Title: Quantum Field Theory for Cosmology - Lecture 4

Speakers: Achim Kempf

Collection: Quantum Field Theory for Cosmology (Kempf)

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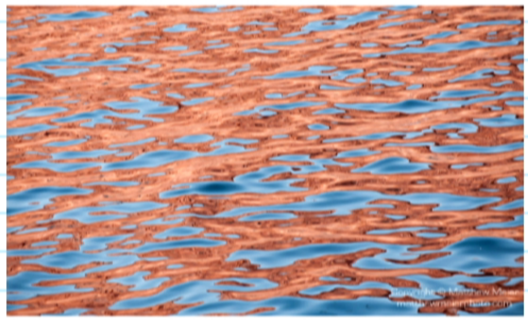
QFT for Cosmology, Achim Kempf, Lecture 4

Note Title

From Heisenberg to Schrödinger picture

Water:

$\phi(x,t)$



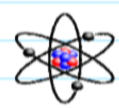
Probe amplitudes, e.g., with a cork:

Quantum field:

$\hat{\phi}(x,t)$

How to visualize an operator-valued field ?

Probe amplitudes, e.g., with atoms (lecture 8):



For now...

Assume we have some means to measure

$$\hat{\phi}(x,t)$$

at a time t for all $x \in \mathbb{R}^3$.

Q: Why possible in principle?

A: Because $\phi^+(x,t) = \phi(x,t)$ and $[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0 \quad \forall x, x' \in \mathbb{R}^3$

Note: The $\hat{\phi}(x,t)$ $\forall x \in \mathbb{R}^3$ are a maximal set of commuting observables.

$\Rightarrow$ At each x obtain real-valued measurement outcome, $\phi(x)$.

Analogous to measuring $\hat{q}_a$ and obtaining measurement outcomes q_a.

Definition: Assume that $\phi : \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary function. Then we define

