Title: Holographic Uhlmann Holonomy and the Entanglement Wedge Symplectic Form

Speakers:

Series: Quantum Fields and Strings

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Abstract: Subregion duality is an idea in holography which states that every subregion of the boundary theory has a corresponding subregion in the bulk theory, called the 'entanglement wedge', to which it is dual. In the classical limit of the gravity theory, we expect the fields in the entanglement wedge to permit a Hamiltonian description involving a phase space and Poisson brackets. In this talk, I will describe how this phase space arises from the point of view of the boundary theory. In particular, I will explain how it emerges from measurements of a certain quantum information-theoretic quantity known as the 'Uhlmann phase', in the boundary subregion.

Holographic Uhlmann Holonomy and the Entanglement Wedge Symplectic Form

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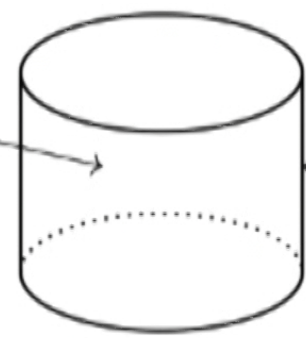
Josh Kirklin

University of Cambridge

Holography

Setting of the talk:

"bulk"		"boundary"
$(d+1)$-dimensional theory with gravity asymptotically AdS $\mathcal{M}$	dual ⟷	d-dimensional theory without gravity CFT lives on $\partial\mathcal{M}$



Holography

Loosely speaking, there are two main avenues of research in holography:

1. Translate hard $\begin{Bmatrix} \text{gravity} \\ \text{QFT} \end{Bmatrix}$ question $\rightarrow$ easy $\begin{Bmatrix} \text{QFT} \\ \text{gravity} \end{Bmatrix}$ question.
   - This translation uses the 'holographic dictionary' – the collection of 1-to-1 maps between concepts in the two theories.
2. Write new entries in the dictionary.
   - Deepens our understanding of holography.
   - Widens scope for potential applications.

Holographic subregion duality

A particular section of the dictionary has emerged in the last few years which relates locality in the two theories.

If A is a subregion of the boundary CFT, then $\exists$ some subregion of the bulk which is dual to A.

Any question we could ask about A can be rephrased as a question about its dual subregion, and vice versa.

Holographic subregion duality

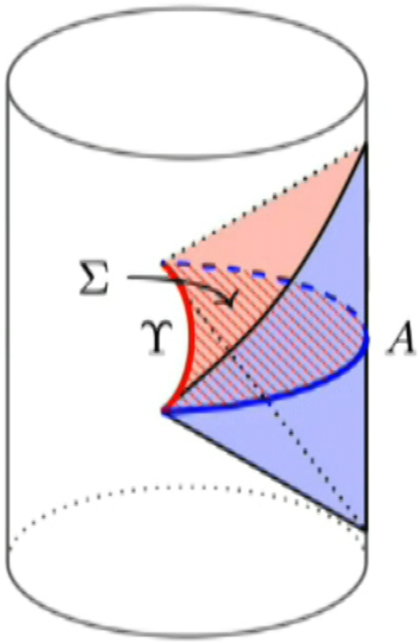
The dual subregion is the 'entanglement wedge'.

A: partial Cauchy surface on the boundary.

Υ: extremal area surface in the bulk homologous to A.

Σ: surface interpolating between A and Υ.

Entanglement wedge: domain of dependence of Σ.



Holographic subregion duality

Area of Υ gives leading order entanglement entropy [Ryu-Takayanagi (2006), Hubeney-Rangamani-Takayanagi (2007)].

Quantum corrections to this come from bulk fluctuations on Σ [Faulkner-Lewkowycz-Maldacena (2013)].

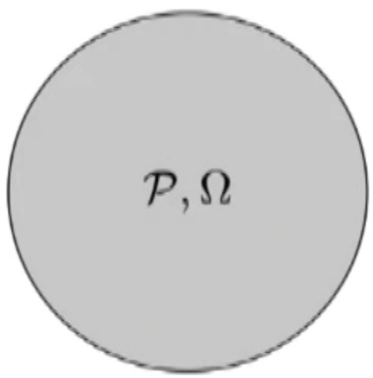
The boundary state in A is dual to the bulk state in the entanglement wedge [Jafferis-Lewkowycz-Maldacena-Suh (2015)].

Any bulk operator $\mathcal{O}_{\text{bulk}}$ acting on fields in entanglement wedge can be reconstructed as boundary operator $\mathcal{O}_A$ acting in A [Dong-Harlow-Wall (2016)].

There are many other developments in subregion duality supporting these ideas.

In this talk, I give an argument for a new entry in the subregion duality section of the holographic dictionary.

In the classical limit of bulk gravity theory ($N \to \infty$) we should get a phase space + Poisson brackets. Mathematically: a symplectic manifold.



Manifold $\mathcal{P}$. Each $x \in \mathcal{P}$ is a classical configuration.

Symplectic form Ω: closed and non-degenerate 2-form.

$$\Omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$$

Ω gives Poisson brackets.

Covariant phase space

Theory of fields ϕ described by action $S = \int L[\phi]$.

A linear variation of the fields gives

$$\delta L[\phi] = L[\phi + \delta\phi] - L[\phi] = E[\phi] \cdot \delta\phi + \mathrm{d}(\theta[\phi, \delta\phi]) \, .$$

Phase space $\mathcal{P}$ is space of solutions to $E[\phi] = 0$.

Pick a Cauchy surface Σ. Define

$$\Theta[\phi, \delta\phi] = \int_\Sigma \theta[\phi, \delta\phi].$$

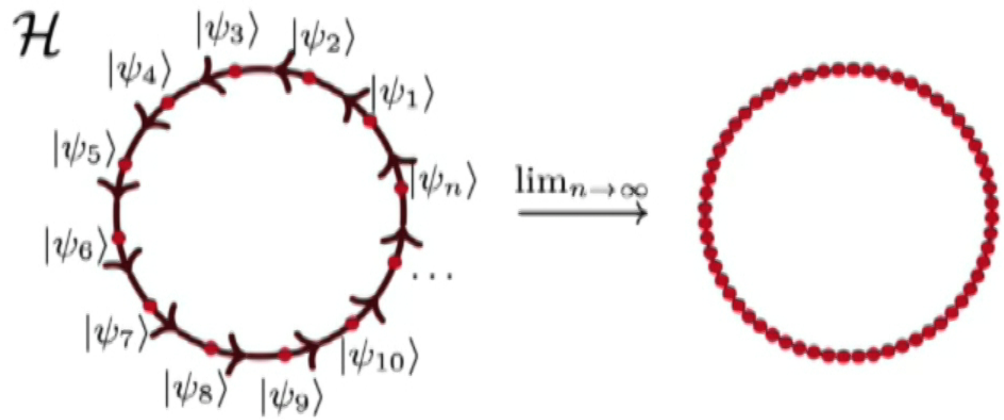
$\delta\phi$ is a vector on $\mathcal{P}$ $\implies$ Θ is a 1-form on $\mathcal{P}$.

Define Ω as exterior derivative of Θ. Ω is symplectic form.

Consider the case of a field theory on an asymptotically AdS spacetime, holographically dual to a CFT.

Then: the symplectic form Ω of the covariant phase space is dual to the Berry curvature of the boundary Hilbert space [Belin-Lewkowycz-Sárosi (2018)].

Berry curvature



Hilbert space $\mathcal{H}$. Consider a closed curve $C : S^1 \to \mathcal{H}$ of normalised states. Choose states ordered along C.

Transition amplitude:

$$\langle\psi_1|\psi_n\rangle\ \langle\psi_n|\psi_{n-1}\rangle \ldots \langle\psi_3|\psi_2\rangle\ \langle\psi_2|\psi_1\rangle \longrightarrow \exp(i\gamma)$$

Berry curvature

Berry phase:

$$\gamma = i \int_C \langle \psi(s) | \frac{\mathrm{d}}{\mathrm{d}s} | \psi(s) \rangle \, \mathrm{d}s = \int_C a,$$

where $a = i \langle \psi | \mathrm{d} | \psi \rangle$ is the Berry connection.

The curvature of this connection

$$f = \mathrm{d}a = i \, \mathrm{d} \langle \psi | \wedge \mathrm{d} | \psi \rangle$$

is the Berry curvature.

