Title: Indefinite causal order without post-selection

Speakers: Katja Ried

Collection: Indefinite Causal Structure

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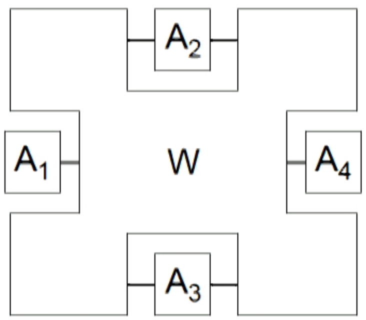
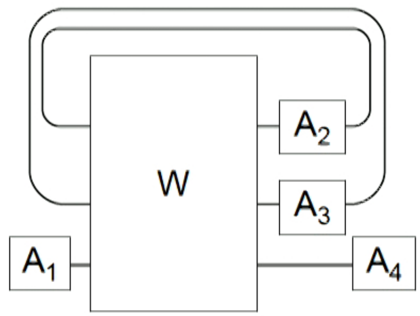
Abstract: The possibility of indefinite causal order has garnered considerable interest in recent years, both for its promise as a resource, e.g. for communication, and for its role in exploring the fundamental physical constraints on causal structure. In order to gain a better understanding of the phenomenon, one approach is to design experiments that implement â€“ or at least simulate â€“ scenarios with indefinite causal order. While post-selection is one way to simulate exotic causal structures, this approach may not provide the desired insights. Instead, I will discuss how one might go about implementing indefinite causal order without post-selection.

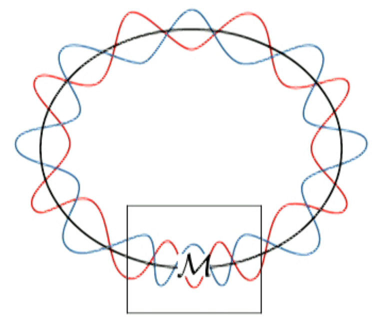
# Indefinite Causal Structure without Post-Selection

Katja Ried

University of Innsbruck



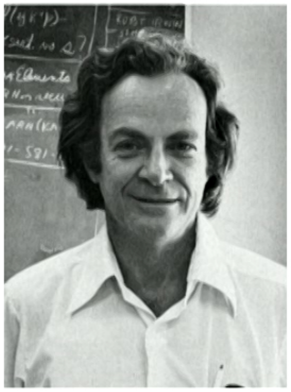




CAUTION

# Motivation and Outline

**Motivation**

- practical: indefinite causal relations are an intriguing resource
  - Perfect discrimination of no-signalling channels via quantum superposition of causal structures [PRA 86, 040301 (2012)]
  - Computational Advantage from Quantum-Controlled Ordering of Gates [PRL 113, 250402 (2014)]
  - …
- fundamental: better understanding of possible causal relations
  - Active learning machine learns to create new quantum experiments [PNAS 115, 1221 (2018)]



"What I cannot create, I do not understand."
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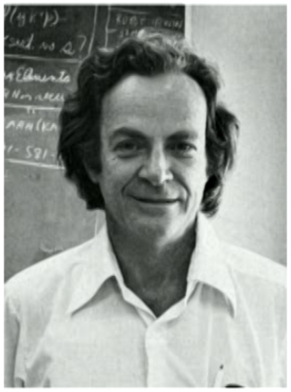

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**Questions**

- Can we implement analogues of processes with indefinite causal structure?
- How to read off consistent states and measurement resulting statistics?
- How to check whether a given map corresponds to a valid process?

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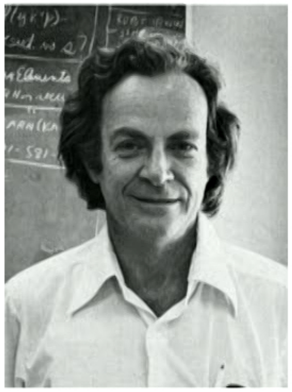

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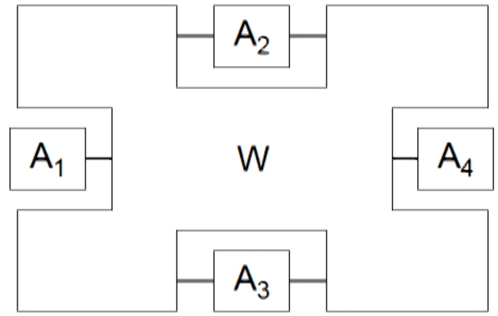
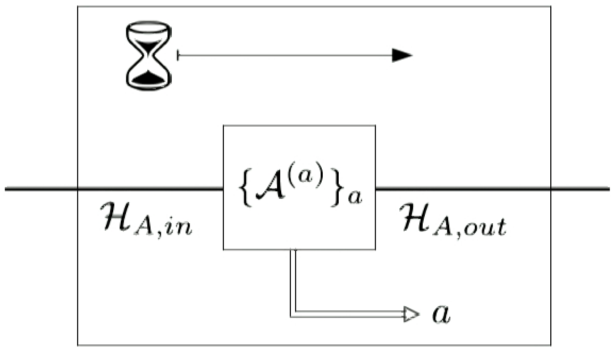
### Outline

- Introduction: process matrices
- Resonators as analogues of causal loops
- Validity conditions for process matrices
- Identifying consistent states

# The framework of process matrices

The process matrix W relates inputs and outputs of laboratories (or parties) $A_i$, without demanding that they satisfy a well-defined global **causal order**.

