

Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 11

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Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

Date: December 13, 2019 - 1:00 PM

URL: <http://pirsa.org/19120058>

Today: Generalized coh. thys.
in particular: "supercdh"

G discrete gp

Recollection: Cat of spaces $= \{ \text{simplicial sets satisfying Kan condition} \}$

$\supset BG$
 $\parallel$
 G^{*0}

Given ab gp A , it makes sense to talk about

$Z^n(-, A), H^n(-, A) \colon \text{Spaces}^{\text{or}} \rightarrow \text{AbGp}$

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$C^n(-, A)$, $Z^n(-, A)$, $H^n(-, A) \stackrel{\circ}{=} \text{Spaces}^{ab} \rightarrow \text{AbGp}$

Recollection: Cat of spaces $= \{ \text{simplicial sets satisfying Kan condition} \} \ni BG$

$$\begin{array}{c} \text{ii} \\ G^{\times n} \end{array}$$

Given ab gp A , it makes sense to talk about

$$\begin{array}{ccccc} C^n(-; A) & , & \underline{Z^n(-; A)} & , & H^n(-; A) \\ \text{coc chain} & & \text{co cycle} & & \text{cohomology} \end{array} \quad \text{Spaces} \xrightarrow{\text{or}} \text{AbGp.}$$

↑ is representable

Given ab gp A , it makes sense to talk about

$$\underbrace{C^n(-; A)}_{\text{cochain}}, \underbrace{Z^n(-; A)}_{\text{cocycle}}, \underbrace{H^n(-; A)}_{\text{is representable, cohomology}} \cong \text{Spaces}^{\text{op}} \rightarrow \text{AbGp.}$$

DK corresp

$$\underbrace{\text{Ch.}}_{\text{connective chain complexes}} \xrightarrow{\sim} \text{AbGp} \begin{matrix} \rightarrow \text{Set} \\ \searrow \cup \\ \text{Spaces} \end{matrix}$$

$$A[n] \mapsto K(A, n) \quad \text{"n"th Eilenberg-MacLane space}$$

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$$K(G, 1) = BG$$

"bar complex"

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any gp G has
 $m: G \times G \rightarrow G$ map of sets.
 it is a map of gpi iff
 G is abelian.

$$K(G, 1) = BG \in \text{Spaces even if } G \text{ nonabelian}$$

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$\uparrow$ is representable $\uparrow$ is homology-rep'd.

DK corresp

$$\underbrace{\text{Ch.}}_{\text{connective chain complexes}} \xrightarrow{\sim} \text{S AbGp} \begin{matrix} \rightarrow \text{S Set} \\ \searrow \\ \text{Spaces} \end{matrix}$$

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any gp G has
 $m: G \times G \rightarrow G$ as a map of gpi
 it is a map of gpi iff
 G is abelian

$$K(G, 1) = BG \in \text{Spaces even if } G \text{ nonabelian}$$

"bar complex"

$$Z^n(-; A)$$

$$\cong \text{maps}(-, K(A, n))$$

there is a
 canonical form.

$\uparrow$ you could find it w/ no
 further info.

Given ab gp A , it makes sense to talk about

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is homology-rep'd.

DK corresp

$$\underbrace{Ch.}_{\text{connective chain complexes}} \xrightarrow{\sim} s\text{AbGp}$$

$$\xrightarrow{\quad} s\text{Set} \cup \text{Spaces}$$

$$[X, Y] = \text{homotopy classes of maps } X \rightarrow Y$$

$$A[n] \mapsto K(A, n)$$

" n th Eilenberg-MacLane space"

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"bar complex"

$$Z^n(-, A)$$

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there is a canonical map.
 you could find it w/ no further info.

$$H^n(-, A) \cong [-, K(A, n)]$$

Given ab gp A , it makes sense to talk about

$$\underbrace{C^n(-; A)}_{\text{cochain}}, \underbrace{Z^n(-; A)}_{\text{cocycle}}, \underbrace{H^n(-; A)}_{\text{cohomology}} \cong \text{Spaces}^{\text{or}} \rightarrow \text{AbGp.}$$

$\uparrow$ is representable $\uparrow$ is homology-rep'd.

DK corresp

$\text{Ch.} \xrightarrow{\sim} \text{S AbGp.}$
 connective chain complexes

$\searrow$
 S Set
 $\cup$
 Spaces

$[X, Y] =$
 homotopy classes
 of maps $X \rightarrow Y$

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 canonical map.
 $\uparrow$ you could find it w/ no
 further info.

$$H^n(-; A) \cong [-, K(A, n)]$$

$$\text{if } C_\bullet = C_0 \xrightarrow{d} C_1 \xrightarrow{d} C_2 \leftarrow \dots$$

$\uparrow$
 Ch

$$\text{then } \pi_n(DK(C_\bullet)) = H_n(C).$$

$X \in \text{Spaces}$, then it makes sense to talk
 ω
 $\circ$ about $\Omega X \in \text{Space}$

$$I := \circ \text{---} !$$

$$\Gamma(W, \Omega X) = \left\{ f: W \times I \rightarrow X \right. \\ \left. \text{st. } f(w, 0) = f(w, 1) = \circ \in X \quad \forall w \in W \right\}$$

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$$\text{maps}(W, \Omega X) = \left\{ f: W \times I \rightarrow X \right. \\ \left. \text{st. } f(w, 0) = f(w, 1) = 0 \in X \quad \forall w \in W \right\}$$

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$$\Omega K(A, n) \simeq K(A, n-1).$$

ΩX has a comp. law
 (concatenate loops).
 go up to homotopy.

$$I := \circ \longrightarrow !$$

$X \in \text{Spaces}$, then it makes sense to talk
 $\circ$ about $\Omega X \in \text{Space}$

$$I := \circ \text{---} \circ$$

go up to homotopy.

$\Rightarrow H^n(-, A)$ is valued
 in ab. gps.

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$$H^n(X, A) = [X, K(A, n)]$$

$X \in \text{Spaces}$, then it makes sense to talk
 $\circ$ about $\Omega X \in \text{Space}$

$$I := \circ \text{---} \circ$$

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$$H^n(X, A) = [X, K(A, n)] = [X, \Omega^2 K(A, n+2)]$$

$$= \pi_0 \underbrace{\text{maps}(X, \Omega^2 K(A, n+2))}_{\in \text{Space}}$$

$$= \pi_0 \Omega^2 \text{maps}(X, K(A, n+2))$$

$$= \pi_0 \text{maps}(X, K(A, n+2))$$

$$\begin{array}{c} \text{maps}(W, \text{maps}(X, Y)) \\ \parallel \\ \text{maps}(W \times X, Y) \end{array}$$

$X \in \text{Spaces}$, then it makes sense to talk
 $\circ$ about $\Omega X \in \text{Space}$

$$I := 0 \text{---} 1$$

go up to homotopy.

