisms for mode excitation / particle creation?

I.e.: What are mechanisms for exciting harmonic oscillators?

▫ **2 types of mechanism:** (here, $\hat{q}(t)$ stands for $\hat{\phi}_k(t)$)

we'll begin with this effect → **a.)** A "driving force" shakes the oscillator:

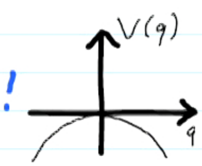
$$\ddot{\hat{q}}(t) = -\omega^2 \hat{q}(t) + \hat{j}(t)$$

e.g.: 

b.) A time dependence of ω affects the oscillator:

$$\ddot{\hat{q}}(t) = -\omega^2(t)\, \hat{q}(t)$$

e.g.: 

And, $\omega^2(t)$ could even go negative! 

e.g.: 

All occur in QFT:

A) Multiple fields enter into the Hamiltonian and into the eqns of motion. Thus, fields provide each other with J terms, e.g.:

$$H(\hat{\phi},\hat{\psi}) = \hat{H}_1(\hat{\phi}) + \hat{H}_2(\hat{\psi}) + \int_{\mathbb{R}^3} \lambda\, \hat{\phi}(x,t)\, \hat{\psi}(x,t)\, d^3x$$

- Wave interpretation: Nontrivial interaction of waves of different types of fields
- Particle interpretation: The collision of particles happens when their mode oscillators drive one other.
  → Collissions can create and annihilate particles.
- Strongest effects?

When oscillator "resonates" with driving force.

E.g.: It takes high energy particles to make high energy particles

B) The presence of gravity can effectively influence the $\omega_k(t)$.

- **Wave interpretation:** * E.g., cosmic expansion stretches the wavelength

  $\Rightarrow$ expect $\omega = \omega(t)$ decreases. True, and also:

  * if wavelength > horizon then $\omega^2(t) < 0$!

  $\Rightarrow$ runaway harmonic mode "oscillators"

  (then: field amplification but no particle interpretation)

- **Particle interpretation:**

  Gravity can excite mode oscillators, i.e. it can create particles from the vacuum.

- **Strongest effects?** When oscillator resonates with $\omega(t)$. This effect is called parametric resonance.

Case A: Particle creation through external driving of mode oscillators.

Example: Production of photons by an antenna:

- We model the electromagnetic field as a Klein Gordon field.

  (The fact that EM fields have polarization and have $m=0$ is not important here)

- Consider an arbitrary mode of the electromagnetic field:

  $$\hat{\phi}_k(t)$$

  should really be quantized too

- We model the electric current as a given classical field $j(x,t)$ whose modes are $j_k(t)$.

  should really be vector-valued

In a rough simplification, the EM k mode obeys:

$$\hat{H}_k = \frac{1}{2}\hat{\pi}_k^+(t)\,\hat{\pi}_k(t) + \frac{1}{2}\omega_k^2\,\hat{\phi}_k^+(t)\,\hat{\phi}_k(t) + \hat{\phi}_k(t)\,j_k(t)$$

⇒ If the current $j(t)$ varies in time it can excite the mode oscillators, thus creating photons.

⇒ **Need to study the quantized driven harmonic oscillator!**

for $\hat{H}_k(t)$ — for $\hat{\pi}_k(t)$ — stands for a field mode $\hat{\phi}_k(t)$

$$\boxed{\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)}$$

stands for a mode $J_k(t)$ of another classical (or better quantum) field.

I Preparation:

- Recall that for all observables $\hat{f}$:

state at initial time

$$\bar{f}(t) = \langle \psi_0 | \hat{U}^+(t)\, \hat{f}_0\, \hat{U}(t) | \psi_0 \rangle$$

operator at initial time

with the time-evolution operator obeying:

$$\hat{U}(t_0) = 1\,, \qquad i\frac{d}{dt}\hat{U}(t) = \hat{U}(t)\,\hat{H}(t)$$

the original Hamiltonian

"Heisenberg Hamiltonian"

- Schrödinger picture? We write, equivalently:

Exercise: check!