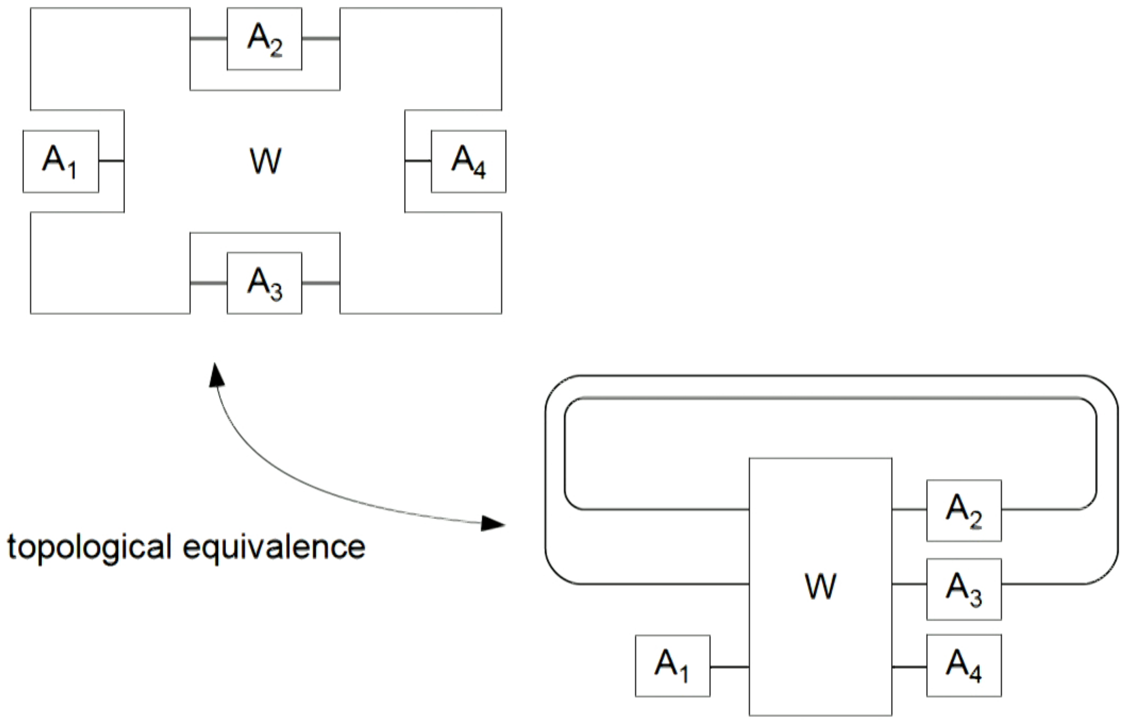



Each **local lab** $A_i$ has

- local causal order
- input and output Hilbert spaces with fixed $d_i$
- a choice of a quantum channel $\mathcal{A}_i$ or, more generally, an instrument $\{\mathcal{A}_i^{(a_i)}\}_{a_i}$ that can yield different outcomes $a_i$, with $\mathcal{A}_i = \sum_{a_i} \mathcal{A}_i^{(a_i)}$

# Alternative representation with loops



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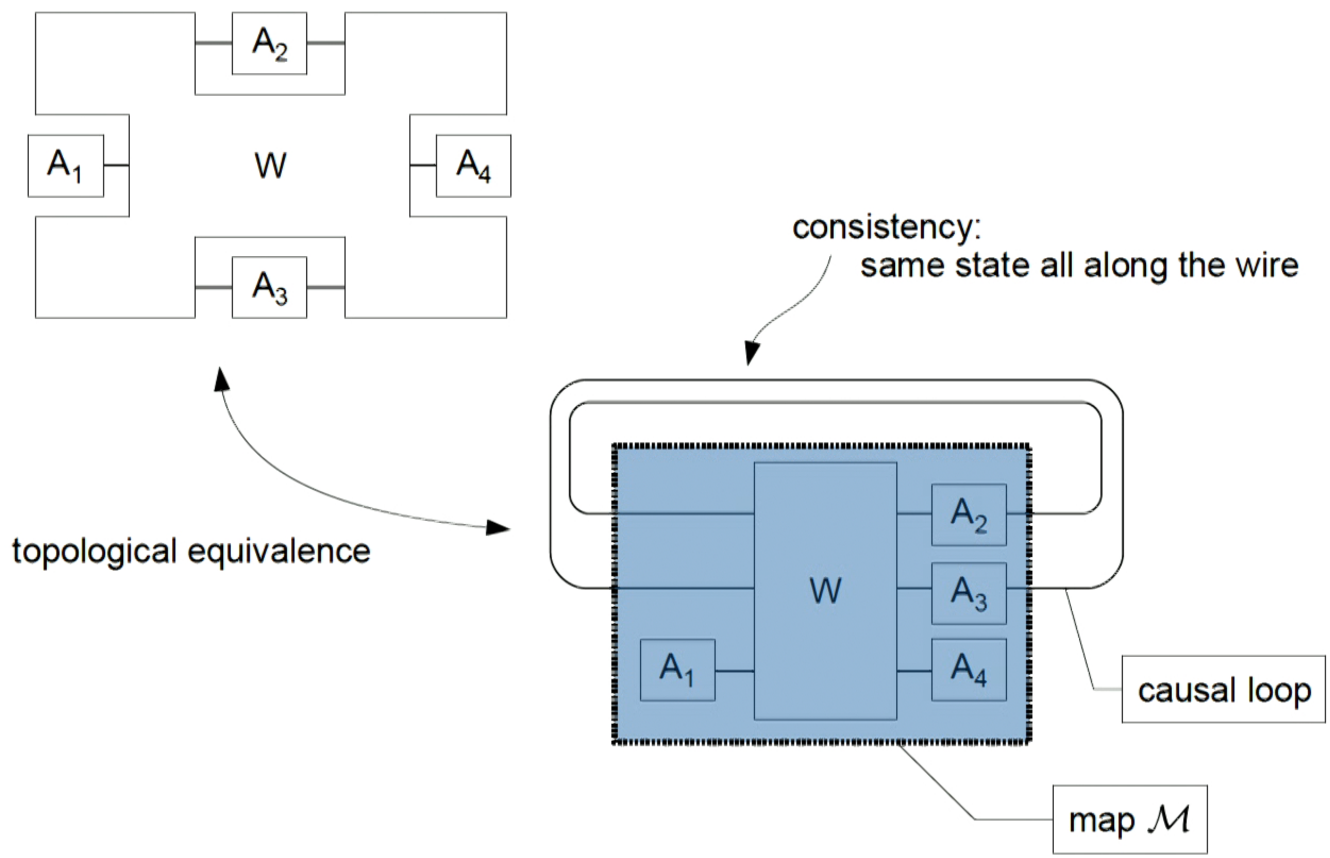
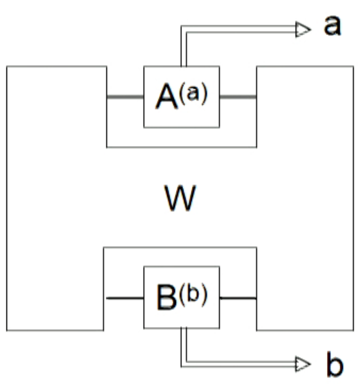