$\Rightarrow H^n(-, A)$ is valued
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$$= \pi_0 \text{maps}(X, K(A, n+2))$$

$$\text{maps}(W, \text{maps}(X, Y))$$

$$\cong \text{maps}(W \times X, Y)$$

many functors are
 rep- $\mathcal{Q}$.

main thing to check:

$\mathbb{Z}$: $\text{Space} \xrightarrow{*} \text{Set}$

$X \in \text{Spaces}$, then it makes sense to talk
 $\circ$
 $\circ$ about $\Omega X \in \text{Space}$

$$I := \circ \text{---} \circ$$

go up to homotopy.

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$$\text{maps}(W, \Omega X) = \left\{ f: W \times I \rightarrow X \right. \\ \left. \text{st. } f(w, 0) = f(w, 1) = \circ \in X \quad \forall w \in W \right\}$$

many functors are
 rep'd.

main thing to check.

$Z: \text{Spaces} \xrightarrow{\text{sp}} \text{Set}$

$$Z\left(\bigcup_{W \subseteq V} U\right)$$

$$\rightarrow Z(U)$$

$$\rightarrow Z(V)$$

$$= \pi_0 \text{maps}\left(X, \Omega^2 K(A, n+2)\right)$$

$\in \text{Space}$

$$= \pi_0 \int^2 \text{maps}\left(X, K(A, n+2)\right)$$

$$= \pi_0 \text{maps}\left(X, K(A, n+2)\right)$$

$$\text{maps}(W \times X, Y)$$

$X \in \text{Spaces}$, then it makes sense to talk
about $\Omega X \in \text{Space}$

$$I := 0 \rightarrow 1$$

go up to homotopy.

$\Rightarrow H^n(-, A)$ is valued
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$$\text{maps}(W, \Omega X) = \left\{ f: W \times I \rightarrow X \mid \begin{array}{l} \text{st. } f(w, 0) = f(w, 1) = 0 \in X \\ \forall w \in W \end{array} \right\}$$

$$H^n(X, A) = [X, K(A, n)] = [X, \Omega^2 K(A, n+2)]$$

$$= \pi_0 \underbrace{\text{maps}(X, \Omega^2 K(A, n+2))}_{\in \text{Space}}$$

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$$= \pi_0 \text{maps}(X, K(A, n+2))$$

$$\begin{array}{c} \text{maps}(W, \text{maps}(X, Y)) \\ \parallel \\ \text{maps}(W \times X, Y) \end{array}$$

many functors are
rep- $\mathcal{Q}$.

main thing to check:

$Z: \text{Spaces} \xrightarrow{\text{rep}} \text{Set}$

$$\begin{array}{ccc} & & Z(W) \\ & \nearrow & \searrow \\ Z(U \xrightarrow{u} V) & & \\ & \searrow & \nearrow \\ & & Z(V) \end{array}$$

Given ab gp A , it makes sense to talk about

$C^n(-; A)$ (cocchain), $Z^n(-; A)$ (cocycle), $H^n(-; A) \doteq \text{Spaces}^{\text{op}} \rightarrow \text{AbGp}$.

$\uparrow$ is representable, $\uparrow$ is homology-rep'd.

Moral: to describe $H^n(-; A)$, all I needed were spaces $K(A, n)$ and eqns $K(A, n) \simeq \mathcal{S}(K(A, n-1))$.

Suppose you have $\overset{0}{\circ} X \rightarrow \overset{0}{\circ} Y$ a map in Spaces .

eg. $X = K(A, n) = K(B, n)$,
and f a cochain operation.

$\text{Cone}(f) \in \text{Spaces}$ defined so that

$\{x \in X \text{ together w/ } f(x) \sim 0\}$ maps to $\text{maps}(W, \text{Cone}(f))$

$= \{x: W \rightarrow X \text{ and } h: W \times I \rightarrow Y \text{ s.t. } h(w, 1) = x(w), h(w, 0) = 0\}$

Given ab gp A , it makes sense to talk about

$$\begin{array}{c} C^n(-; A) \\ \text{cocycle} \end{array}, \underbrace{Z^n(-; A)}_{\text{coZycle}}, H^n(-; A) \stackrel{\circ}{=} \text{Spaces}^{\text{op}} \rightarrow \text{AbGp.}$$

$\uparrow$ is representable, $\uparrow$ cohomology, $\uparrow$ is homotopy-rep'd.

Moral to describe $H^n(-; A)$, all I needed were spaces $K(A, n)$ and eqns $K(A, n) \simeq \mathcal{S}(K(A, n-1))$

Suppose you have $f: X \rightarrow Y$ a map in Spaces .

e.g. $X = K(A, m), Y = K(B, n)$,
and f represents a cohom operation.

$\text{Cone}(f) \in \text{Spaces}$

$\{x \in X \text{ together w/ } f(x) \sim 0\}$

defined so that

$$\begin{aligned} \text{maps}(W, \text{Cone}(f)) &= \{x: W \rightarrow X \text{ and } h: W \times I \rightarrow Y \text{ st. } h(w, 1) = x(w), h(w, 0) = 0\} \end{aligned}$$

Given ab gp A , it makes sense to talk about

$C^n(-; A)$ (cocchain), $Z^n(-; A)$ (cozyde), $H^n(-; A)$ (cohomology)

$\text{Spaces} \xrightarrow{\text{or}} \text{AbGp}$

is representable, is homology-rep'l.

Moral: to describe $H^n(-; A)$, all I needed were spaces $K(A, n)$ and eqns $K(A, n) \simeq \int K(A, n-1)$

Suppose you have $f: X \rightarrow Y$ a map in Spaces .

eg. $X = K(A, m)$, $Y = K(B, n)$,
 f represents a coh. operdn.