This talk: a generalisation of this result to subregions.

Let Σ be a Cauchy surface for the entanglement wedge.

Covariant phase space formalism gives a symplectic form for the entanglement wedge.

I will show how to recover this symplectic form by measuring Uhlmann phase in A.

Uhlmann phase: a generalisation of Berry phase to mixed states. Necessary because state in A is always mixed due to entanglement.

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Plan

- What is Uhlmann phase?
  - Uhlmann holonomy, fidelity
  - Uhlmann's theorem
- Uhlmann holonomy in holographic theories
  - Replica trick
  - Holographic Uhlmann phase $\to$ the symplectic form

Plan

- What is Uhlmann phase?
  - Uhlmann holonomy, fidelity
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- Uhlmann holonomy in holographic theories
  - Replica trick
  - Holographic Uhlmann phase $\to$ the symplectic form
- Comments and future directions

Purifications

Density matrix $\rho : \mathcal{H} \to \mathcal{H}$.

A purification of ρ is a state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}'$ obeying

$$\rho = \mathrm{tr}' \, |\psi\rangle \langle\psi| \, .$$

$\mathcal{H}'$ is any auxiliary Hilbert space.

There are many possible purifications for a given ρ.

Parallel purifications

We measure a system to be initially ρ_1, and then subsequently ρ_2.

We assume $\rho_1, \rho_2 : \mathcal{H} \to \mathcal{H}$ arise as reduced density matrices of $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H} \otimes \mathcal{H}'$.

Transition probability:

$$|\langle\psi_2|\psi_1\rangle|^2$$

Uhlmann's idea: assume this probability is maximised. In a classical regime, this is a good approximation.

Purifications that maximise the transition probability are called 'parallel'.

Parallel purifications

Theorem (Uhlmann, 1976)

Purifications $|\psi_1\rangle, |\psi_2\rangle$ of ρ_1, ρ_2 are parallel if and only if

$$\mathrm{tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right) = |\langle\psi_2|\psi_1\rangle|.$$

(For $\mathcal{O} \geq 0$, $\sqrt{\mathcal{O}}$ is defined spectrally, i.e. if $\mathcal{O}$ has eigenstate $|\alpha\rangle$ with eigenvalue α, then $\sqrt{\mathcal{O}}\,|\alpha\rangle = \sqrt{\alpha}\,|\alpha\rangle$.)

$\mathrm{tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right)$ is called the 'fidelity'.

Parallel purifications

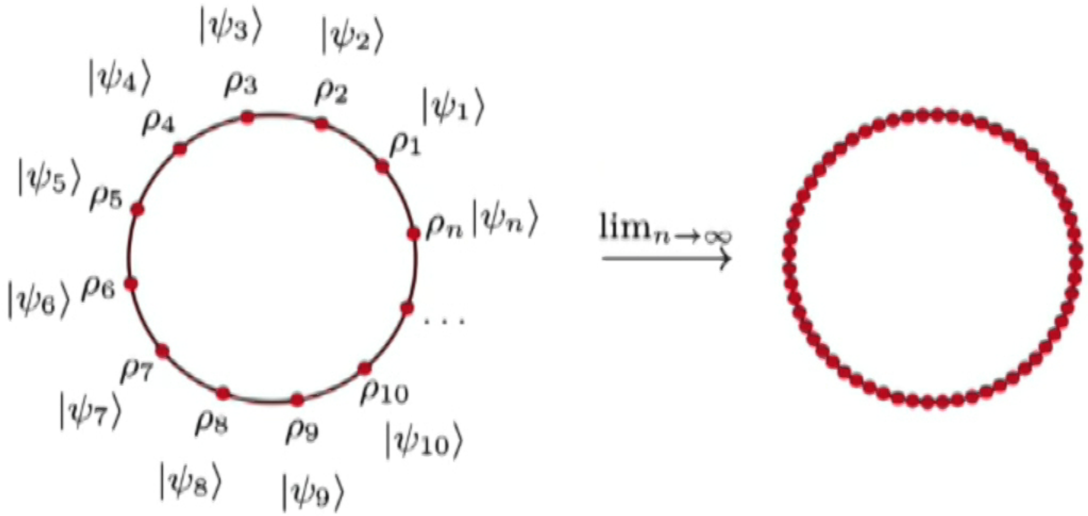
Parallel purifications are not unique:

If $|\psi_1\rangle, |\psi_2\rangle$ are parallel purifications of ρ_1, ρ_2, then so are

$$e^{if_1}(I \otimes U)\,|\psi_1\rangle\,, \quad e^{if_2}(I \otimes U)\,|\psi_2\rangle\,,$$

where $f_1, f_2 \in \mathbb{R}$, and $U^\dagger U = I$.

Parallel lift

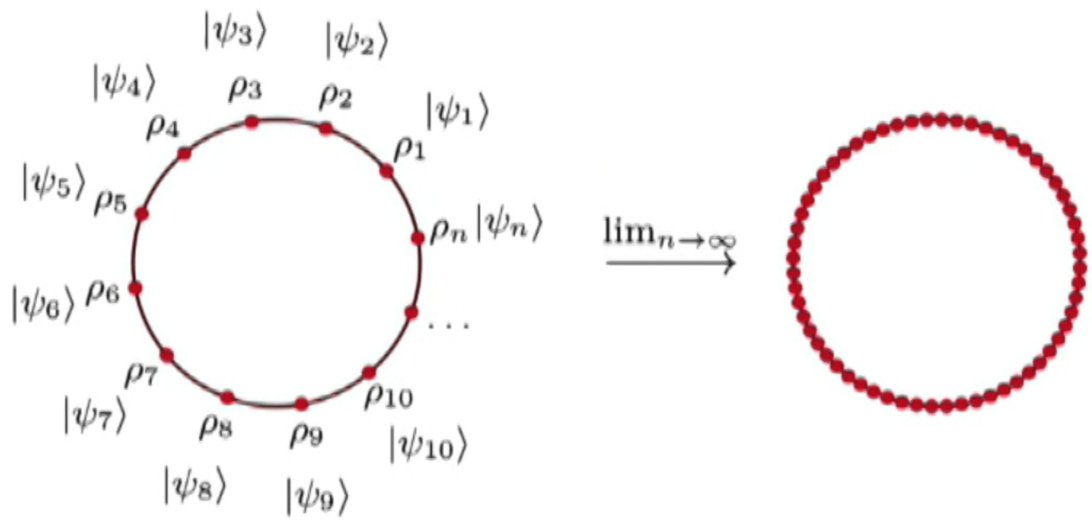


Consider a closed curve C of density matrices. Choose density matrices ρ_i ordered along curve.

For each ρ_i, pick a purification $|\psi_i\rangle$ such that $|\langle\psi_{i+1}|\psi_i\rangle|^2$ is maximal.

As $n \to \infty$, the states $|\psi_i\rangle$ converge to a curve $\bar{C}$ in $\mathcal{H}$.

Parallel lift



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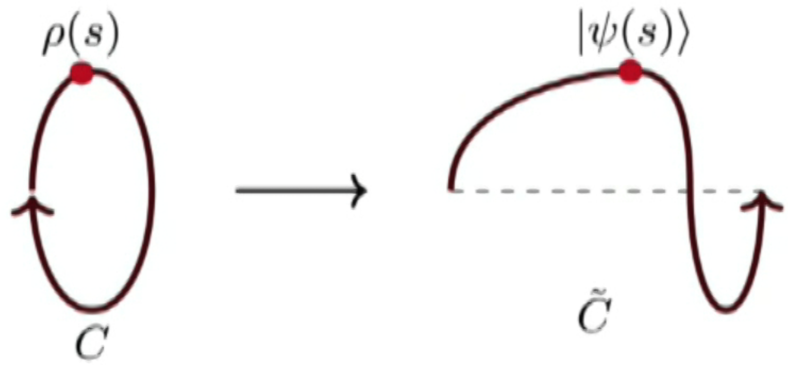
$\tilde{C}$ is a 'parallel lift' of C. Not unique.

Uhlmann holonomy



C is closed, but $\tilde{C}$ doesn't need to be.

$$|\psi(1)\rangle = (I \otimes X)\,|\psi(0)\rangle\,, \quad X^\dagger X = I.$$

Can think of this construction as a function mapping closed curves C to the unitary operator X.