# Validity / Physicality

Formally, the process matrix is a linear supermap from local instruments to outcome probabilities:

$$P(a,b) = Tr\left[W(A^{(a)} \otimes B^{(b)})\right]$$

where $A^{(a)}$denotes the operator that is Choi isomorphic to the element $\mathcal{A}^{(a)}$ of Alice's instrument.



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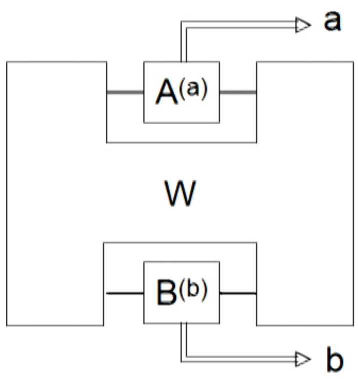



**Physicality condition** on W: the total probability of obtaining any outcomes,

$$P(click|A,B) := \sum_{a,b} P(a,b) = \sum_{a,b} Tr\left[W(A^{(a)} \otimes B^{(b)})\right] = Tr\left[W(A \otimes B)\right]$$

must be unity:

$$P(click|A,B) = \boxed{Tr\left[W(A \otimes B)\right] = 1 \quad \forall A, B}$$

## Parentheses: the Choi-Jamiołkowski isomorphism

… establishes an isomorphism between

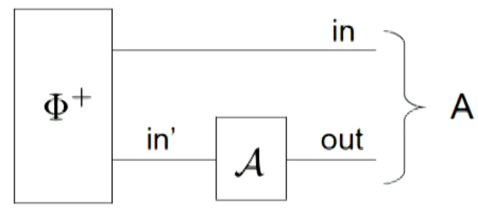
(a) a CPTP map $\mathcal{A} : \mathcal{H}_{in} \rightarrow \mathcal{H}_{out}$

(b) the positive operator

$$A_{in,out} := (I \otimes \mathcal{A})(d \,|\, \phi^+ \rangle \langle \phi^+ \,|)$$

in the sense that

$$\mathcal{A}(\rho) = Tr_{in}[\rho_{in}^T A_{in,out}] \;\forall \rho \in \mathcal{D}(\mathcal{H}_{in})$$




$$|\, \Phi^+ \rangle := \frac{1}{\sqrt{d}} \sum_{n=1}^{d} |\, n, n \,\rangle$$

on some fixed basis $\{|\, n \,\rangle\}$ of $\mathcal{H}_{in}$

Some observations:

- For any Choi operator, the marginal on the input space is the identity operator:

$$Tr_{out} A = \sum |\, n \,\rangle \langle\, n \,|_{in} = \mathbb{I}_{in} \;\Rightarrow\; Tr(A) = d$$

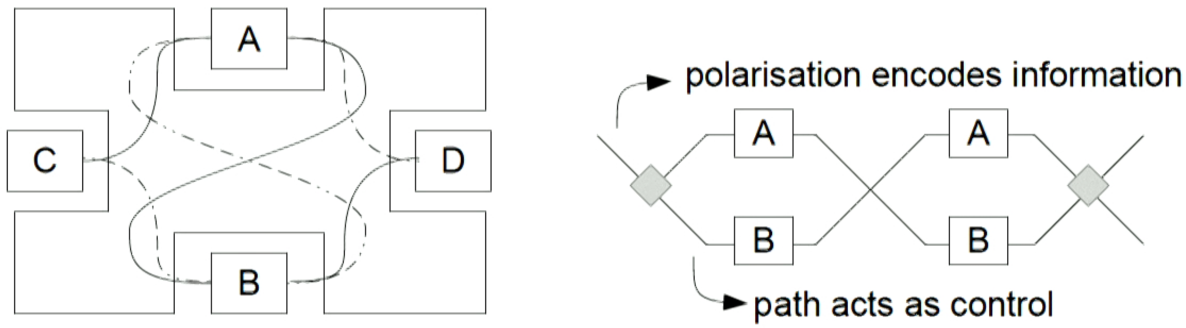
- The identity channel $\mathcal{I}$ is CJ isomorphic to $d \,|\, \Phi^+ \rangle \langle \Phi^+ \,| := I$
- The process matrix operator W is CJ isomorphic to a CPTP map