$\text{Cone}(f) \in \text{Spaces}$ defined so that

$\{x \in X \text{ together w/ } f(x) \sim 0\}$

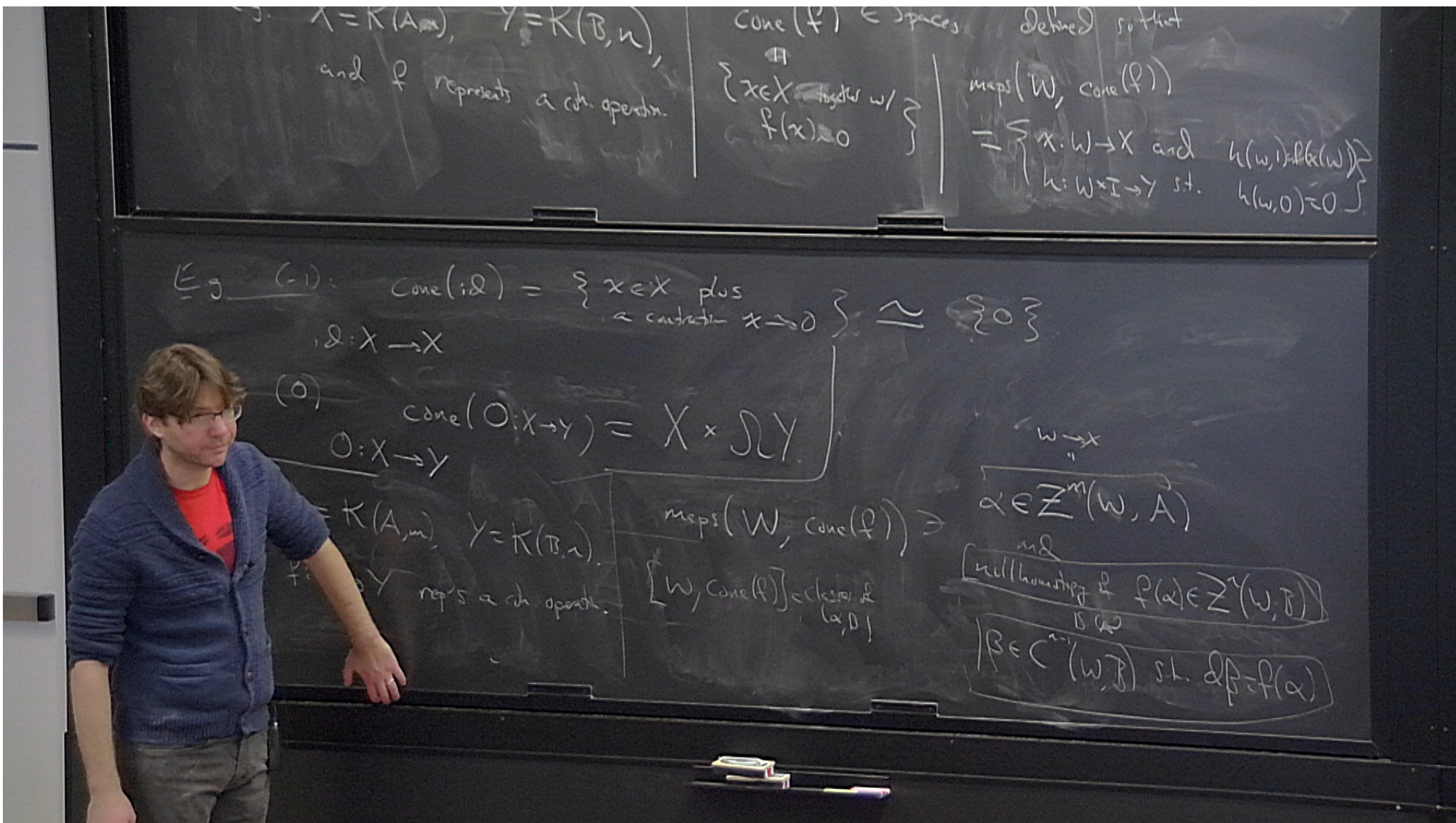
$\text{maps}(W, \text{Cone}(f)) = \{x: W \rightarrow X \text{ and } h: W \times I \rightarrow Y \text{ st. } h(w, 1) = x(w), h(w, 0) = 0\}$

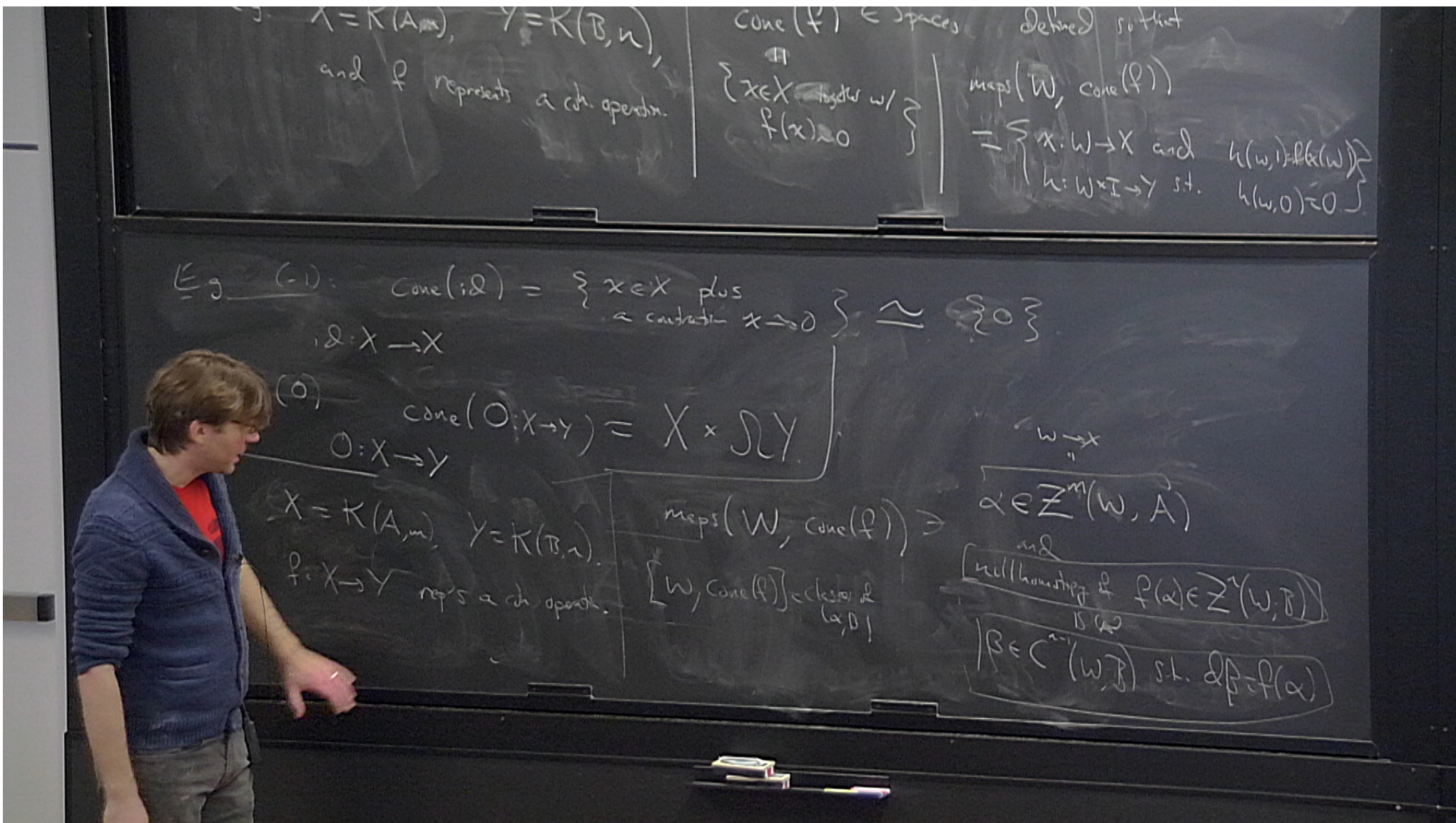
$X = K(A, m)$, $Y = K(B, n)$,
 and f represents a continuous operation.