$$|\phi\rangle \in \mathcal{H}$$

to be the joint eigenvector of all $\hat{\phi}(x,t)$ obeying

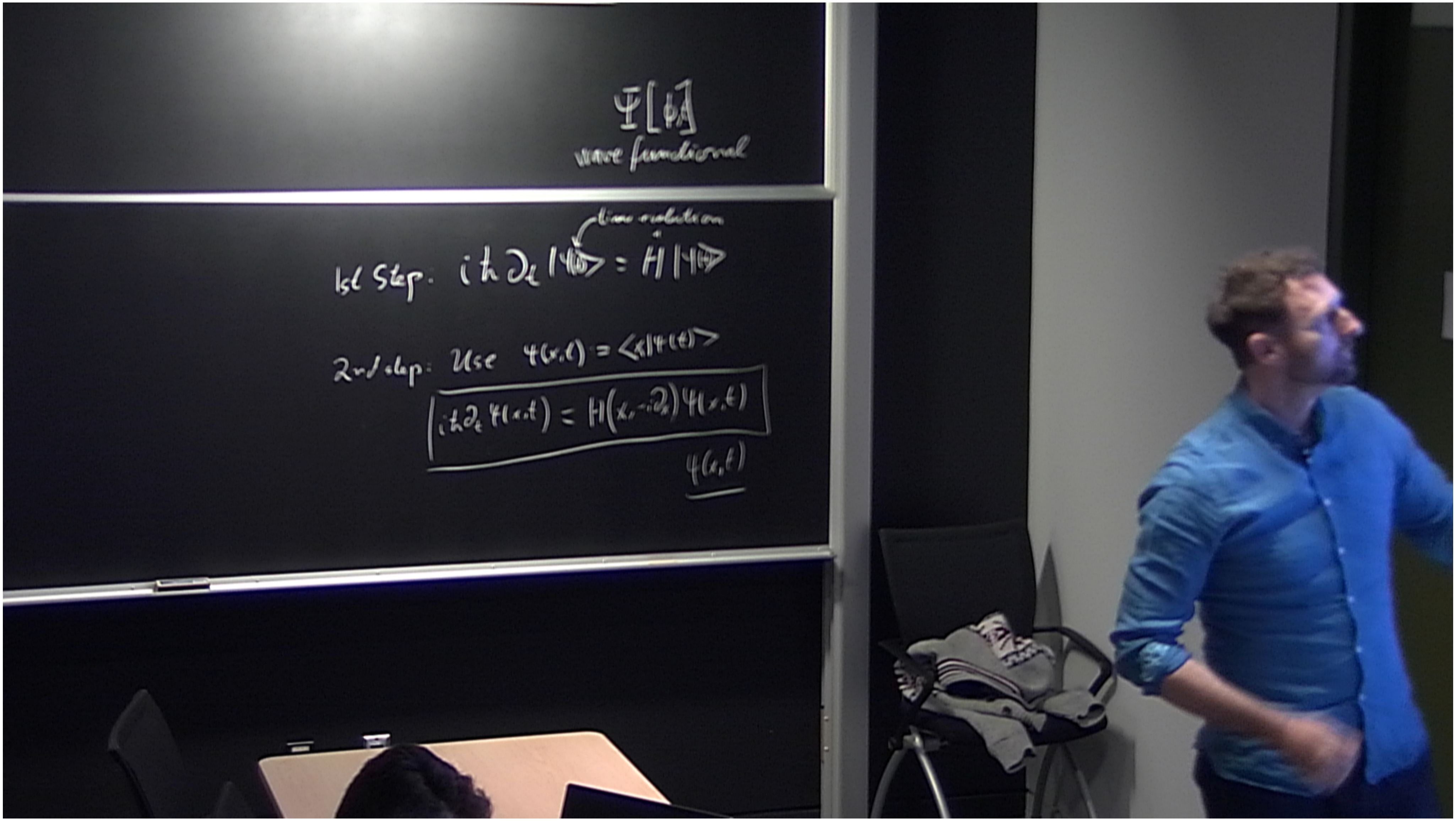
unique up to a phase

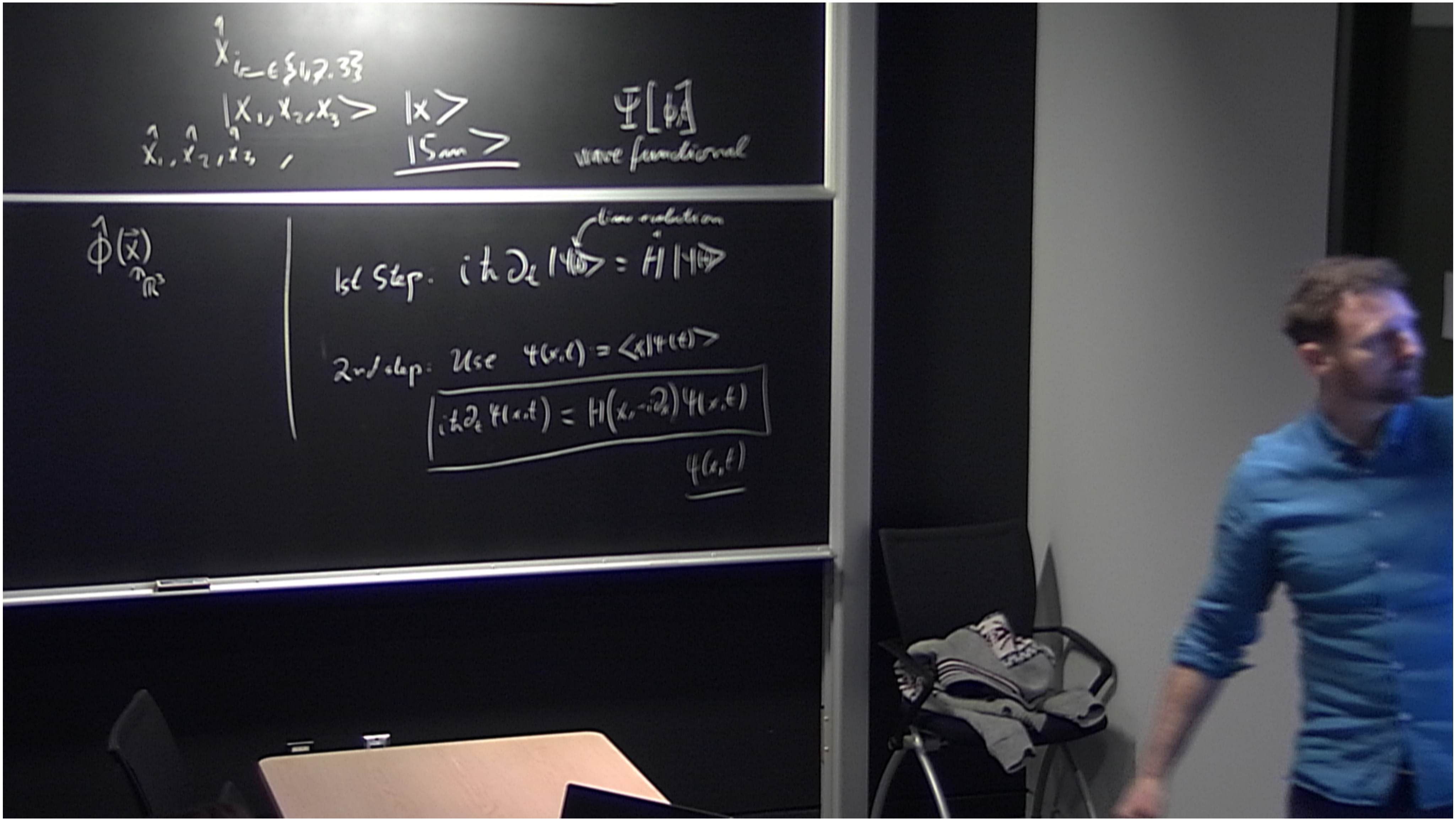
i.e. for all $x \in \mathbb{R}^3$

$$\hat{\phi}(x,t)\,|\phi\rangle = \phi(x)\,|\phi\rangle \qquad \text{for all } x \in \mathbb{R}^3$$