$$\mathcal{W}(\rho) = Tr_{out}[\rho_{out}^T W_{in,out}] \quad (\mathcal{W} : \otimes_i(\mathcal{H}_{i,out}) \rightarrow \otimes_i(\mathcal{H}_{i,in}))$$

*Note that the subscripts in and out denote inputs and outputs of the local labs.

# Existing experimental implementations of the quantum switch

Quantum switch: an information-carrier from the past visits two labs (and undergoes their instruments) in a superposition of causal orders.

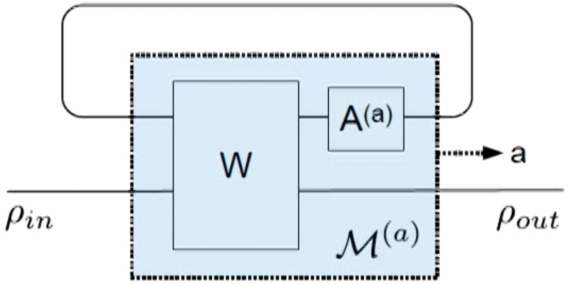



Criticism:
- Conceptual problem of identifying two spacetime events as the same lab
- The quantum switch is causally non-separable, but it (and presumably other processes of a similar nature [Araújo et al]) cannot violate causal inequalities, which are a stronger (device-independent) benchmark of nonclassicality in causal relations.

[1] Rubino et al., Science Advances 3, e1602589 (2017)
[2] Goswami et al., Phys. Rev. Lett. 121, 090503 (2018)
[3] Araújo et al., New J. Phys. 17, 102001 (2015)

## Post-Selection

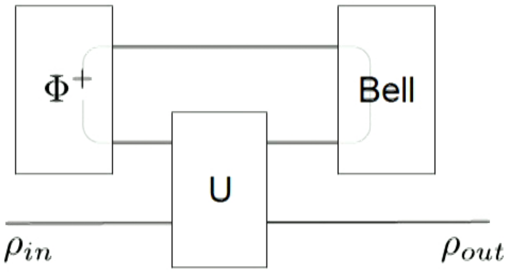
The effect of the process matrix on states ρ or the data seen in local labs can be understood as passing systems interacting with CTCs.




There exist proposals for the implementation of CTCs using teleportation / **post-selection**.

This is problematic because:

- allows one to 'cherry-pick' any result
- muddles causal analysis
- can inflate the efficiency of computational processes



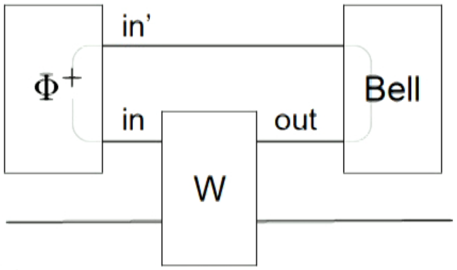

Example (Lloyd): interaction of a passing particle with a CTC succeeds if the subsequent Bell measurement finds $\Phi^+$.

What if $P(\Phi^+) = 0$, e.g. if $U = \mathbb{I} \otimes X$?

$$\rho_{out}^{(\Phi^+)} \cdot \frac{1}{P(\Phi^+)}$$

Examples: Lloyd et al., Phys.Rev.Lett. 106, 040403 (2011); Ringbauer et al., Nat. Comm. 5, 4145 (2014)

## Post-Selection




Additional property of process matrices: for all valid W

$$P(click|\mathcal{A}) = 1 = Tr(WA) \quad \forall A$$

Rewriting the outcome probability of the CTC implementation:

$$P(\Phi^+) = Tr\left[|\ \Phi^+\ \rangle\langle\ \Phi^+\ |_{i'o}(\mathcal{I}\otimes\mathcal{W})(|\ \Phi^+\ \rangle\langle\ \Phi^+\ |_{i'i})\right]$$
$$= Tr\left[\frac{1}{d}I_{i'o}\frac{1}{d}W_{i'o})\right] = \frac{1}{d^2}P(click|\mathcal{I}) = \frac{1}{d^2}$$

Conclusion

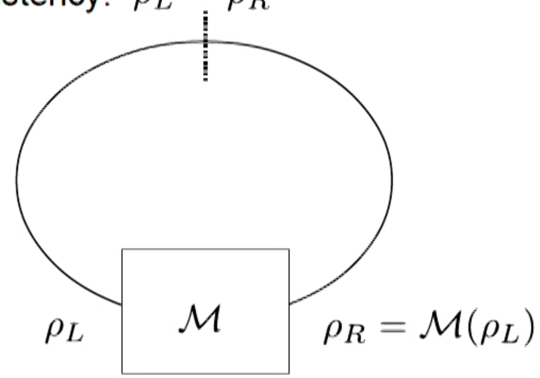
For any valid process matrix, $P(\Phi^+) = \Pi_{parties} d_i^2$

That is, the implementation succeeds with non-zero probability, but decreasing exponentially with the number of parties.