$\text{Cone}(f) \in \text{spaces}$ defined so that
 Π
 $\left\{ x \in X \text{ together w/ } f(x) = 0 \right\}$
 $\text{maps}(W, \text{Cone}(f))$
 $= \left\{ x: W \rightarrow X \text{ and } h(w, 1) = x(w) \right\}$
 $\left\{ h: W \times I \rightarrow Y \text{ st. } h(w, 0) = 0 \right\}$

Eg (1): $\text{Cone}(id) = \left\{ x \in X \text{ plus a contraction } x \Rightarrow 0 \right\} \simeq \{0\}$
 $id: X \rightarrow X$

(2) $\text{Cone}(0: X \rightarrow Y) = X \times \Omega Y$
 $0: X \rightarrow Y$





$X = K(A, m)$, $Y = K(B, n)$,
 and f represents a ch. operon.

$\text{Cone}(f) \in \text{spaces}$ defined so that
 Π
 $\{x \in X \text{ together w/ } f(x) = 0\}$
 $\text{maps}(W, \text{Cone}(f))$
 $= \{x: W \rightarrow X \text{ and } h(w, 1) = f(x(w))\}$
 $h: W \times I \rightarrow Y \text{ s.t. } h(w, 0) = 0\}$

Eg (1): $\text{Cone}(id) = \{x \in X \text{ plus a contraction } x \Rightarrow 0\} \simeq \{0\}$
 $id: X \rightarrow X$

(0) $\text{Cone}(0: X \rightarrow Y) = X \times \Omega Y$
 $0: X \rightarrow Y$

$X = K(A, m)$, $Y = K(B, n)$,
 $f: X \rightarrow Y$ rep's a ch. operon.

$\text{maps}(W, \text{Cone}(f)) \cong$
 $[W, \text{Cone}(f)] = \text{class of } (\alpha, D)$

$W \rightarrow X$
 $\alpha \in Z^m(W, A)$
 and
 $[nullhomotopy f \text{ } f(\alpha) \in Z^n(W, B)]$
 $\beta \in C^{n-1}(W, B) \text{ s.t. } d\beta = f(\alpha)$

$X \in \text{Spaces}$, then it makes sense to talk
 $\circ$ about $\Omega X \in \text{Space}$

$$I := \circ \text{---} \circ$$

go up to homotopy.

$\Rightarrow H^n(-, A)$ is valued
 in ab. gps.

$$\text{maps}(W, \Omega X) = \left\{ f: W \times I \rightarrow X \mid \begin{array}{l} \text{st. } f(w, 0) = f(w, 1) = \circ \in X \\ \forall w \in W \end{array} \right\}$$

Ex. (1) $\overset{\text{Seq}}{\text{Bockstein}}: K(\mathbb{F}_2, n) \rightarrow K(\mathbb{F}_2, n+1)$

$$(H^n(-, \mathbb{F}_2) \rightarrow H^{n+1}(-, \mathbb{F}_2))$$

$X \in \text{Spaces}$, then it makes sense to talk
 $\circ$ about $\Omega X \in \text{Space}$

$$I := \circ \text{---} \circ$$

go up to homology.

$\Rightarrow H^n(-, A)$ is valued
in ab. gps.

$$\text{maps}(W, \Omega X) = \left\{ f: W \times I \rightarrow X \mid \begin{array}{l} \text{st. } f(w, 0) = f(w, 1) = \circ \in X \\ \forall w \in W \end{array} \right\}$$

Ex. 1 Bockstein: $K(\mathbb{Z}_2, n) \rightarrow K(\mathbb{Z}_2, n+1)$

$$(H^n(-, \mathbb{Z}_2) \rightarrow H^{n+1}(-, \mathbb{Z}_2))$$

$\leftarrow$ α cycle.

$$\tilde{\alpha}(\Delta) = \alpha(\Delta) \in \begin{matrix} \{0, 1\} \subset \mathbb{Z}_2 \\ \uparrow \\ \{0, 1, 2, 3\} \subset \mathbb{Z}_4 \end{matrix}$$

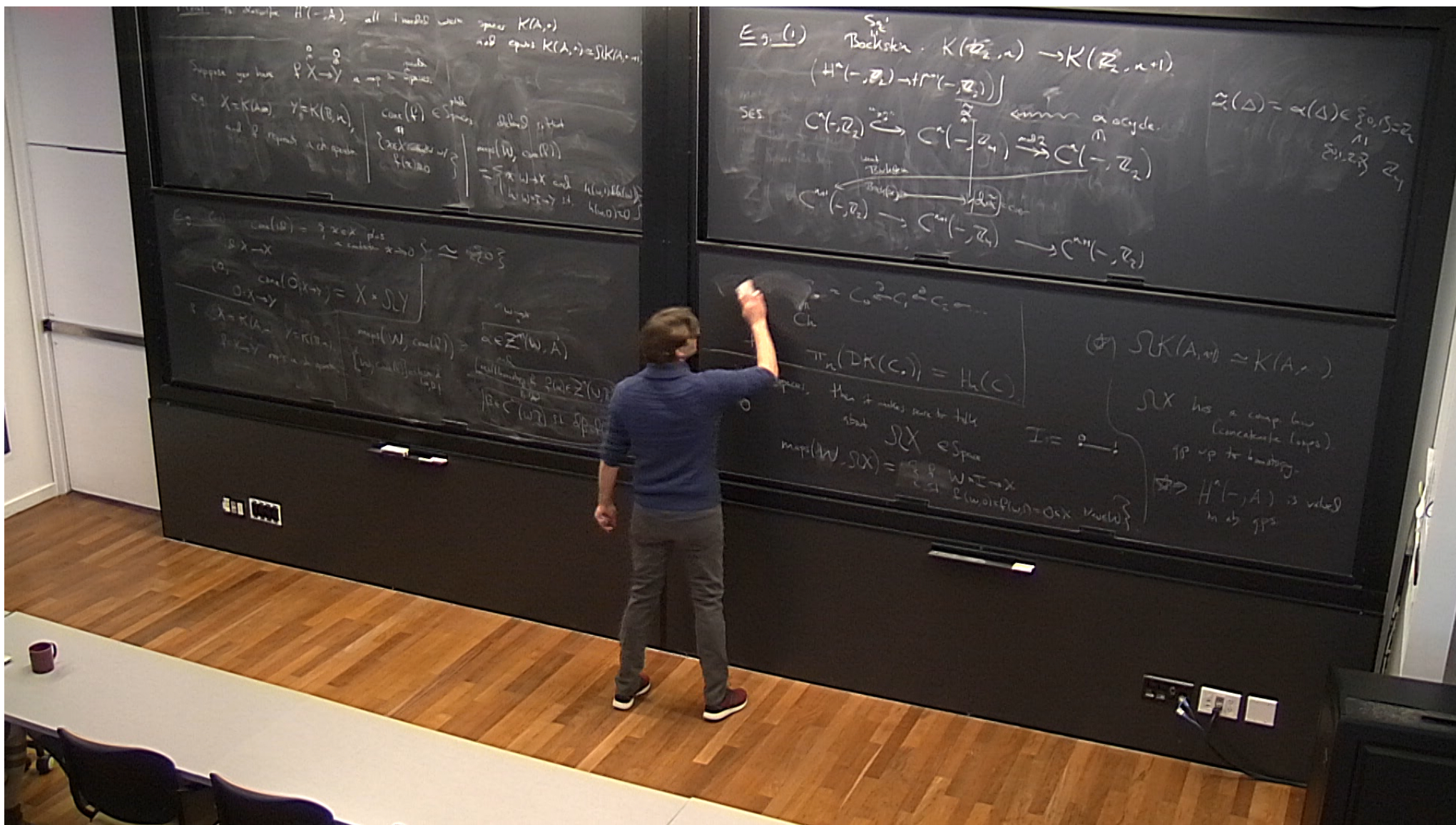
SES.