This is an example of holonomy/parallel transport.

Uhlmann phase

Suppose a curve $C : s \mapsto \rho(s)$ has parallel lift $\tilde{C} : s \mapsto |\psi(s)\rangle$.

Consider Berry phase along $\tilde{C}$:

$$\langle\psi_1|\psi_n\rangle \langle\psi_n|\psi_{n-1}\rangle \dots \langle\psi_3|\psi_2\rangle \langle\psi_2|\psi_1\rangle \longrightarrow \exp(i\gamma)$$

RHS does not depend on the choice of parallel lift.

γ is the 'Uhlmann phase' of C. For a curve of pure states

$$\rho(s) = |\psi(s)\rangle \langle\psi(s)| \,,$$

it reduces to Berry phase.

Holographic Uhlmann holonomy

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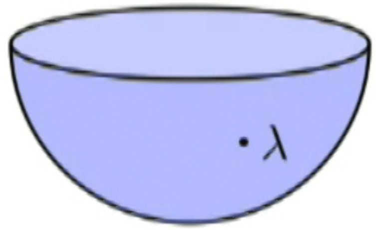
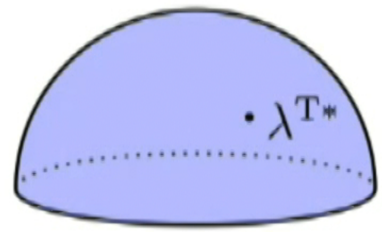
Holographic states



$|\lambda\rangle$



$\langle\lambda|$

Euclidean path integral on half-sphere gives CFT vacuum $|0\rangle$.

$$|\lambda\rangle = \mathrm{T}\exp\Big(-\int_{\tau<0} \mathrm{d}\tau\, \mathrm{d}^{d-1}x\, \lambda(\tau,x)\cdot\mathcal{O}(\tau,x)\Big)\,|0\rangle\,.$$

λ sources operators $\mathcal{O}$.

Dual states obtained by time reflection and complex conjugation:

$$\langle\lambda| = \langle 0|\,\mathrm{T}\exp\Big(-\int_{\tau>0} \mathrm{d}\tau\, \mathrm{d}^{d-1}x\, \lambda^{*}(-\tau,x)\cdot\mathcal{O}^{\dagger}(\tau,x)\Big).$$

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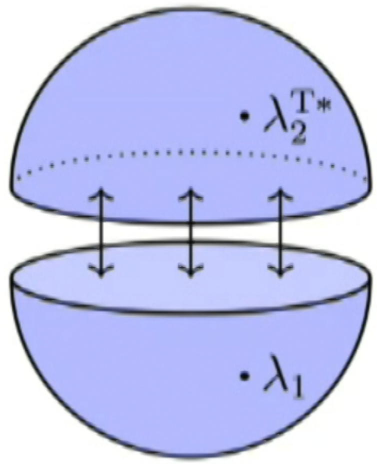
Holographic states

To get inner product:



$$\langle\lambda_2|\;|\lambda_1\rangle$$

Holographic states

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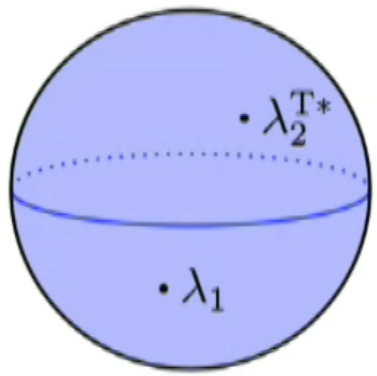


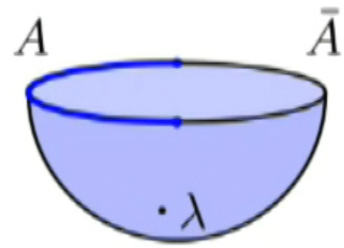
$$\langle\lambda_2|\lambda_1\rangle$$

In a large N limit, the holographic dictionary $\Longrightarrow$

$$\langle\lambda_2|\lambda_1\rangle = e^{-S[\phi]},$$

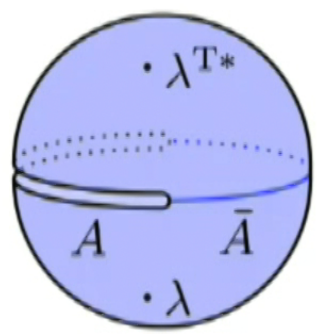
where $S[\phi]$ is the bulk action for the on-shell bulk field configuration ϕ matching the boundary conditions set by $\lambda_1, \lambda_2^{\mathrm{T}*}$.

Subregion state



Decompose boundary Hilbert space as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. State in A for sources λ:

$$\rho(\lambda) = \mathrm{tr}_{\bar{A}} \frac{|\lambda\rangle \langle\lambda|}{\langle\lambda|\lambda\rangle}$$



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Fidelity from a replica trick

Consider two such states $\rho_1 = \rho(\lambda_1)$, $\rho_2 = \rho(\lambda_2)$.

Define for any $k \in \mathbb{C}$ the 'replicated fidelity'

$$F_k = \mathrm{tr}\Big((\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^k\Big).$$

Normal fidelity is $F_{\frac{1}{2}} = \mathrm{tr}\Big(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\Big)$.

Fidelity from a replica trick

Can compute

$$\mathrm{tr}\Big((\rho_1\rho_2)^k\Big) = \mathrm{tr}\Big(\underbrace{\rho_1\rho_2\rho_1\rho_2\ldots\rho_1\rho_2}_{2k}\Big)$$

with a path integral.

Fidelity from a replica trick

Carlson's theorem $\implies F_k$ uniquely determined in the range $\mathrm{Re}\, k \geq \frac{1}{2}$ by the values it takes on the positive integers $k \in \mathbb{Z}_+$.

Strategy to find fidelity: compute F_k for $k \in \mathbb{Z}_+$, and then analytically continue back to $k = \frac{1}{2}$. This is a form of replica trick.

$k \in \mathbb{Z}_+$ is easier to compute than other values of k, because

$$F_k = \mathrm{tr}\Big((\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^k\Big) = \mathrm{tr}\Big((\rho_1\rho_2)^k\Big)$$

has no square roots.

Fidelity from a replica trick

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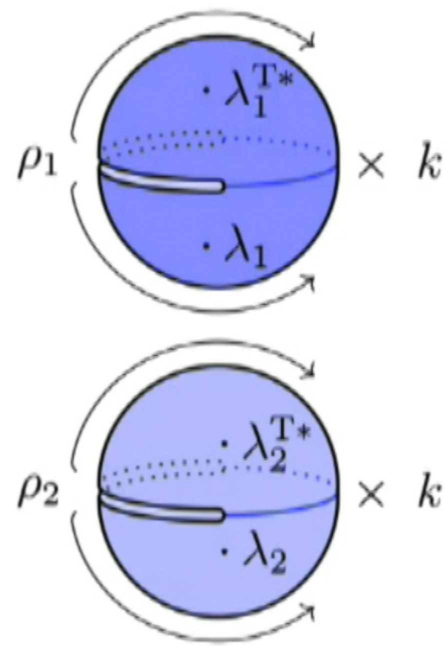

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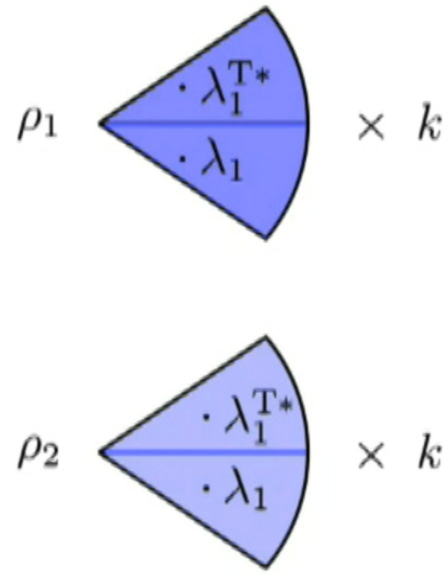
Composition of density matrices ≈ gluing each spacetime to the next.

Taking the trace ≈ gluing the last spacetime to the first.