# Basic construction of resonators

Version of process matrix
with a loop in time

consistency: $\rho_L = \rho_R$



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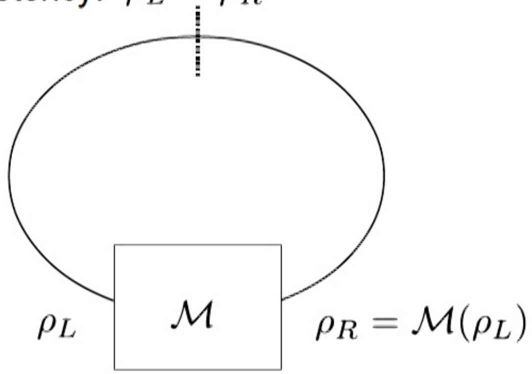
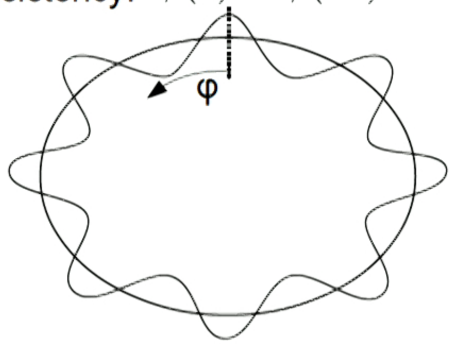



Stationary state of particle on a ring in space

consistency: $\psi(0) = \psi(2\pi)$



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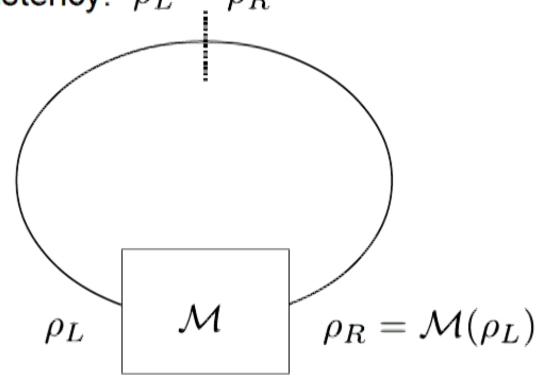


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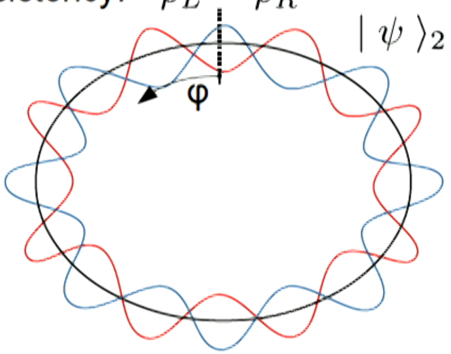
$$|\,\psi\,\rangle_2 = \begin{pmatrix} \psi_r \\ \psi_b \end{pmatrix}$$




Modifications to spatial ring:

1. two modes, with Ĥ such that the total probability does not change with φ

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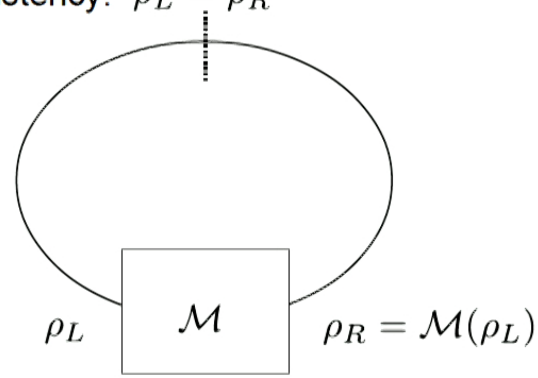


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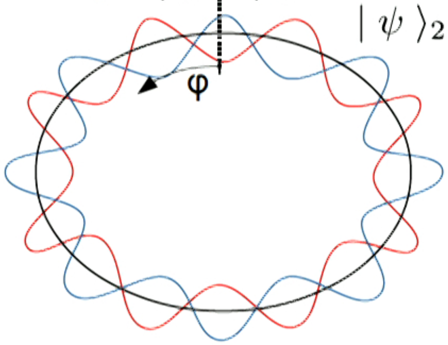


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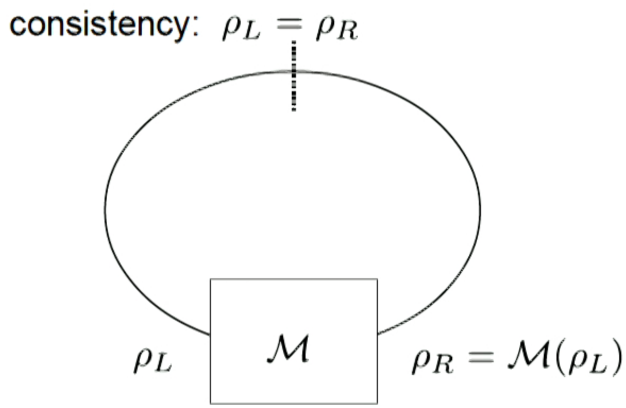
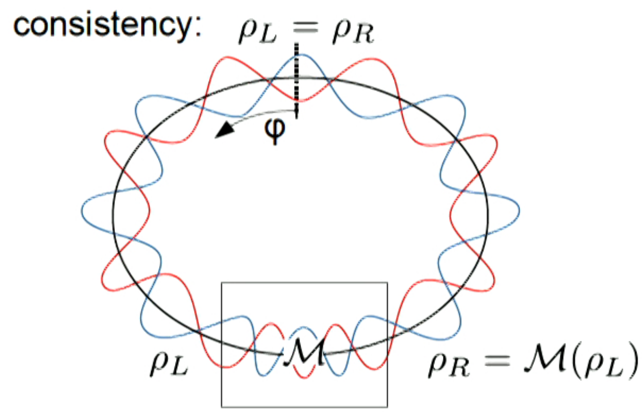