Analogous to: $\hat{q}_a(t)\,|q\rangle = q_a\,|q\rangle \qquad$ for all $a = 1, \dots, 3N$

# of particles


$\Psi[\phi]$
wave functional
time evolution
1st Step: $i\hbar\partial_t|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$
2nd step: Use $\psi(x,t) = \langle x|\psi(t)\rangle$
$\boxed{i\hbar\partial_t\psi(x,t) = H(x,-i\hbar\partial_x)\psi(x,t)}$
$\psi(x,t)$


$\hat{X}_{i \in \{1,2,3\}}$
$|X_1, X_2, X_3\rangle$
$|x\rangle$
$\hat{X}_1, \hat{X}_2, \hat{X}_3$
$|5\,\mathrm{cm}\rangle$
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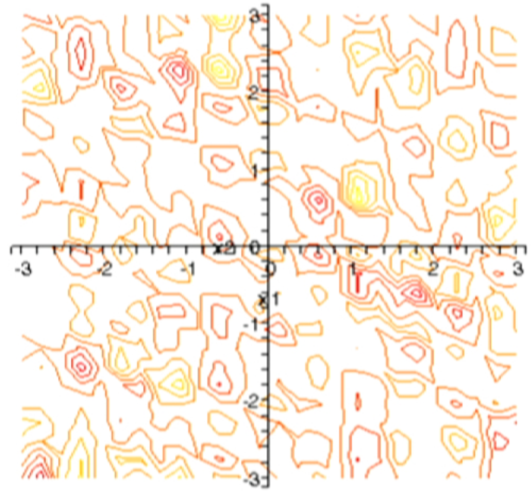
Example: Assume system is in the vacuum state $|\Omega\rangle$.

What is then a typical measurement outcome $\phi(x)$?

Shown are the level curves of a typical measurement outcome $\phi(x)$.



The measurement collapses the system into the new state:

$$|\phi\rangle \in \mathcal{H}$$

Analogous to a state $|q\rangle = |q_1, q_2, q_3, \dots\rangle$

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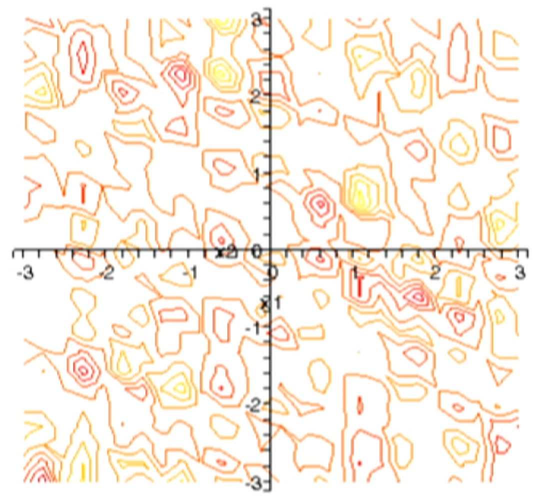

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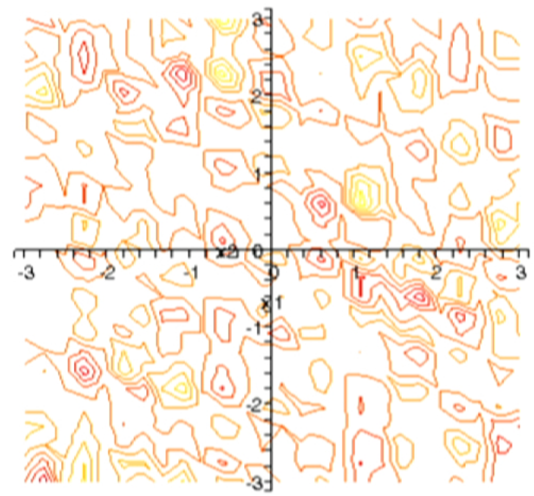