$$\bar{f}(t) = \left(\langle \psi_0 | \hat{U}^+(t)\right) \hat{f}_0 \underbrace{\left(\hat{U}(t) | \psi_0 \rangle\right)}_{= |\psi(t)\rangle}$$

$$= \langle \psi(t) | \hat{f}_0 | \psi(t) \rangle$$

The dynamics is $i\frac{d}{dt}|\psi(t)\rangle = \hat{H}_s(t)|\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_s(t) = \hat{U}(t)\hat{H}(t)\hat{U}^+(t)$

Exercise: check

- We will use, equivalently, the Heisenberg picture:

$$\bar{f}(t) = \langle\psi_0|\underbrace{\left(\hat{U}^+(t)\,\hat{f}_0\,\hat{U}(t)\right)}_{=}|\psi_0\rangle$$

$$= \langle\psi_0|\;\hat{f}(t)\;|\psi_0\rangle$$

with dynamics:

$$i\frac{d}{dt}\hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

The dynamics is $i \frac{d}{dt} |\psi(t)\rangle = H_s(t) |\psi(t)\rangle$

with Schrödinger Hamiltonian: $\hat{H}_s(t) = \hat{U}(t) \hat{H}(t) \hat{U}^+(t)$

Exercise: check

We will use, equivalently, the Heisenberg picture:

$$\bar{f}(t) = \langle\psi_0| \underbrace{\left(\hat{U}^+(t)\, \hat{f}_0\, \hat{U}(t)\right)}_{=} |\psi_0\rangle$$

$$= \langle\psi_0|\ \hat{f}(t)\ |\psi_0\rangle$$

with dynamics:

$$i \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}(t)]$$

II Aspects of the Heisenberg picture:

- The state of the quantum system stays the same Hilbert space vector, say $|\psi\rangle \in \mathcal{H}$ (from measurement to measurement).

- The observables, say $\hat{H}(t)$, $\hat{f}(t)$, etc, are time-dependent operators in Hilbert space.

- Important implication:

The eigenbases and the eigenvalues of observables, such as $\hat{H}(t)$ and any $\hat{f}(t)$ depend on time!

$$\hat{f}(t)\,|f_n(t)\rangle = f_n(t)\,|f_n(t)\rangle$$

$$\hat{H}(t)\,|E_n(t)\rangle = E_n(t)\,|E_n(t)\rangle$$

Example: * Assume the driven harmonic oscillator starts out at time t_1 in n'th energy state, say $|\psi\rangle = |E_n(t_1)\rangle$:

$$\hat{H}(t_1)|E_n(t_1)\rangle = E_n(t_1)|E_n(t_1)\rangle$$

* State vector of the system stays $|\psi\rangle$ for $t > t_1$.

* But at later times, say $t > t_1$ the Hamiltonian and its eigenvectors and eigenvalues are

$$\hat{H}(t)|E_m(t)\rangle = E_m(t)|E_m(t)\rangle$$

and we generally have

$$E_n(t) \neq E_n(t_1) \quad , \quad |E_n(t)\rangle \neq |E_n(t_1)\rangle$$

$\Rightarrow$ At time t_2 system is still in state $|\chi\rangle$ and still

$$|\chi\rangle = |E_n(t_1)\rangle$$

but $|\chi\rangle$ is generally no longer n'th (or any other) energy eigenstate!

In particular:

* Assume system starts out at t_1 in lowest energy state (i.e., in vacuum): $|\chi\rangle = |E_0(t_1)\rangle$

* Then if $|\chi\rangle = |E_0(t_1)\rangle \neq |E_0(t_2)\rangle$

  $\Rightarrow$ At t_2 the system's state $|\chi\rangle$ is not the ground state i.e. not the vacuum state, i.e. particles (e.g. photons) exist at time t_2.