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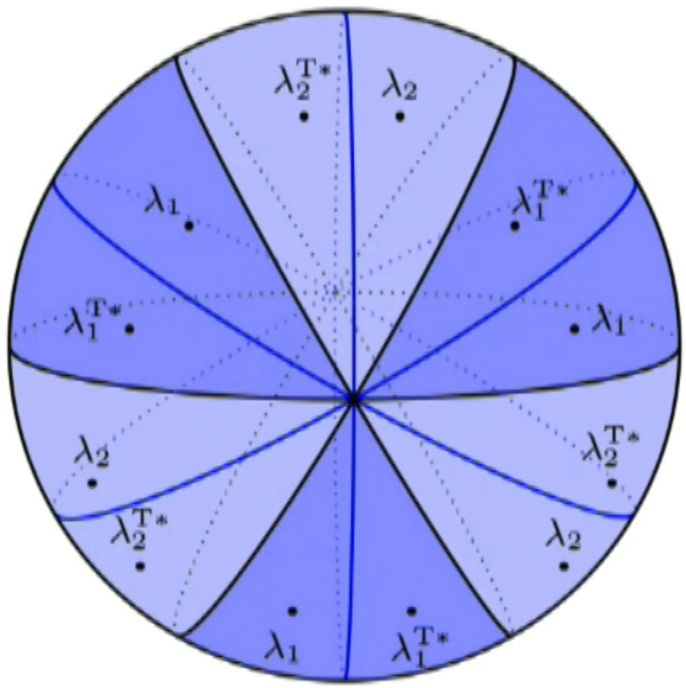
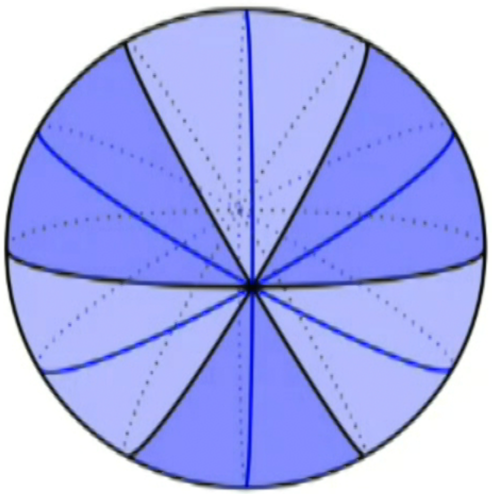
Fidelity from a replica trick

$$\mathrm{tr}\left(\rho_1\rho_2\rho_1\rho_2\rho_1\rho_2\right)$$



Fidelity from a replica trick

At large N: this manifold gives the boundary conditions for a bulk on-shell action.



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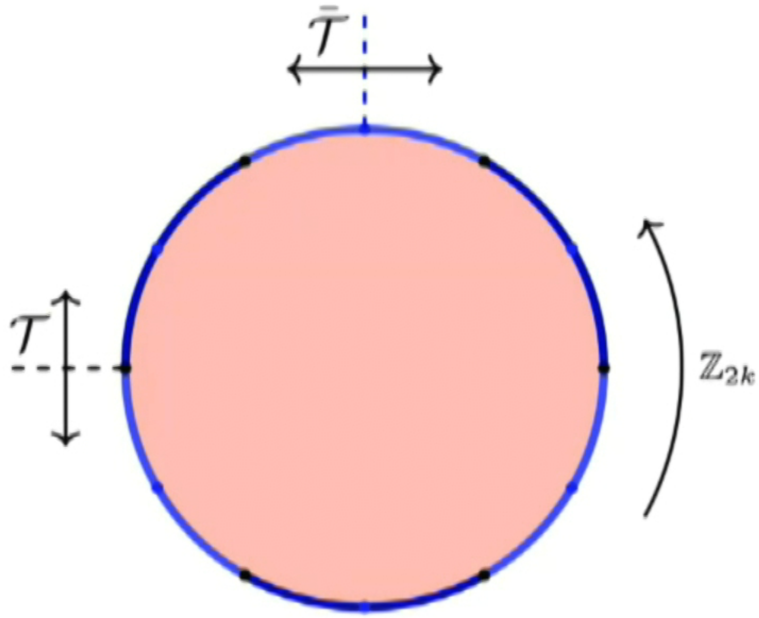

Symmetries: $\bar{\mathcal{T}}$, $\mathbb{Z}_k$

When $\lambda_1 \approx \lambda_2$: $\mathcal{T}$, $\mathbb{Z}_{2k}$

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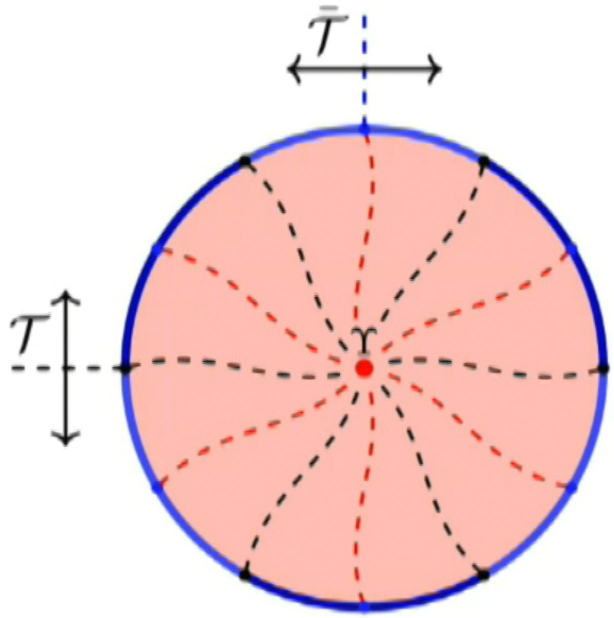

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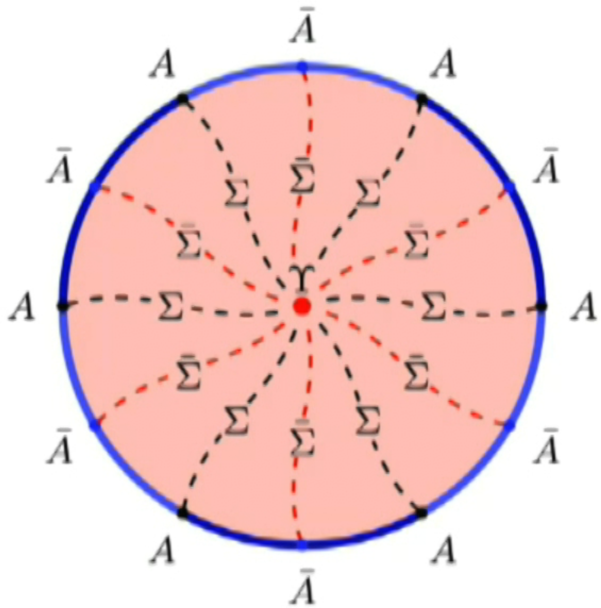

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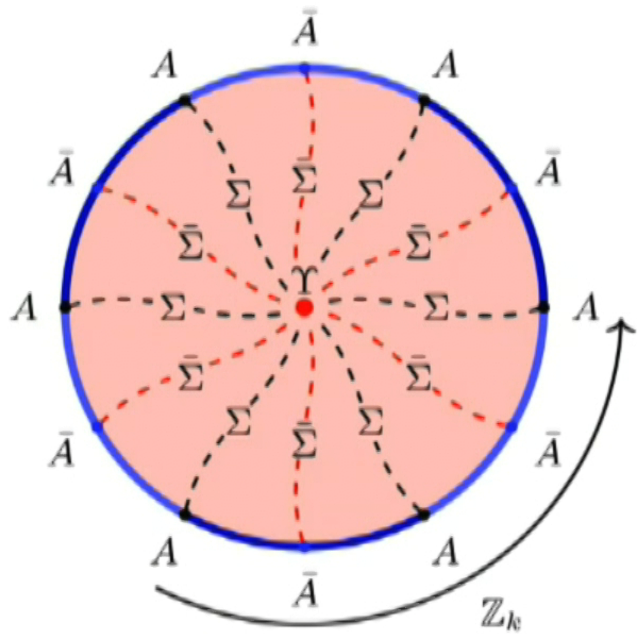
Using $\bar{\mathcal{T}}$ and $\mathbb{Z}_k$ symmetries, only need consider action on a part of this manifold.