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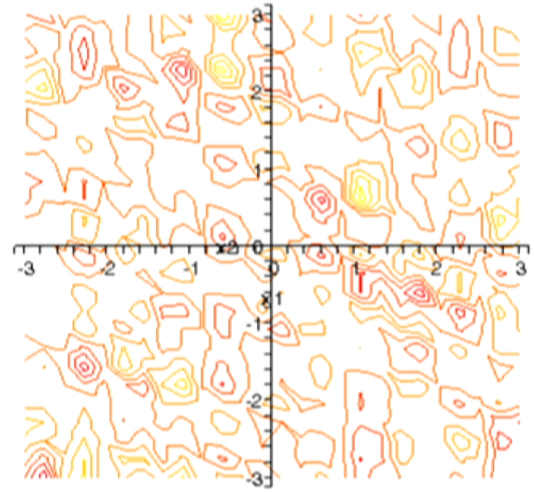

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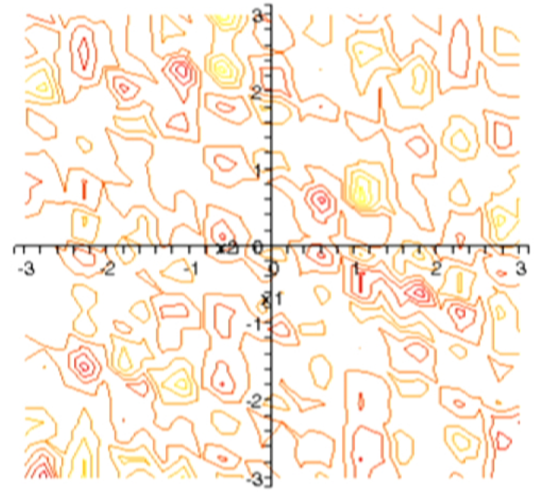

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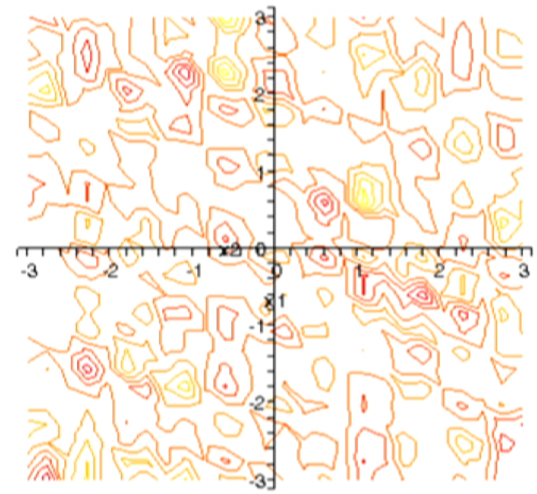

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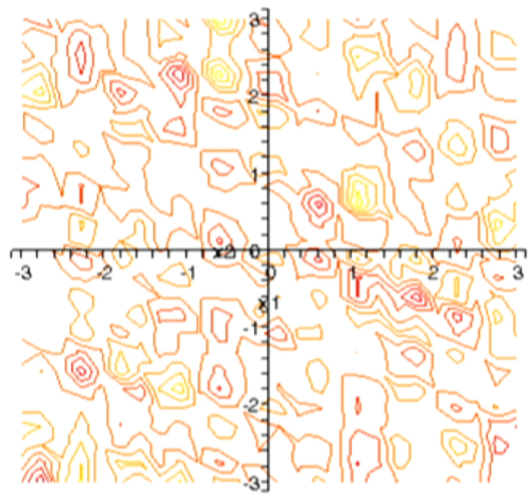

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How to calculate in the Schrödinger picture?

Preparations:

Hilbert basis: The set $\{|\phi\rangle\}$

of all joint eigenvectors of the $\hat{\phi}(x,t)$ for all $x \in \mathbb{R}^3$ can be used to form a "complete ON basis's" of $\mathcal{H}$. (up to functional analytic subtleties).

Resolution of the identity:

$\Rightarrow$ For any $|\Psi\rangle \in \mathcal{H}$ we have:

$$|\Psi\rangle = \int_{\phi \in L^2(\mathbb{R}^3)} |\phi\rangle\langle\phi|\Psi\rangle \, D[\phi]$$

$\leftarrow$ it's more subtle really

analogous to:

$$|\psi\rangle = \int |q\rangle \underbrace{\langle q|\psi\rangle}_{=\psi(q)} d^3q$$