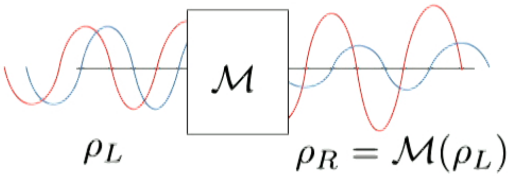

Modifications to spatial ring:

1. two modes, with $\hat{H}$ such that the total probability does not change with φ
2. $\hat{H}_{ring}$ such that φ introduces at most a single global phase on both modes
3. region with $\hat{H}_{map}$ that can change relative phase and amplitude of the two modes, enforcing arbitrary boundary conditions at its ends such that $\rho_R = \mathcal{M}(\rho_L)$

→ For a given $\mathcal{M}$, the ~~stationary~~ ground states on the equivalent spatial ring reveal the consistent states on the temporal loop.

* Caveat: plane waves in φ!

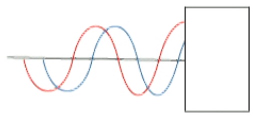
# Analogues of Gates and Maps




Basic idea: penalize 'wrong' states with potential energy.

Trace: don't penalize any state

$$\hat{H} \mid 0 \rangle = 0$$
$$\hat{H} \mid 1 \rangle = 0$$



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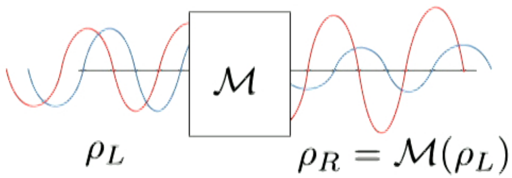


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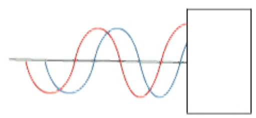
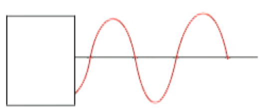


Preparation: penalize all but one state
(and renormalize*)

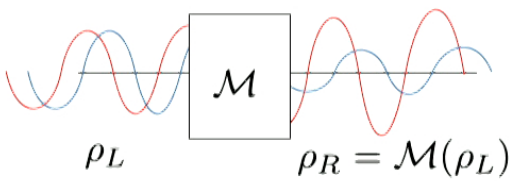


$$\hat{H}\,|\,0\,\rangle = 0$$
$$\hat{H}\,|\,1\,\rangle = E_p > 0$$

… of mixed states: with ancilla

$$\hat{H}\,(\cos\theta\,|\,0\,\rangle\,|\,0\,\rangle + \sin\theta\,|\,1\,\rangle\,|\,1\,\rangle) = 0$$

# Analogues of Gates and Maps

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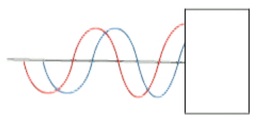
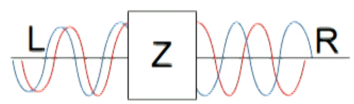


Unitary: allow pairs $\{|\,j\,\rangle_L \otimes (U\,|\,j\,\rangle)_R\}_j$




$$\hat{H}\,|\,00\,\rangle_{LR} = 0$$
$$\hat{H}\,|\,11\,\rangle_{LR} = 0$$

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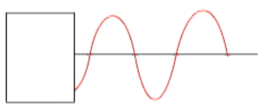
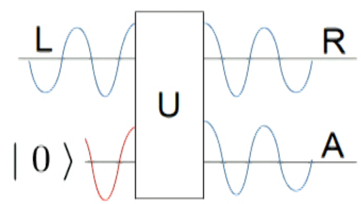


$$\hat{H}\,|\,0\,\rangle = 0$$
$$\hat{H}\,|\,1\,\rangle = E_p > 0$$

Measurements: von Neumann construction



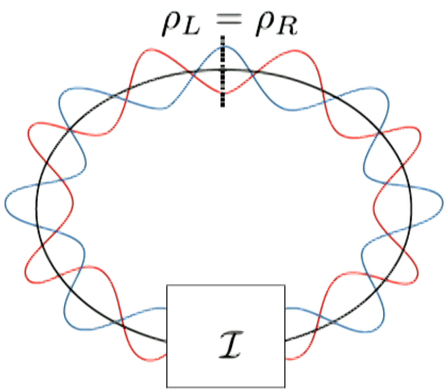

$$\hat{H}\,|\,000\,\rangle_{LRA} = 0$$
$$\hat{H}\,|\,111\,\rangle_{LRA} = 0$$

… of mixed states: with ancilla

$$\hat{H}\,(\cos\theta\,|\,0\,\rangle\,|\,0\,\rangle + \sin\theta\,|\,1\,\rangle\,|\,1\,\rangle) = 0$$