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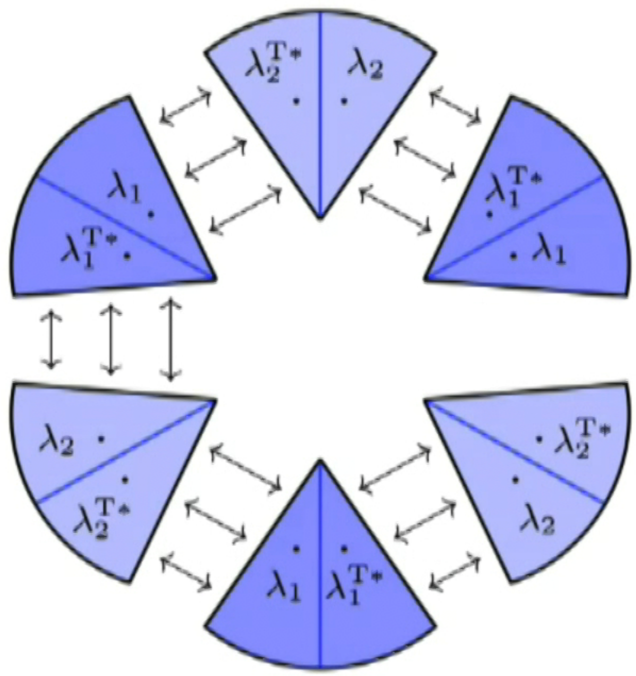
$\rho_1\rho_2\rho_1\rho_2\rho_1\rho_2$



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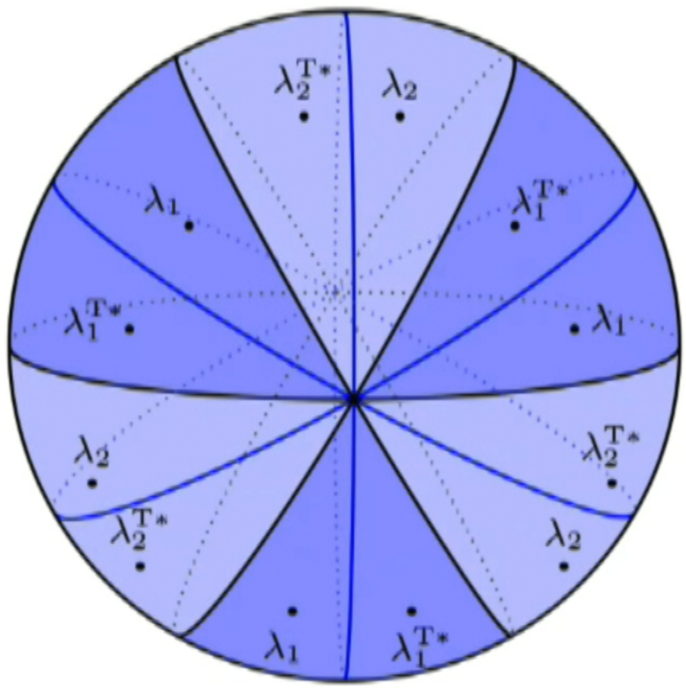

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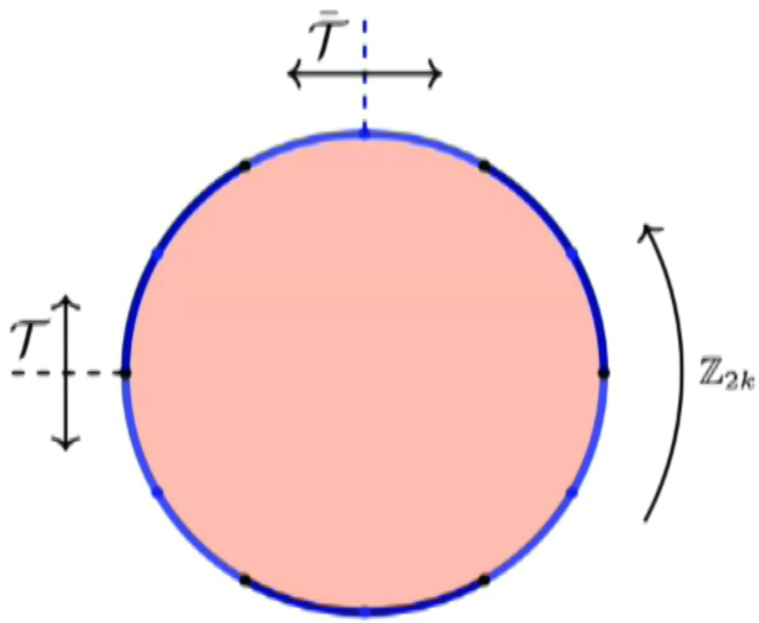

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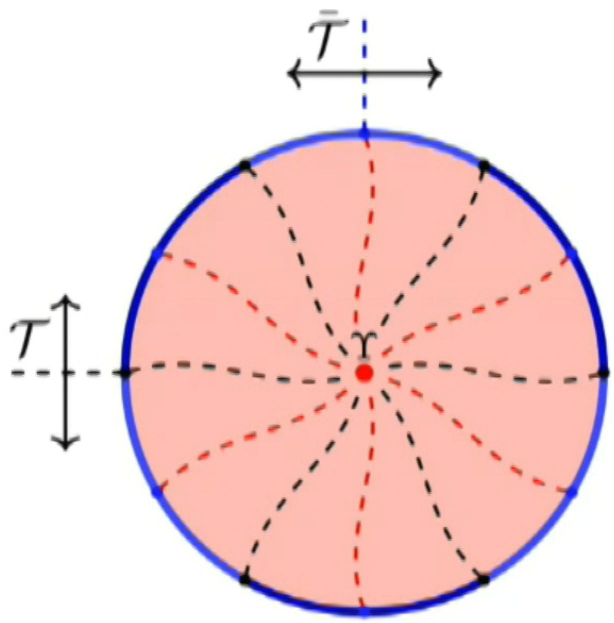

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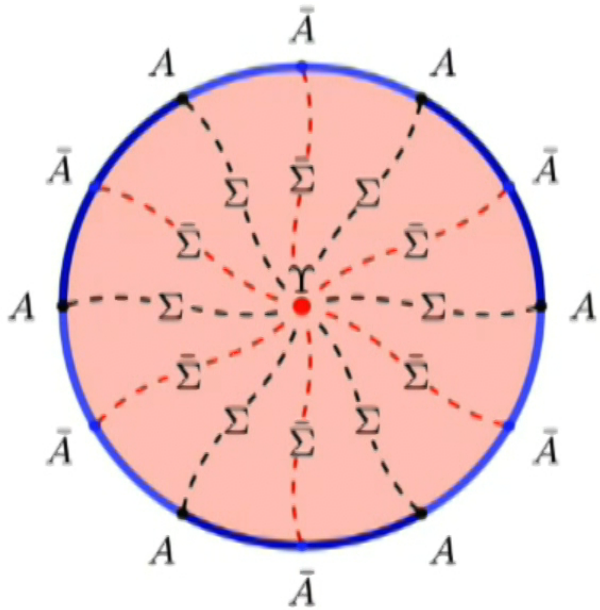

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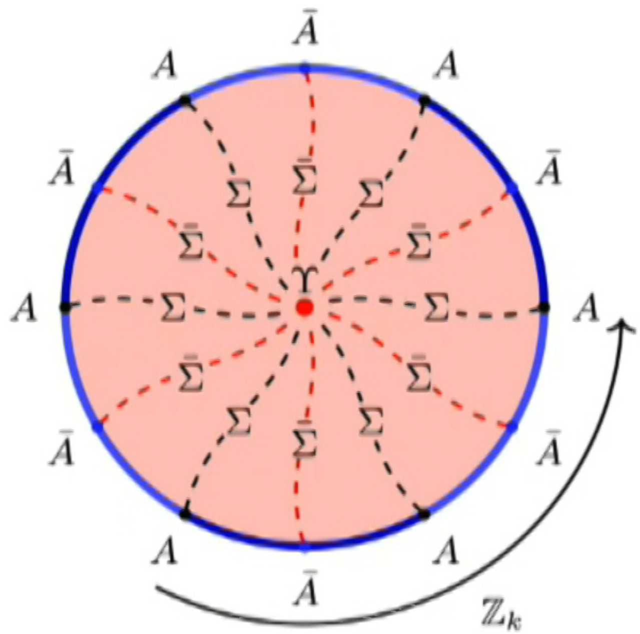