Wave function: $\psi(q)$

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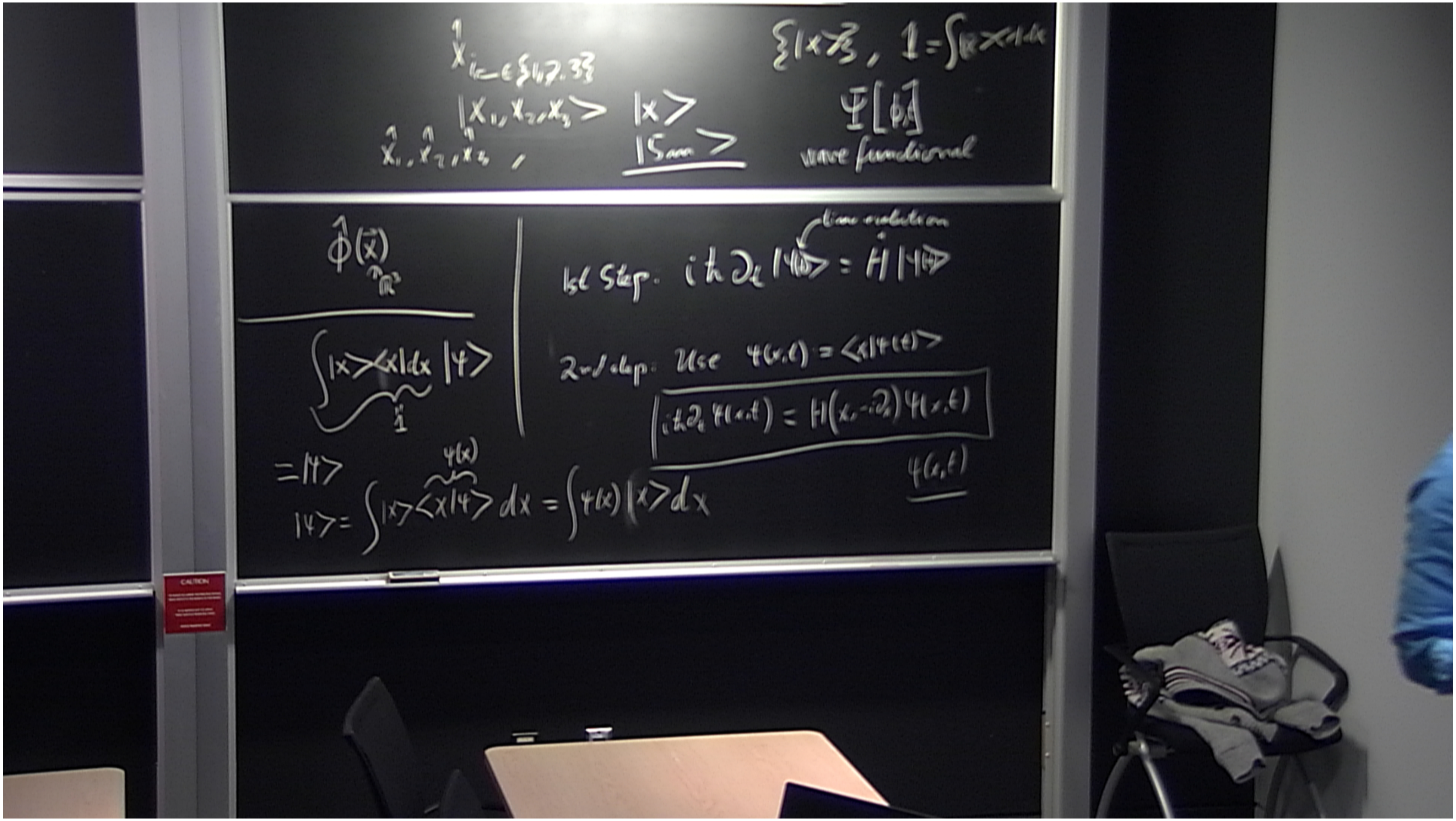