$$C^n(-) \rightarrow C^n(-, \mathbb{Z}_4) \xrightarrow{\text{mod } 2} C^n(-, \mathbb{Z}_2)$$

with
Bockstein

$$\text{Bock}(\alpha) \rightarrow \boxed{d\tilde{\alpha}} = \text{clon}$$

$$\rightarrow C^{n+1}(-, \mathbb{Z}_4) \rightarrow C^{n+1}(-, \mathbb{Z}_2)$$



Ex. (1) Seq^1 Bockstein $K(\mathbb{Z}_2, n) \rightarrow K(\mathbb{Z}_2, n+1)$

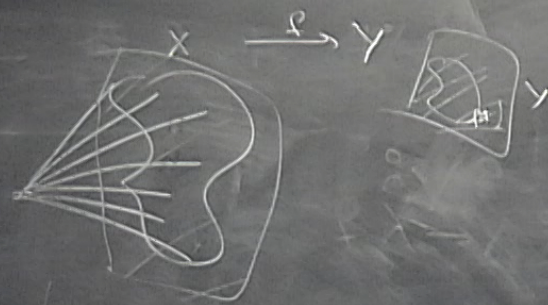
$$(\alpha, \beta) \in \text{Cone}(\text{Bock} : \mathbb{Z}^n(-; \mathbb{Z}_2) \rightarrow \mathbb{Z}^{n+1}(-; \mathbb{Z}_2)) \xrightarrow{\sim} \mathbb{Z}^n(-; \mathbb{Z}_4)$$

$$\left. \begin{array}{l} \alpha \in \mathbb{Z}^n(-; \mathbb{Z}_2) \\ \beta \in \mathbb{C}^n(-; \mathbb{Z}_2) \text{ st.} \end{array} \right\} (\alpha := \gamma \bmod 2, \beta := \frac{\gamma - \tilde{\alpha}}{2}) \xleftarrow{\quad} \gamma$$

solve $2\beta = \text{Bock}(\alpha)$

$$\text{then } 2(2\beta + \tilde{\alpha}) \in \mathbb{C}^n(-; \mathbb{Z}_4)$$

$$= 2\text{Bock}(\alpha) + 2\text{Bock}(\tilde{\alpha}) = 0.$$



I was using

$$\begin{aligned} \ker(f) &\rightarrow X_f \\ \parallel & \\ \ker(f) & \end{aligned}$$

in topology, books

$\text{core}(f)$

"core"

$$X \xrightarrow{f} Y \rightarrow \text{core}(f)$$

$$\text{me (Boch: } K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2))$$

$$\parallel \\ K(n, \mathbb{Z}_2) \cong \text{as sets.}$$

Boch is stable.

$$\text{Boch: } K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$$

$$\parallel \quad \parallel \\ \text{JBoch: } J(K(n+1, \mathbb{Z}_2)) \rightarrow J(K(n+1, \mathbb{Z}_2))$$

"stable"

Because Boch: $K(\cdot, \mathbb{Z}_2) \rightarrow K(\cdot+1, \mathbb{Z}_2)$

$ne(f)$

$$X \xrightarrow{f} Y \rightarrow \text{Cone}(f)$$

"Coker"

$$\text{Bock}_n: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$$

"stable"

$$\Omega \text{Bock}_n: \Omega K(n+1, \mathbb{Z}_2) \rightarrow \Omega K(n+1, \mathbb{Z}_2)$$

Because $\text{Bock}_n: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$
is stable,

$$X^n = ne(\text{Bock}_n: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2))$$

$$\text{and } \Omega X^{n+1} \simeq X^n$$

So $[-, X^n]$ is valued in ab gr.

" π " $X \rightarrow Y$
 $ne(f)$

$$X \xrightarrow{f} Y \rightarrow \text{Coker}(f)$$

"Coker"

$$\text{Back}_i: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$$

"stable"

$$\mathcal{K} \text{Back}_i: \mathcal{K}(K(n, \mathbb{Z}_2)) \rightarrow \mathcal{K}(K(n+1, \mathbb{Z}_2))$$

Because $\text{Back}_i: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$
 is stable,

$$X^n = ne(\text{Back}_i: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)) \simeq K(n, \mathbb{Z}_2)$$

and $\mathcal{K}X^{n+1} \simeq X^n$

So $[-, X^n]$ is valued in ab grps
 $H^n(-, \mathbb{Z}_2)$

this $\mathbb{Z}_2$ is also stable!

Suppose you didn't know about $\mathbb{Z}_4$,
but you did know about Boekstein.

$m(0, \alpha_2) = m(\alpha_1, 0) = 0 \dots$
really want.

$$(\beta_1, \alpha_1) \text{ s.t. } d\beta_1 = \text{Boek}(\alpha_1)$$

$$(\beta_2, \alpha_2)$$

try to define $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$

$$= (\beta_1 + \beta_2 + m(\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$$

$$\begin{aligned} d(\beta_1 + \beta_2 + m(\alpha_1, \alpha_2)) &= d\beta_1 + d\beta_2 + dm(\alpha_1, \alpha_2) \\ &= \text{Boek}(\alpha_1 + \alpha_2) = \text{Boek}(\alpha_1) + \text{Boek}(\alpha_2) + dm(\alpha_1, \alpha_2) \end{aligned}$$

Suppose you don't know about $\mathbb{Z}_4$,
but you did know about Boekstein.

$m(0, \alpha_2) = m(\alpha_1, 0) = 0 \dots$
really want.