III Strategy for solving quantized driven harmonic oscillator

Problem:

* CCR: $[\hat{q}(t), \hat{p}(t)] = i\,1$
* Hermiticity: $\hat{q}^+(t) = \hat{q}(t), \quad \hat{p}^+(t) = \hat{p}(t)$
* Hamiltonian: $\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\hat{q}(t)$
* Heisenberg eqns $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ yield:

$$\dot{\hat{q}}(t) = \hat{p}(t)$$
$$\dot{\hat{p}}(t) = -\omega^2\hat{q}(t) + J(t)$$

This is a good strategy with and without a driving force

Strategy:

* Combine

is operator even though no "hat".

$$a(t) := \alpha\,\hat{q}(t) + i\beta\,\hat{p}(t)$$

(analogous to "real" & "imaginary" parts)

Mukhanov calls it $a^-(t)$

* Choose α, β so that $\hat{H}(t)$ and eqn of motion simplify.

IV Determine α and β:

- Notice first that once we have $a(t)$ we immediately obtain $\hat{q}(t)$, $\hat{p}(t)$: Use of $a^+(t) = \alpha\,\hat{q}(t) - i\beta\,\hat{p}(t)$ yields:

$$\hat{q}(t) = \frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

$$\hat{p}(t) = \frac{i}{2\beta}\left(a^+(t) - a(t)\right)$$

- Use this to express $[\hat{q},\hat{p}] = i$ in terms of new variable $a(t)$:

$$\Rightarrow \qquad [a(t), a^+(t)] = 2\alpha\beta$$

For simplicity, we choose $\beta = \frac{1}{2\alpha}$ so that:

$$\boxed{[a(t), a^+(t)] = 1}$$

Now express $\hat{H}(t)$ in terms of new variable $a(t)$:

$$\hat{H}(t) = -\frac{1}{2}\alpha^2 \left(a^+(t) - a(t)\right)^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2}\left(a^+(t) + a(t)\right)^2$$
$$- J(t)\frac{1}{2\alpha}\left(a^+(t) + a(t)\right)$$

We notice that the terms $\sim a^+(t)^2$ and $\sim a(t)^2$ drop out if we choose:

$$-\frac{1}{2}\alpha^2 + \frac{\omega^2}{2}\frac{1}{4\alpha^2} = 0$$

Thus, we choose: $\alpha = \sqrt{\frac{\omega}{2}}$ and therefore $\beta = \frac{1}{\sqrt{2\omega}}$

Thus, by definition:

$$\boxed{a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)}$$

- The Hamiltonian simplifies to become:

$$\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - J(t)\frac{1}{\sqrt{2\omega}}\left(a^+(t) + a(t)\right)$$

V Solve for $a(t)$:

- The Heisenberg equation $i\dot{\hat{f}}(t) = [\hat{f}(t), \hat{H}(t)]$ reads for $a(t)$:

$$i\,\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}} J(t)$$

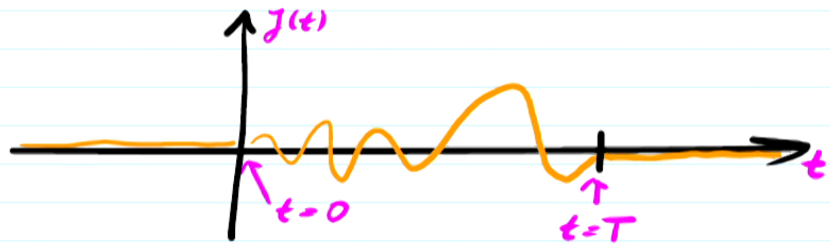
- Let us give $a(t=0)$ a name: $a_{in} = a(0)$. Then:

Exercise: verify.

$$\boxed{a(t) = a_{in}\, e^{-i\omega t} + 1\,\frac{i}{\sqrt{2\omega}} \int_0^t J(t')\, e^{i\omega(t'-t)}\, dt'}$$

VI Case of force of finite duration

- Assume $J(t) = 0$ for all $t \notin [0,T]$



- Define $J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$

- Then:

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0,T] \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } t > T \end{cases}$$

- Define $$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t')\, e^{i\omega t'}\, dt'$$

- Then: $$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t < 0 \\ \text{see above} & \text{for } t \in [0,T] \\ (a_{in} + J_0)\, e^{-i\omega t} & \text{for } t > T \end{cases}$$

Next:

<u>Implications in terms of particle (e.g. photon) production?</u>