## Examples

Example 1: Identity




→ full two-dimensional space of zero-energy states

→ all states are consistent

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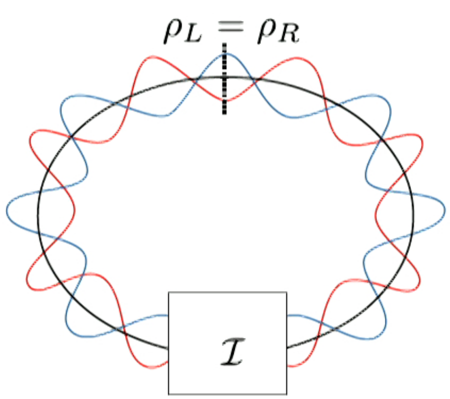


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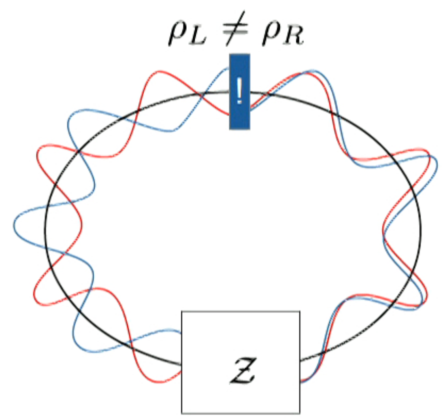
Example 2: Z Gate




→ only 0 state has zero energy

→ only 0 state is consistent

# Validity Conditions for Process Matrices

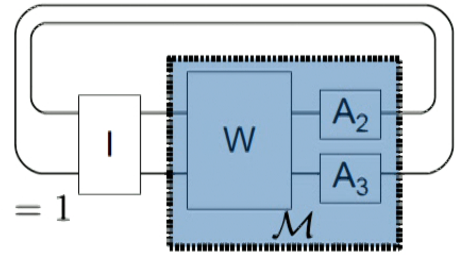
# Validity condition in terms of matrix elements

Rewrite the success probability

$$P(click|A,B) = Tr\left[W(A \otimes B)\right] = 1 \quad \forall A, B$$

in terms of the identity channel:

$$P(click|\mathcal{M}) = Tr(MI) = \sum_{j,k} \langle\, j \mid \mathcal{M}\,(\mid j\,\rangle\langle\, k \mid) \mid k\,\rangle = 1$$



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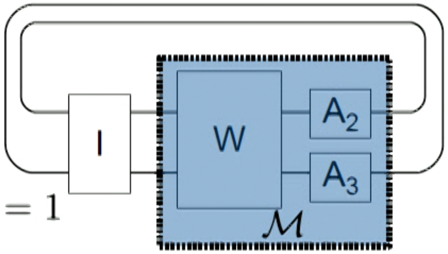


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Compare with the condition for trace-preserving maps:

$$Tr(M(\rho)) = \sum_{j} \langle\, j \,|\, \mathcal{M}\,(\rho)\,|\, j\,\rangle = 1 \; \forall \rho$$

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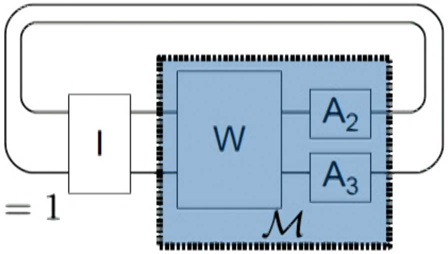


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in terms of the identity channel:

$$P(click|\mathcal{M}) = Tr(MI) = \sum_{j,k} \langle\, j \mid \mathcal{M}\left(\mid j\,\rangle\langle\, k \mid\right) \mid k\,\rangle = 1$$

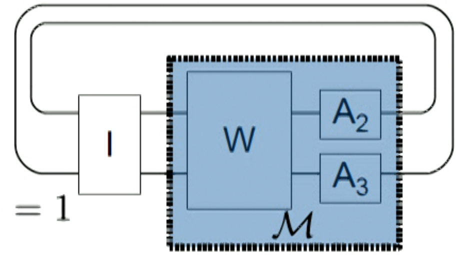
Compare with the condition for trace-preserving maps:

$$Tr(M(\rho)) = \sum_{j} \langle\, j \mid \mathcal{M}\left(\rho\right) \mid j\,\rangle = 1 \;\forall \rho$$

Example 1: single-qubit Z gate

$$P(click|\mathcal{Z}) = \langle 0|0\rangle\langle 0|0\rangle + \langle 1|1\rangle\langle 1|1\rangle - \langle 0|0\rangle\langle 1|1\rangle - \langle 1|1\rangle\ \langle 0|0\rangle = 0$$

## Validity condition in terms of matrix elements

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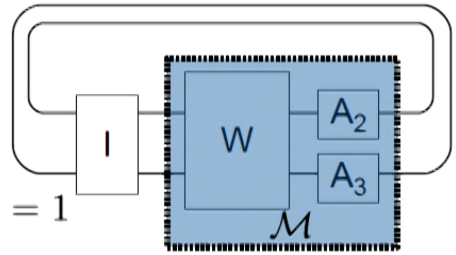


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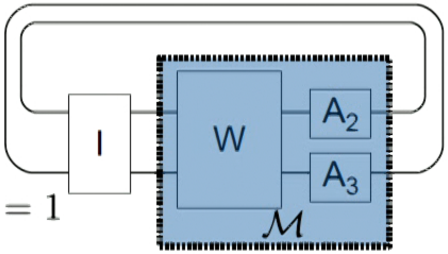