(β_1, α_1) s.t. $d\beta_1 = \text{Boek}(\alpha_1)$
 (β_2, α_2)

try to define $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$
 $= (\beta_1 + \beta_2 + m(\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$

$$d(\beta_1 + \beta_2 + m(\alpha_1, \alpha_2)) = d\beta_1 + d\beta_2 + dm(\alpha_1, \alpha_2)$$

$$= \text{Boek}(\alpha_1 + \alpha_2) = \text{Boek}(\alpha_1) + \text{Boek}(\alpha_2) + dm(\alpha_1, \alpha_2)$$

$$m(\alpha_1, \alpha_2) \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) = \alpha_1 \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \cdot \alpha_2 \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right)$$

↑
green cycles

in topology, books

$$\begin{array}{c} \ker(f) \rightarrow X \xrightarrow{f} Y \\ \parallel \\ \text{ne}(f) \end{array}$$

$$X \xrightarrow{f} Y \rightarrow \text{Coker}(f)$$

"coker"

Back is stable.

$$\begin{array}{ccc} \text{Back: } K(n, \mathbb{Z}_2) & \rightarrow & K(n+1, \mathbb{Z}_2) \\ \parallel & & \parallel \\ \text{Back: } J(K(n, \mathbb{Z}_2)) & \rightarrow & J(K(n+1, \mathbb{Z}_2)) \end{array}$$

"stable"

$$(2) \ S_f^2: K(n, \mathbb{Z}_2) \rightarrow K(n+2, \mathbb{Z}_2)$$

$$[-, \text{ne}(S_f^2)] = \left\{ \begin{array}{l} \alpha \in \mathbb{Z}^n(-, \mathbb{Z}_2) \\ \text{and } \beta \in \end{array} \right.$$

in topology, both

$$\begin{array}{c} \text{ker}(f) \rightarrow X \xrightarrow{f} Y \\ \parallel \\ \text{ne}(f) \end{array}$$

"coker"

$$X \xrightarrow{f} Y \rightarrow \text{coker}(f)$$

Back is stable.

Back: $K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$

is

Back: $J(K(n, \mathbb{Z}_2)) \rightarrow J(K(n+1, \mathbb{Z}_2))$

"stable"

(2) $S_{\mathbb{Z}}^2: K(n-2, \mathbb{Z}_2) \rightarrow K(n, \mathbb{Z}_2)$

$S_{\mathbb{Z}}^n := \text{ne}(\quad)$

$[-, S_{\mathbb{Z}}^n]$

$\ker(f) \rightarrow X \xrightarrow{f} Y \rightarrow \text{coker}(f)$
 "kernel" $\rightarrow$ "coker"
 $\text{ne}(f)$

Back is stable.

$\text{Back: } K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$
 $\downarrow$ $\downarrow$
 $\text{Back: } \mathcal{K}(K(n, \mathbb{Z}_2)) \rightarrow \mathcal{K}(K(n+1, \mathbb{Z}_2))$
 "stable"

(2) $SZ^n: K(n, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$

$SZ^n := \text{ne}(\quad)$

$\text{maps}(-, SZ^n) = \left\{ \begin{array}{l} \alpha \in Z^{n-1}(-, \mathbb{Z}_2) \\ \text{or } \beta \in C^n(-, \mathbb{Z}_2) \text{ s.t. } d\beta = SZ^n \alpha \end{array} \right\} = Z^n(-; SZ^n)$

Because $SZ^n \circ SZ^n = SZ^n$
 $\mathcal{K}SZ^{n+1} = SZ^n$

$SH^*(-) = H(-, SZ^n) = [-, SZ^n]$

$\ker(f) \rightarrow X \xrightarrow{f} Y \rightarrow \text{coker}(f)$
 $\parallel$
 $\text{ne}(f)$

Back is stable.

$\text{Back: } K(n, \mathbb{Z}_2) \rightarrow SK(n+1, \mathbb{Z}_2)$
 $\parallel$
 $\text{Back: } J(K(n+1, \mathbb{Z}_2)) \rightarrow J(K(n+1, \mathbb{Z}_2))$
 "stable"

$(2) \quad S_{\mathbb{Z}}^2: K(n+1, \mathbb{Z}_2) \rightarrow K(n+1, \mathbb{Z}_2)$
 $S_{\mathbb{Z}}^n := \text{ne}(\quad)$

real
"supercocycles"

$\text{maps}(-, S_{\mathbb{Z}}^n) = \left\{ \begin{array}{l} \alpha \in \mathbb{Z}^{n-1}(-, \mathbb{Z}_2) \\ \text{or } \beta \in C^1(-, \mathbb{Z}_2) \text{ s.t. } d\beta = S_{\mathbb{Z}}^2 \alpha \end{array} \right\} = \mathbb{Z}^n(-; S_{\mathbb{Z}}^n)$

Because $J S_{\mathbb{Z}}^2 = S_{\mathbb{Z}}^2$
 $J S_{\mathbb{Z}}^{n+1} = S_{\mathbb{Z}}^n$

$SH^*(-) = H^*(-, S_{\mathbb{Z}}) = [-, S_{\mathbb{Z}}]$
 real
"supercoh"

to define $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$

$$= (\beta_1 + \beta_2, \alpha_1 + \alpha_2)$$

$$m(\alpha_1, \alpha_2)(\text{triangle}) = \alpha_1(\text{triangle}) \cdot \alpha_2(\text{triangle})$$

$$= \alpha_1 \cup \alpha_2$$

$$\mathbb{Z}_2 = \{\pm 1\} = \text{real unitary numbers} \subseteq \mathbb{R}^\times$$

cat of real unitary lines
1-dim v space

$$\text{Sym} \otimes \text{cat} = \mathbb{B} \mathbb{Z}_2$$

$$\mathbb{Z}_2 \leftrightarrow S_{\mathbb{Z}}^1$$

cat of real unitary super lines
two objects bos, ferm

$$\text{Sym} \otimes \text{cat} \\ d_5 = \mathbb{Z}_2 \\ \mathbb{Z} \text{-mod} = \mathbb{Z}_2$$

$$\text{Koszul(Sym, mod)} \leftrightarrow S_{\mathbb{Z}}^2$$

$$Z(\alpha_1, \alpha_2) = (\beta_1 + \beta_2, \alpha_1 + \alpha_2)$$

$$m(\alpha_1, \alpha_2) = \alpha_1 \cup \alpha_2$$

degree 1 cocycles

$$E_S: SH^2(-)$$

cocycle is (α, β) where

$$\alpha \in Z^1(-, \mathbb{Z}_2)$$

$$\beta \text{ solves } 2\beta = S_S^2 \alpha.$$

$$Z(\alpha, \beta) = (\beta_1 + \beta_2 + \alpha_1, \alpha_1 + \alpha_2)$$

$$m(\alpha_1, \alpha_2) = \alpha_1 \cup \alpha_2$$

degree 1 cocycle

any:

$$E.S.: SH^2(X) \xrightarrow[\text{not } \cong]{\text{Set}} H^1(X, \mathbb{Z}_2) \times H^2(X, \mathbb{Z}_2)$$

Cocycle is (α, β) where

$$\alpha \in Z^1(X, \mathbb{Z}_2)$$

$$\beta \text{ solves } d\beta = S_{\mathbb{Z}_2}^2 \alpha.$$

as a gp,

$$(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$$

$$= (\beta_1 + \beta_2 + \alpha_1 \cup \alpha_2, \alpha_1 + \alpha_2)$$

$$\text{on } H^1, S_{\mathbb{Z}_2}^2 \alpha = 0$$

for dg. reason.