{|x>}, 1 = ∫|x><x|dx
|x1, x2, x3> |x>
Ψ[φ]
wave functional
φ̂(x)
∫|x><x|dx |ψ>
= |ψ>
|ψ> = ∫|x><x|ψ> dx = ∫ψ(x)|x>dx
time evolution
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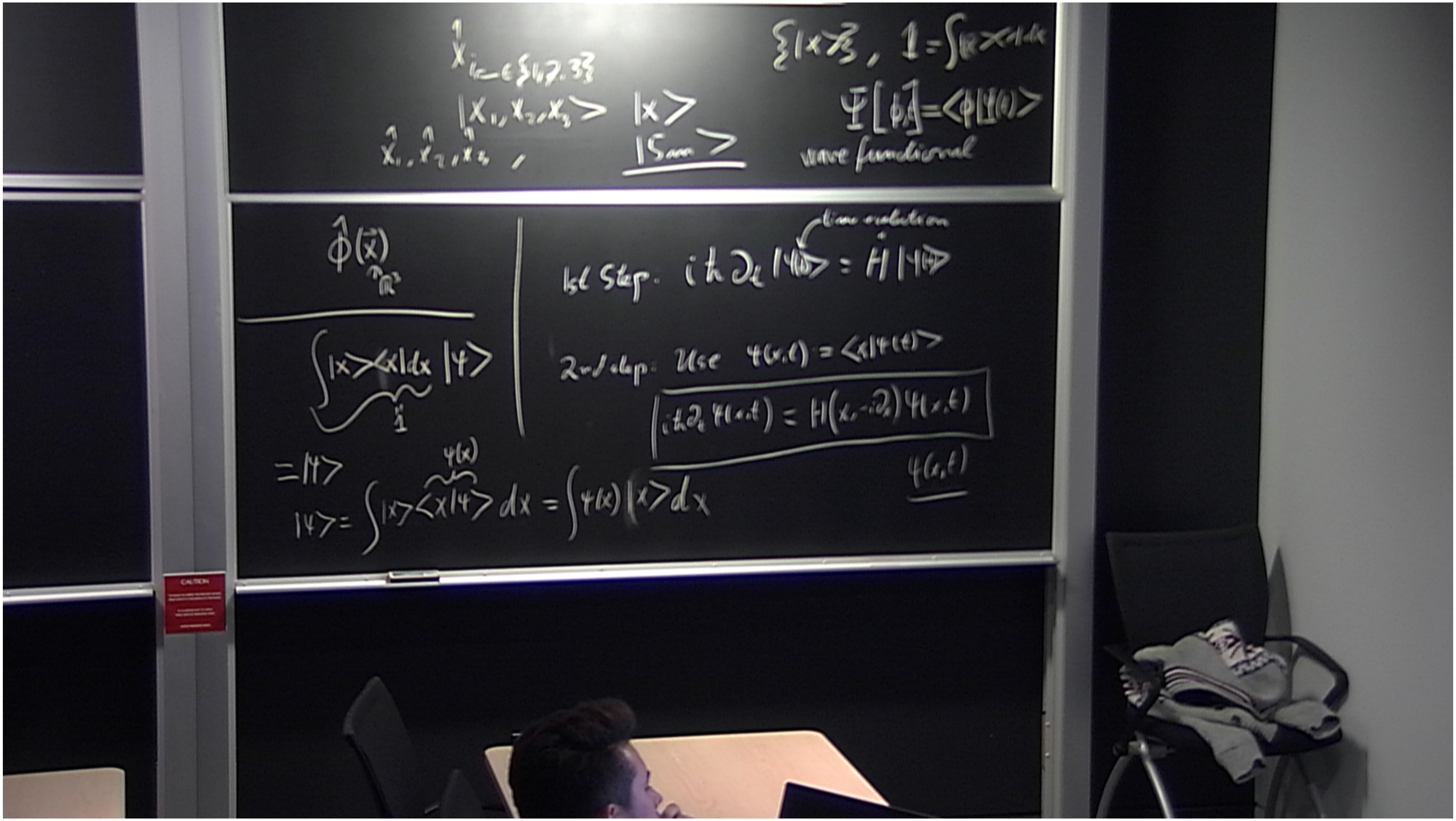

$\hat{X}_{i=} \in \{1,2,3\}$
$|X_1, X_2, X_3\rangle$ $|x\rangle$
$\hat{X}_1, \hat{X}_2, \hat{X}_3$,
$|S_{\dots}\rangle$
$\{|x\rangle\}$, $1 = \int |x\rangle\langle x| dx$
$\Psi[\phi_x] = \langle \phi | \Psi(t) \rangle$
wave functional
$\hat{\phi}(\vec{x})$
$\mathbb{R}^3$
$\int |x\rangle\langle x| dx \, |\psi\rangle$
1
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The "Wave functional"

Recall QM:

- Assume $\{\hat{q}_i\}_{i=1}^{N}$ is compl. set of commuting observables, with joint eigenvectors $|q\rangle$ obeying: $\hat{q}_i|q\rangle = q_i|q\rangle$.

- Then the function ψ, given by $\psi(q) = \langle q|\psi\rangle$ is called the "**wave function**" of $|\psi\rangle$ in the $\{\hat{q}_i\}$ basis.

Example: $\{\hat{p}_i\}$ yield mom. wave functions $\psi(p) = \langle p|\psi\rangle$

$p = \{p_1, p_2, \ldots, p_N\}$

In QFT:

← or, e.g., also the $\{\hat{\pi}(x)\}$.

E.g., $\{\hat{\phi}(x)\}_{x\in\mathbb{R}^3}$ is compl. set of com. observables

with joint eigenvectors $|\phi\rangle$ obeying $\hat{\phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$.

▫ Then, Ψ, given by

$\{|\phi\rangle\}$ form field ON eigen basis

(Convention: square bracket because argument is a function)

$$\Psi[\phi] := \langle\phi|\Psi\rangle$$

(called a "functional" because argument is a function)

alternatively could use e.g. joint eigen basis of the $\hat{\pi}(x,t)$.

is called the "**wave functional**".

Interpretation of $\Psi[\phi]$?

e.g., vacuum $|\Psi_0\rangle$

- Assume the system is in an arbitrary state $|\Psi\rangle \in \mathcal{H}$ at t.