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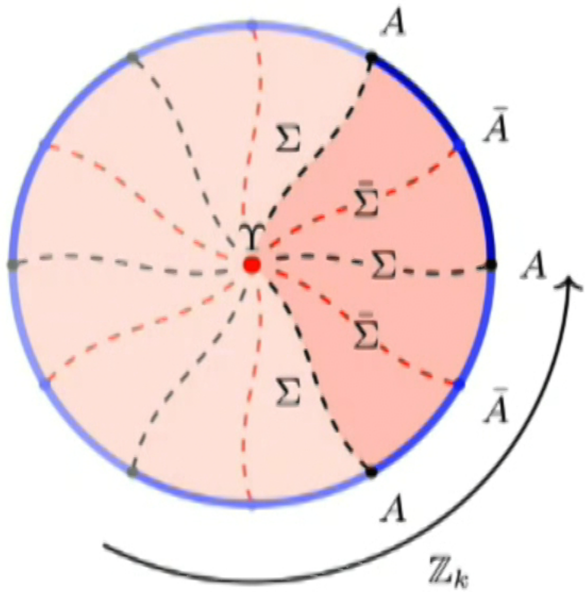

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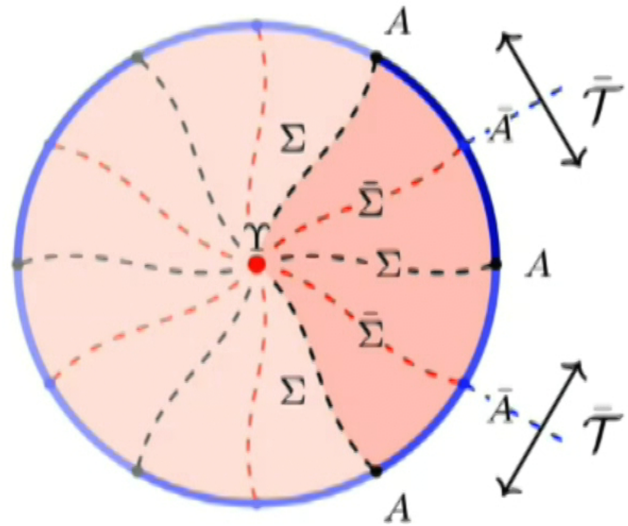

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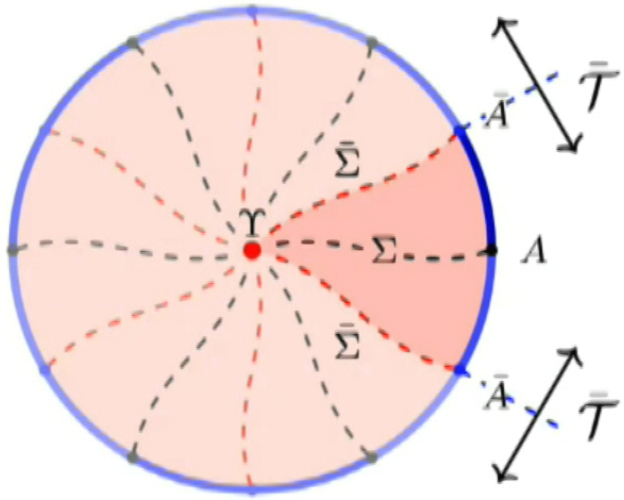
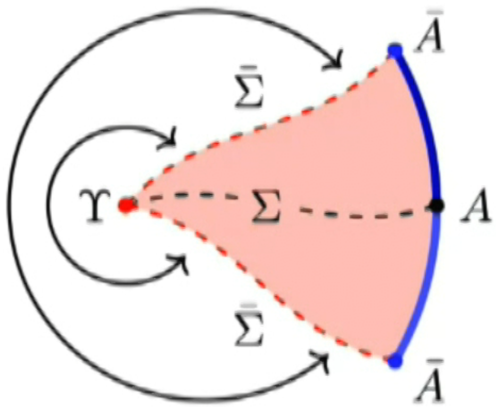


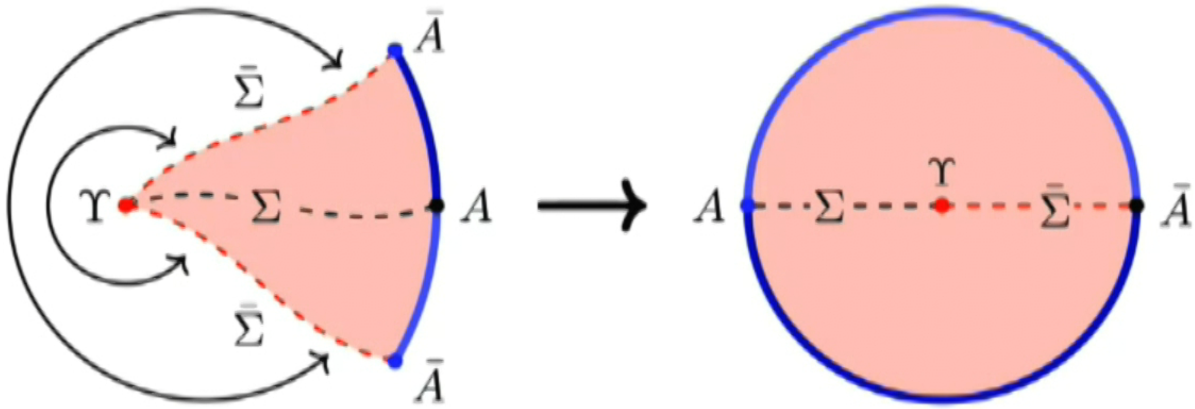
$$S_{\text{total}} = 2k\,\text{Re}\left(S_{\text{part}}\right)$$

Fidelity from a replica trick



Identify the two $\bar{\Sigma}$s.

Fidelity from a replica trick

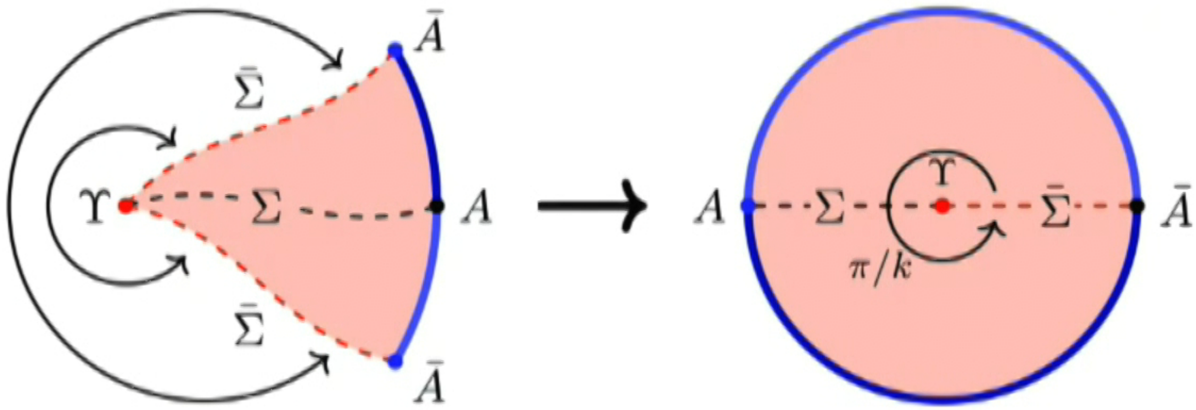


Identify the two $\bar{\Sigma}$s.

Resulting manifold:

- Fields discontinuous at $\bar{\Sigma}$.

Fidelity from a replica trick



Identify the two $\bar{\Sigma}$s.

Resulting manifold:

- Fields discontinuous at $\bar{\Sigma}$.
- Conical defect at Υ of opening angle π/k.