In fact, Steenrod's formula $\Rightarrow S_{\mathbb{Z}_2}^2 \alpha = 0$ as cocycle.

e.s. SH

σ is defined $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$ | $m(\alpha_1, \alpha_2) = \alpha_1 \cup \alpha_2$ any:
 $= (\beta_1 + \beta_2, \alpha_1 + \alpha_2)$ ↑ degree of cycles

$E.S.: SH_{\mathbb{R}}^2(X) \xrightarrow[\text{not } \cong]{\text{Set}} H^1(X; \mathbb{Z}_2) \times H^2(X; \mathbb{Z}_2)$

cocycle is (α, β) where
 $\alpha \in Z^1(X; \mathbb{Z}_2)$
 β solves $2\beta = S_{\mathbb{Z}}^2 \alpha$.

$\sim$ as a gp.
 $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$
 $= (\beta_1 + \beta_2 + \alpha_1 \cup \alpha_2, \alpha_1 + \alpha_2)$

$S_{\mathbb{Z}}^2 \alpha = 0$
 deg. reason
 odd $\Rightarrow S_{\mathbb{Z}}^2 \alpha = 0$ as cocycle

e.s. $SH_{\mathbb{R}}^2(\mathbb{Z}_2) = \mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
 H.W. $\{ \text{non-trivial } \alpha \} \cong \mathbb{Z}_2$ under β .

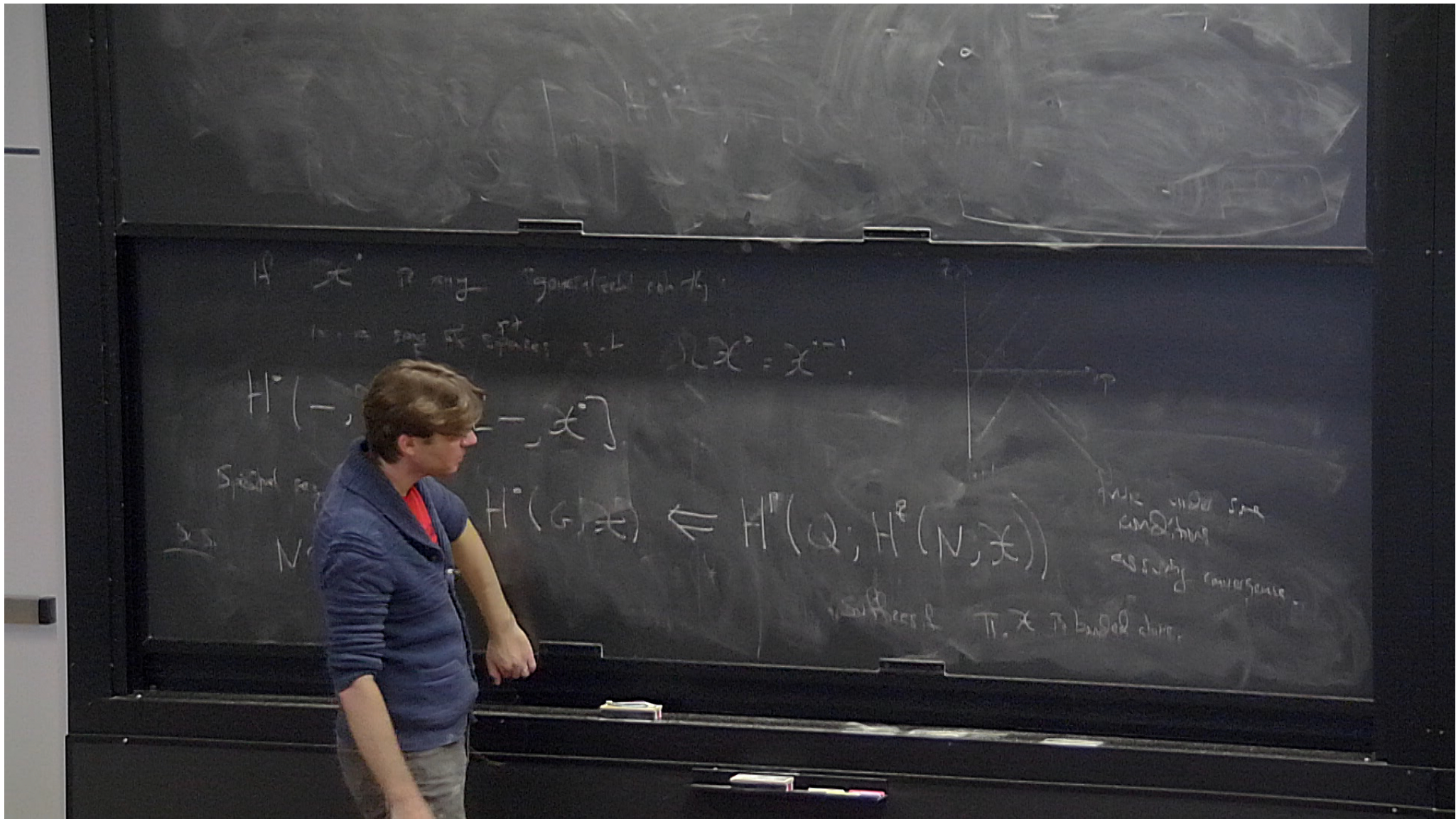
If X is any "generalized relativity"

is a set of spaces $\Rightarrow \Omega X = X^{-1}$.

$$H^*(-, X) = [-, X]$$

Spectral sequence

$$NSG \rightarrow Q$$



to define $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$

$$= (\beta_1 + \beta_2 + (\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$$

$$m(\alpha_1, \alpha_2) = \alpha_1 \cup \alpha_2$$

degree of cycles

carry:

Ex: $N = \mathbb{Z}, G = \mathbb{Q}$

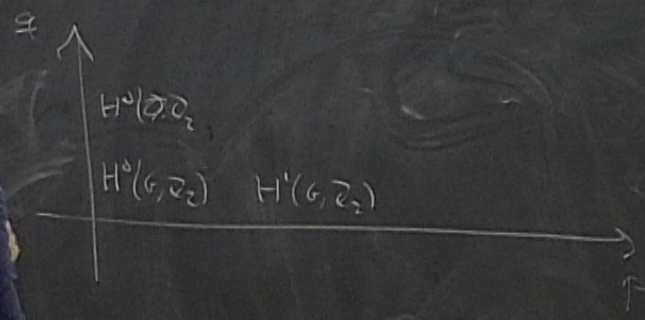
stable of deg $n-m$

$$\chi = \text{nc}(K(A; m) \rightarrow K(B; n))$$

$$H^*(G; \chi) \leftarrow H^*(G; H^*(pt, \chi))$$

for $SH^*_{\mathbb{R}}$, $H^*(pt) = \begin{cases} A & \varepsilon = n-m \\ B & \varepsilon = 0 \end{cases}$