- If measuring now $\hat{\phi}(x,t)$ at all $x \in \mathbb{R}^3$ what is the probability amplitude for finding, say, the values $\phi(x)$?

Answer: $\text{prob}[|\Psi\rangle \to |\phi\rangle] = |\langle\phi|\Psi\rangle|^2 = |\Psi[\phi]|^2$

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- For every quantum theory, we have in the Schrödinger picture of the time evolution:

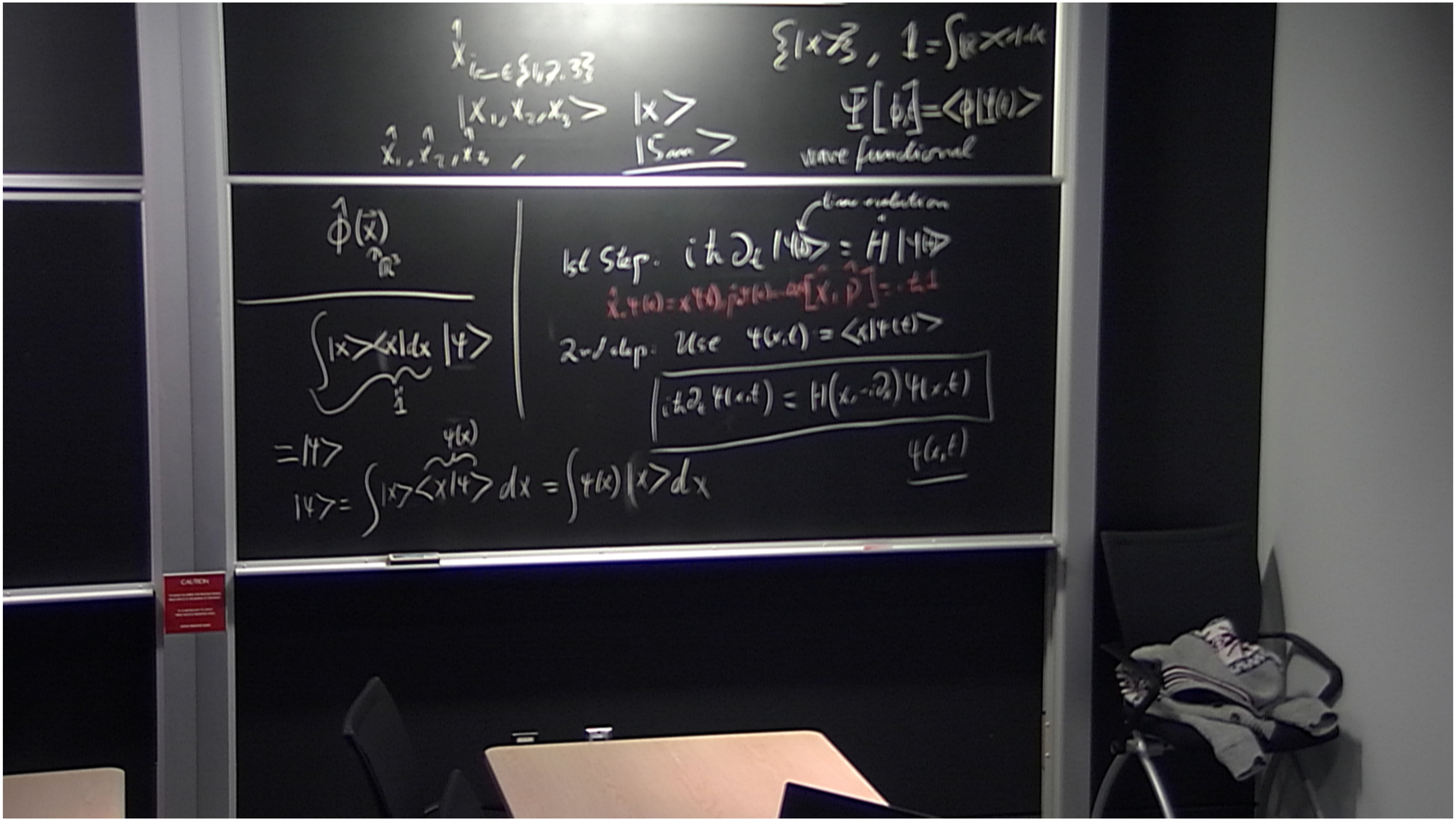
$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$$

- Which form does it take for $\Psi[\phi, t]$?

- Here in QFT:

now independent of time!

$$\hat{H} = \int \frac{1}{2} \left(\hat{\pi}^2(x) + \hat{\phi}(x) \left(-\Delta + m^2 \right) \hat{\phi}(x) \right) d^3x$$


wave functional
time evolution
1st Step.
2nd step. Use
CAUTION

- But how do $\hat{\phi}(x)$ and $\hat{\pi}(x)$ act on wave functionals $\Psi[\phi,t]$?