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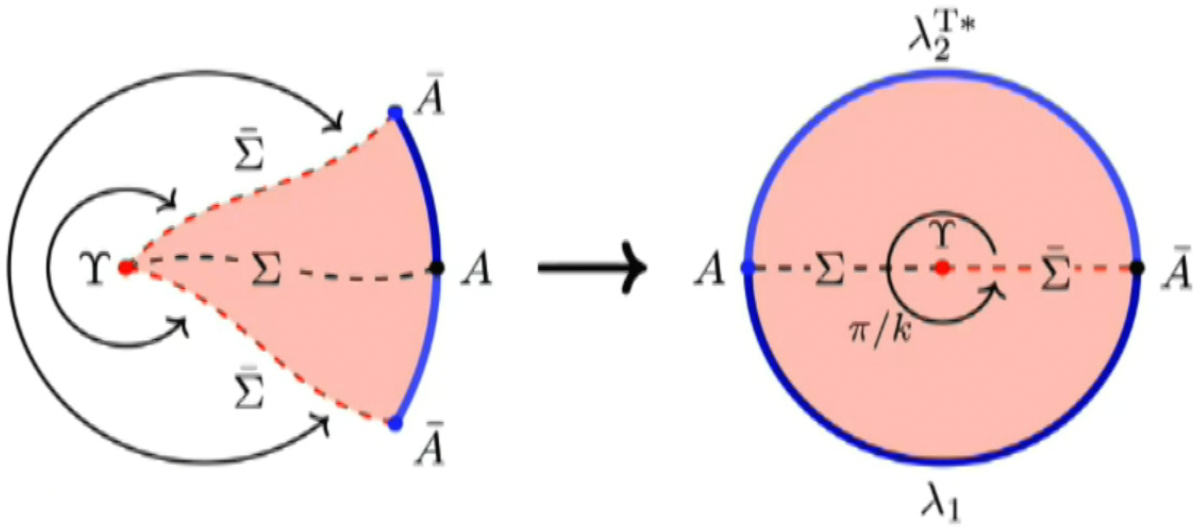


Identify the two $\bar{\Sigma}$s.

Resulting manifold:

- Fields discontinuous at $\bar{\Sigma}$.
- Conical defect at Υ of opening angle π/k.
- Boundary conditions $\lambda_1, \lambda_2^{T*}$.

Fidelity from a replica trick

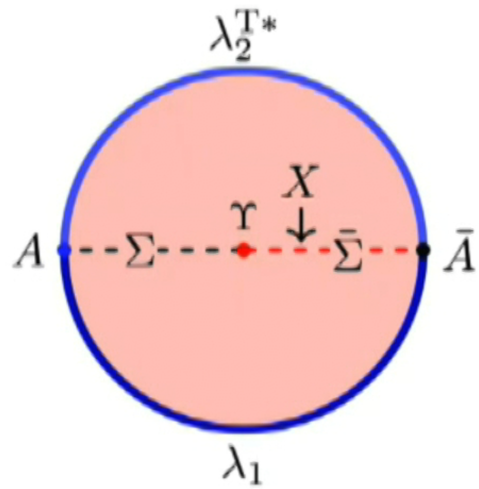
Now analytically continue $k \to \frac{1}{2}$.

Opening angle at Υ becomes 2π, so no more conical defect.

Location of Υ still important. It becomes minimal surface homologous to A [Maldacena-Lewkowycz, 2013].

Field discontinuities at $\bar{\Sigma}$ are constrained by approximate $\mathcal{T}$ symmetry. This can be understood as $k \to \frac{1}{2}$.

End up with a fidelity in terms of bulk action on:



Note that boundary conditions are the same as for $\langle \lambda_2 | \lambda_1 \rangle$.

Only difference is discontinuity at $\bar{\Sigma}$.

Can think of discontinuity as arising from operator insertion:

$$\langle \lambda_2 | X | \lambda_1 \rangle$$

where X is an operator acting at $\bar{\Sigma}$.

It is possible to construct X using bulk perturbation theory.

Symmetries of replica path integral ensure $X^\dagger X = I$.

X acts at $\bar{\Sigma}$. Bulk reconstruction [Dong-Harlow-Wall] $\implies$

$$X = I_A \otimes X_{\bar{A}},$$

where I_A is identity on $\mathcal{H}_A$, and $X_{\bar{A}}$ is unitary acting on $\mathcal{H}_{\bar{A}}$.

Define

$$|\psi_1\rangle = \frac{|\lambda_1\rangle}{\sqrt{\langle\lambda_1|\lambda_1\rangle}}, \quad |\psi_2\rangle = \frac{X^\dagger |\lambda_2\rangle}{\sqrt{\langle\lambda_2|\lambda_2\rangle}}.$$

$|\psi_1\rangle$ is a purification of ρ_1, and $|\psi_2\rangle$ is a purification of ρ_2.

$\langle\psi_2|\psi_1\rangle$ contains $\langle\lambda_2|X|\lambda_1\rangle$, so we can relate it to the fidelity.

Holographic parallel purifications

The precise relationship between the two bulk actions then translates to:

$$\mathrm{tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right) = |\langle\psi_2|\psi_1\rangle|.$$

So Uhlmann's theorem $\implies$ $|\psi_1\rangle$, $|\psi_2\rangle$ are parallel.

Thus we have constructed parallel purifications of the holographic states ρ_1, ρ_2.

Holographic parallel lift

Consider closed curve $\rho(s) = \rho(\lambda(s))$ given by reducing $|\lambda(s)\rangle$, where $\lambda(s)$ is a curve of boundary conditions.

$$\rho(s) = \mathrm{tr}_{\bar{A}} \frac{|\lambda(s)\rangle \langle\lambda(s)|}{\langle\lambda(s)|\lambda(s)\rangle}.$$

Pick $\rho_1, \rho_2, \ldots \rho_n$ along curve.

Using the above results, may construct sequence of parallel purifications

$$|\psi_1\rangle = |\lambda_1\rangle \Big/ \sqrt{\langle\lambda_1|\lambda_1\rangle}, \quad |\psi_2\rangle = X_1^\dagger |\lambda_2\rangle \Big/ \sqrt{\langle\lambda_2|\lambda_2\rangle}, \quad \ldots$$

$$|\psi_i\rangle = X_1^\dagger X_2^\dagger \ldots X_{i-1}^\dagger |\lambda_i\rangle \Big/ \sqrt{\langle\lambda_i|\lambda_i\rangle}$$

As $n \to \infty$ these states converge to a parallel lift of $\rho(s)$.

Holographic Uhlmann holonomy

Each X_i inserts operator on $\bar{\Sigma}$, so may be written

$$X_i = I_A \otimes X_{\bar{A},i}.$$

Uhlmann holonomy is given by

$$\lim_{n\to\infty} X_{\bar{A},1} X_{\bar{A},2} \dots X_{\bar{A},n}.$$

This is a unitary operator acting on $\mathcal{H}_{\bar{A}}$.

In principle the action of this operator contains a large amount of information about the entanglement between A and $\bar{A}$.

Holographic Uhlmann phase

Uhlmann phase given by

$$\langle\psi_1|\psi_n\rangle\ \langle\psi_n|\psi_{n-1}\rangle \ldots \langle\psi_3|\psi_2\rangle\ \langle\psi_2|\psi_1\rangle \longrightarrow \exp(i\gamma)$$

Not hard to compute this by writing correlators $\langle\lambda_{i+1}|X_i|\lambda_i\rangle$ in terms of bulk action.

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Not hard to compute this by writing correlators $\langle\lambda_{i+1}|X_i|\lambda_i\rangle$ in terms of bulk action.

One must be careful about the final correlator $\langle\psi_1|\psi_n\rangle$, because this involves

$$\langle\lambda_1|X_1^\dagger X_2^\dagger \ldots X_{n-1}^\dagger|\lambda_n\rangle\,.$$

But this can be done again in terms of bulk action.

Holographic Uhlmann phase

One finds:

$$\gamma = i \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\Sigma} \theta[\phi_i, \delta\phi_i]$$

where:

- ϕ_i = dominant bulk field configuration for λ_i.
- $\delta\phi_i = \phi_{i+1} - \phi_i$.
- $\delta L[\phi] = E[\phi] \cdot \delta\phi + \mathrm{d}(\theta[\phi, \delta\phi])$.

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The integral is only done over Σ because of some auspicious cancellations (related to discontinuities at $\bar{\Sigma}$).