$\Rightarrow \mathbb{R} \otimes$



To define $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$

$$= (\beta_1 + \beta_2 + (\alpha_1, \alpha_2), \alpha_1 + \alpha_2)$$

$$m(\alpha_1, \alpha_2) = \alpha_1 \cup \alpha_2$$

degree of cycles

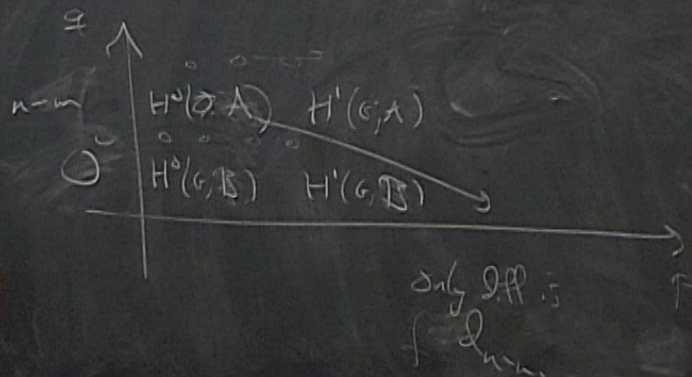
any:

Ex: $N = \mathbb{Z}, G = Q$

$\mathcal{X} = \text{ne}(K(A; m)) \xrightarrow{\text{stable at deg } n-m} K(B; n)$

$$H^*(G; \mathcal{X}) \leftarrow H^*(G; H^*(pt, \mathcal{X}))$$

for $SH_{\mathbb{R}}^*$, $H^*(pt) = \begin{cases} A & \varepsilon = n-m \\ B & \varepsilon = 0 \\ 0 & \text{else} \end{cases}$



try to define $(\beta_1, \alpha_1) \cdot (\beta_2, \alpha_2)$

$$= (\beta_1 + \beta_2, \alpha_1 + \alpha_2)$$

$$m(\alpha_1, \alpha_2) = \alpha_1 \cup \alpha_2$$

Ex: $N = \mathbb{Z}, G = \mathbb{Q}$

$$X = \text{nc}(K(A; n) \xrightarrow{\text{stable of deg } n-m} K(B; n))$$

$$H^*(G; X) \leftarrow H^*(G; H^*(pt, X))$$

$$\begin{array}{ccc} & H^0(G; A) & H^1(G; A) \\ \circ & \downarrow & \downarrow \\ & H^0(G; B) & H^1(G; B) \end{array}$$

only diff is $f = d_{n-m}$

$$H^*(pt) = \begin{cases} A & \varepsilon = n-m \\ B & \varepsilon = 0 \end{cases}$$

$$\Rightarrow \text{LES: } H^1 \hookrightarrow SH^*_{\mathbb{R}} \rightarrow H^{n-1}$$

$$(2) (-1)^{S_Z^2} : K(n-1, \mathbb{Z}_2) \rightarrow K(n+1, U(1))$$

$$S_Z^n := \text{ne}(\quad)$$

(3) extended SH_C d_2 diff. eq

2	$\mathbb{Z}_2$	$\begin{array}{c} \nearrow S_Z^2 \\ \searrow (-1)^{S_Z^2} \end{array}$
1	$\mathbb{Z}_2$	
0	$U(1)$	
	0	

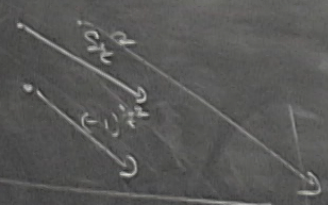
$$S_Z^2 \circ S_Z^2 \neq 0.$$

$$\text{but } (-1)^{S_Z^2} S_Z^2 = 0 \text{ on homology.}$$

choose a chain reasm $\leadsto d_3$ diff.

$$\partial_{\text{tot}} = \begin{pmatrix} \partial_{U(1)} \circ (-1)^{S_Z^2} & \boxed{?} \\ 0 & \partial_{\mathbb{Z}_2} & S_Z^2 \\ 0 & 0 & \partial_{U(1)} \end{pmatrix}$$

2	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
0	$U(1)$
0	
0	



$$S_2^2 \circ S_2^2 \neq 0.$$

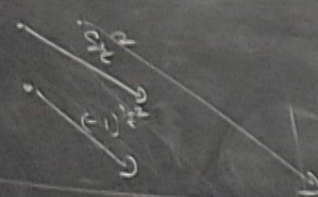
$$\text{but } (-1)^{S_2^2 \circ S_2^2} = 0 \text{ on homology.}$$

choose action reason why ∂_3 diff.

$\mathbb{Z}$
$\mathbb{Z}_2$
$\mathbb{Z}_2$
$U(1)$

- ∂_2 differential
- $\rightarrow S_2^2 \circ (\text{mod } 2)$
 - $\rightarrow S_2^2$
 - $\rightarrow (-1)^{S_2^2}$

2	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
0	$U(1)$
0	0
0	0



$$S_2^2 \circ S_2^2 \neq 0.$$

but $(-1)^{S_2^2 \circ S_2^2} = 0$ on homology.

choose action reason with d_3 diff.

$\mathbb{Z}$
$\mathbb{Z}_2$
$\mathbb{Z}_2$
$\rightarrow U(1)$

d_2 diff. $\rightarrow S_2^2 \pmod{2}$
 $\rightarrow S_2^2$
 $\rightarrow (-1)^{S_2^2}$

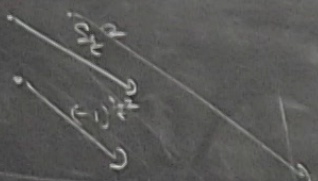
maybe $d_2 \pmod{2}$.

invert 2 $\rightarrow$

$\mathbb{Z}$
0
0
$U(1)$

$$d_4 \text{ diff} = \exp\left(\frac{2\pi i}{3} P'_{\text{res}} \pmod{3}\right)$$

2 $\mathbb{Z}_2$
 1 $\mathbb{Z}_2$
 0 $U(1)$
 0
 0



$$S_2^2 \circ S_2^2 \neq 0.$$

but $(-1)^{S_2^2 \circ S_2^2} = 0$ on homology.

choose addition reason with d_3 d_4 .

beyond $\rightarrow \mathbb{Z}$
 $\mathbb{Z}_2$
 $\mathbb{Z}_2$
 $\mathbb{Z}_2$
 $U(1)$

d_2 different $\rightarrow$
 $S_2^2 \pmod{2}$
 S_2^2
 $(-1)^{S_2^2}$

in $\mathbb{Z}_2$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ $\mathbb{Z}_2$

invert 2 $\rightarrow$

$\mathbb{Z}$
 0
 0
 0
 $U(1)$

$$d_4 d_4 = \exp\left(\frac{2\pi i}{3} P' \cdot 0 \pmod{3}\right)$$