(Exercise: check)

- A valid representation of $[\hat{\phi}(x),\hat{\pi}(x')] = i\delta^3(x-x')$ is:

$$\hat{\phi}(x).\ \Psi[\phi,t] = \phi(x)\,\Psi[\phi,t]$$

Analogous to: $\hat{q}_a . \Psi(q,t) = q_a \Psi(q,t)$

$$\hat{\pi}(x).\ \Psi[\phi,t] = -i\frac{\delta}{\delta\phi(x)}\,\Psi[\phi,t]$$

$\hat{p}_a . \Psi(q,t) = -i\frac{\partial}{\partial q_a}\Psi(q,t)$

functional derivative, as in variational principle used to derive Euler Lagrange equations.

- Therefore:

$$\hat{H} = \int \frac{1}{2}\left(-\frac{\delta^2}{\delta\phi^2(x)} + \phi(x)(-\Delta+m^2)\phi(x)\right)$$

inconvenient

- It is more convenient to use infrared-regularized momentum space:

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▫ It is more convenient to use infrared-regularized momentum space:

- We now need to represent

$$[\hat{\phi}_k, \hat{\pi}_{k'}] = i\delta_{k,-k'}$$

on the wave functionals $\Psi[\tilde{\phi}, t]$.

($\tilde{\phi}_k$ is Fourier transform of $\phi(x)$)

- As you should verify, this works:

$$\hat{\phi}_k \cdot \Psi[\tilde{\phi}, t] = \tilde{\phi}_k \Psi[\tilde{\phi}, t]$$

$$\hat{\pi}_k \cdot \Psi[\tilde{\phi}, t] = -i \frac{\partial}{\partial \phi_{-k}} \Psi[\tilde{\phi}, t]$$

Note: Ordinary derivatives here because set of variables $\{\tilde{\phi}_k\}$ is discrete, since $k = \frac{2\pi}{L}(n_1, n_2, n_3)$, $\vec{n} \in \mathbb{Z}^3$.

Schrödinger equation:

$i\partial_t |\Psi\rangle = \hat{H}|\Psi\rangle$ becomes:

$$i\partial_t \Psi[\tilde{\phi},t] = \sum_k \frac{1}{2}\left(-\frac{\partial}{\partial\tilde{\phi}_k}\frac{\partial}{\partial\tilde{\phi}_{-k}} + (k^2+m^2)\tilde{\phi}_k\tilde{\phi}_{-k}\right)\Psi[\tilde{\phi},t]$$

Recall: For QM harm. osc., ground state Schrödinger wave function is:

$$\psi(x,t) = N\, e^{-\frac{1}{2}\omega x^2 - i\omega t}$$

Exercise: check it. Can you solve for excited states?

$\Rightarrow$ Ground state solution in QFT reads, similarly:

$$\Psi[\tilde{\phi},t] = N\, e^{-\sum_k \frac{1}{2}\left(\omega_k \tilde{\phi}_k\tilde{\phi}_{-k} - i\omega_k t\right)}$$

$\omega_k = (\vec{k}^2+m^2)^{1/2}$

Exercise: verify

... which we had already seen before.

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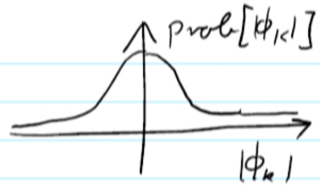
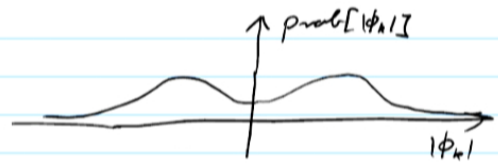
function spreads out – classically its amplitude of oscillation would increase.

⇒ If a mode k is excited then the prob. distribution of the ϕ_k spreads:



ground state

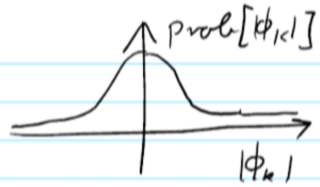


example of excited state

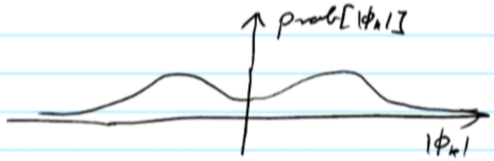
The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a ϕ_k with large modulus $|\phi_k|$.

of oscillation would increase.

⇒ If a mode k is excited then the prob. distribution of the ϕ_k spreads:



ground state



example of excited state

The more a mode k is excited, the more likely is a measurement of $\hat{\phi}_k$ to yield a ϕ_k with large modulus $|\phi_k|$.

⇒ If, e.g., a mode k is very highly excited