Holographic Uhlmann phase

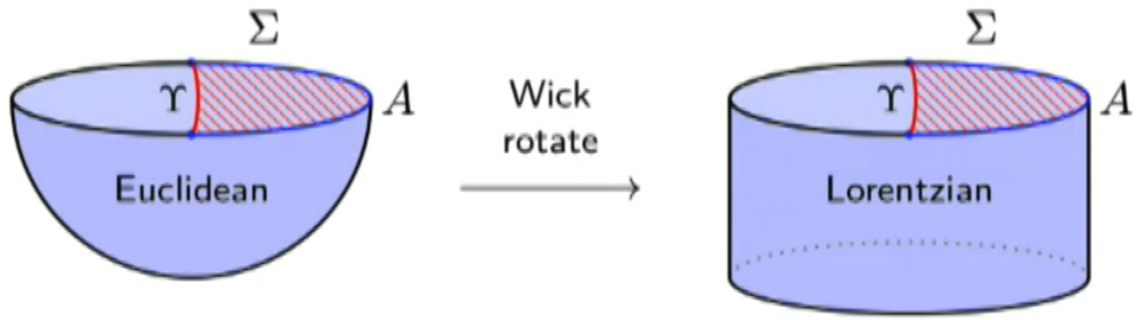
For $n \to \infty$, we may replace the sum by an integral:

$$\gamma = \int_C i\Theta, \text{ where } \Theta = \int_\Sigma \theta.$$

C is a curve $\phi(s)$ in field space, determined by the boundary conditions $\lambda(s)$.

$\Theta[\phi, \delta\phi]$ is a 1-form on field space, because $\delta\phi$ is a field space vector.

The Uhlmann phase γ is the integral of the connection $a = i\Theta$.



$$a = i\Theta[\phi, \delta\phi] = i\int_\Sigma \theta[\phi, \delta\phi]$$

This is a Euclidean expression.

To get a in terms of Lorentzian fields, Wick rotate $\tau \to -it$.

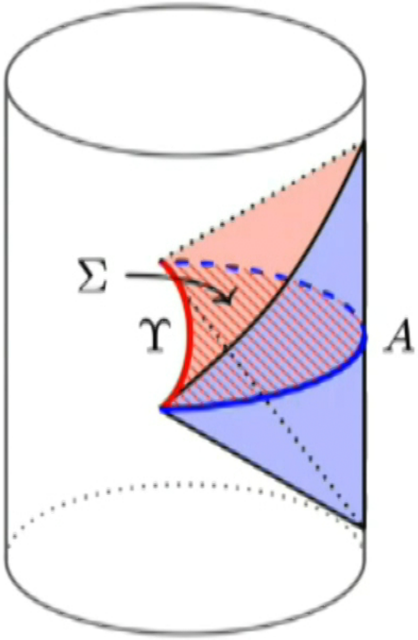
$$a = \Theta[\phi_{\text{Lor}}, \delta\phi_{\text{Lor}}] = \int_\Sigma \theta[\phi_{\text{Lor}}, \delta\phi_{\text{Lor}}]$$

ϕ_{Lor} are the Wick-rotated fields. Factor of i goes away because there are an odd number of time derivatives.

In Lorentzian spacetime, domain of dependence of Σ is the entanglement wedge.

Curvature of $a = \Theta$ is equal to exterior derivative of Θ.

Covariant phase space formalism $\implies$ exterior derivative of Θ is Ω: the symplectic form of fields in the entanglement wedge.



Theorem

The curvature of the Uhlmann phase is holographically dual to the symplectic form of the entanglement wedge.

Comments

When A contains the entire boundary:

- Entanglement wedge contains all of bulk.
- Uhlmann phase reduces to Berry phase.

So we recover the result of [Belin-Lewkowycz-Sárosi]: Berry curvature is dual to bulk symplectic form.

The result presented here is thus a generalisation of theirs.

Comments

This result provides a previously absent context to the meaning of emergent classical bulk subregion physics:

Classical bulk subregion physics emerges in measurements of the Uhlmann phase.

Uhlmann phase is a genuine observable [Åberg-Kult-Sjöqvist-Oi (2007), Viyuela-Rivas-Gasparinetti-Wallraff-Filipp-Martin-Delgado (2016)].

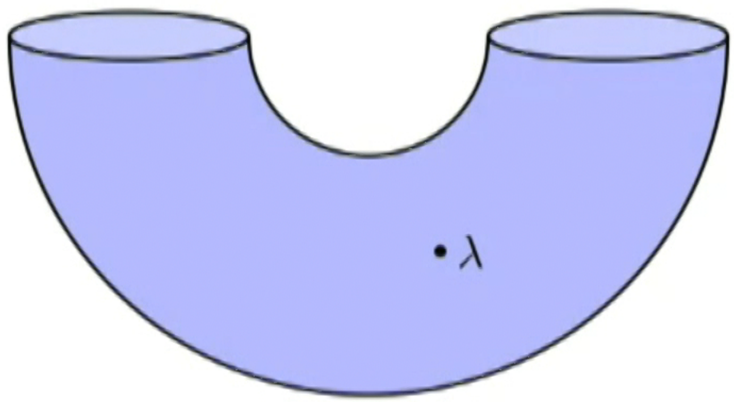
Would be useful to understand more details of this context.

Comments

We considered states built on the vacuum $|0\rangle$.

$$|\lambda\rangle = \mathrm{T}\exp\Big(-\int_{\tau<0} \mathrm{d}\tau\, \mathrm{d}^{d-1}x\, \lambda(\tau,x)\cdot\mathcal{O}(\tau,x)\Big)\,|0\rangle\,.$$

But we can pick more general $|0\rangle$, and the same argument works. For example, thermofield double in two copies of the CFT.



Path integral on $S^{d-1}\times[0,\beta/2]$. Corresponds to a black hole in the bulk.

Comments

There is an ambiguity in the covariant phase space formalism.

$\delta L = E \cdot \delta\phi + \mathrm{d}\theta$ only defines θ up to

$$\theta[\phi, \delta\phi] \to \theta[\phi, \delta\phi] + \mathrm{d}(K[\phi, \delta\phi]) \,.$$

i.e.

$$\Theta = \int_\Sigma \theta \to \int_\Sigma \theta + \int_{\partial\Sigma} K.$$

However, Uhlmann phase is unambiguous. Therefore, this result provides a resolution to the ambiguity.

Comments

We should expect pure states to give a classical phase space in the classical limit.

But it is not so clear that there should be a classical phase space for *mixed* states.

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Clearly there is one for the holographic states considered here. Why?

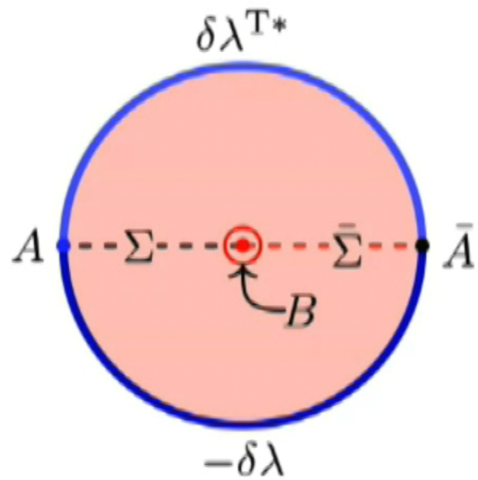
Can run this backwards: quantise the phase space. Pure states in subregions?

Other future directions

- This result is for Einstein gravity (+ other fields), at leading order in $1/N$. Would like to understand both higher derivative corrections, and quantum/statistical corrections.
- [Belin-Lewkowycz-Sárosi] used their result to explore holographic complexity. Would be useful to extend their methods to subregion complexity.
- Holographic renormalisation.
- Connections with modular theory.
- Would like to apply these ideas in combination with recent work on replica wormholes and black hole information paradox.
- Topological phases of holographic condensed matter.

Thank you!

Resolution of the covariant phase space ambiguity



Consider field variation $\tilde{\delta}\phi$:

- Obeys variation of boundary conditions $-\delta\lambda, \delta\lambda^{T*}$
- $\tilde{\delta}\phi$ changes sign when crossing $\bar{\Sigma}$.

Let B be a surface tightly enclosing Υ. We enforce

$$\int_B \theta(\phi, \tilde{\delta}\phi) = 0.$$

This can be done by appropriate redefinition $\theta \to \theta + dK$.

One can show that this is consistent with the Uhlmann phase, and that it resolves the ambiguity.

(extra slides) 53 / 52