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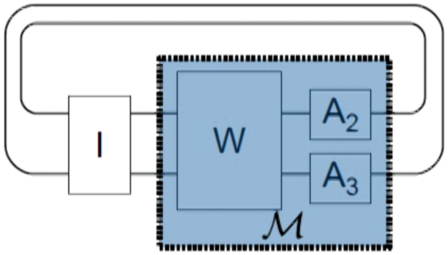
Example 3: single-qubit Z measurement

$$P(click|meas.Z) = \langle 0|0\rangle\langle 0|0\rangle + \langle 1|1\rangle\langle 1|1\rangle + 0 + 0 = 2$$

# Validity condition in terms of Pauli operators

For a single qubit, the Choi state of the identity channel can be written

$$I = (|\,00\,\rangle + |\,11\,\rangle)(\langle\,00\,| + \langle\,11\,|)$$
$$= \frac{1}{2}\mathbb{I}\otimes\mathbb{I} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z$$




hence we can rewrite

$$Tr(MI) = Tr[\mathcal{M}(\frac{\mathbb{I}}{2})] + \sum_j Tr\,[\sigma_j\mathcal{M}(\sigma_j)]$$

so the validity condition becomes

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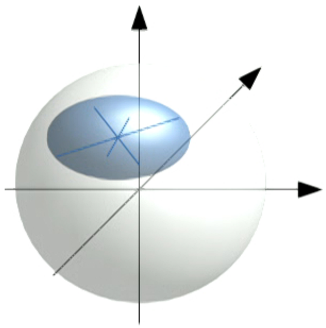

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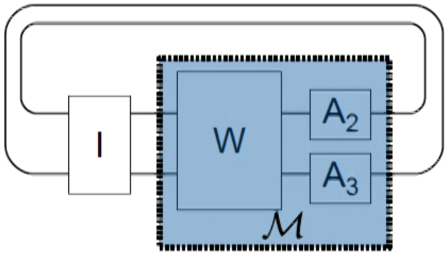


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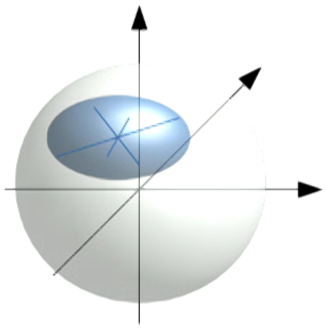
In order for this to hold under combination with arbitrary unitaries,

$$\mathcal{M}(\sigma_j) = 0\ \forall j$$

the map may only admit a single fixed point.

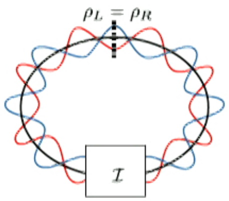


# Identifying Consistent States

# Unitary maps

Case 1: identity

- $\hat{H} \mid \psi \rangle = 0 \mid \psi \rangle \forall \mid \psi \rangle$ → all states are ground states and consistent



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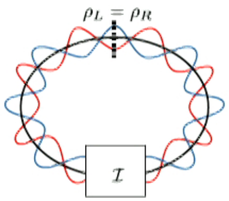


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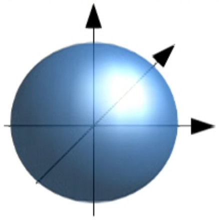
- ‘overcomplete’ → not a valid process



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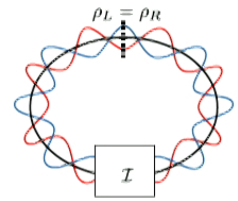


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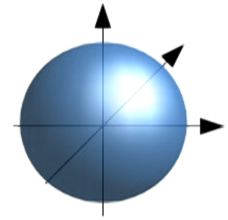
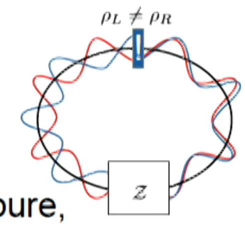
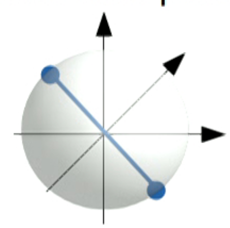




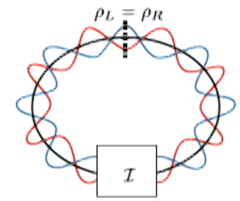

Case 2: single-qubit Z gate:

- $\hat{H} \mid 0 \rangle = 0 \mid 0 \rangle$ but $\hat{H} \mid 1 \rangle = E_M \mid 1 \rangle$ → only one ground/consistent state
- $\mathcal{M}(\mid j \rangle \langle j \mid) = \mid j \rangle \langle j \mid (j = 0, 1)$ → two discrete fixed points given by pure, orthogonal states and their mixtures



→ not every fixed state is a consistent state!

## Unitary maps

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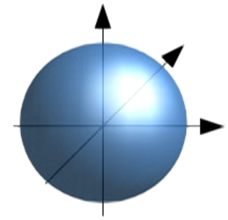

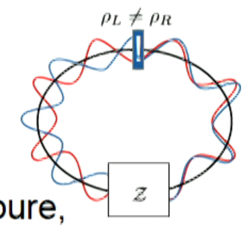


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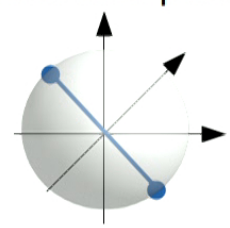

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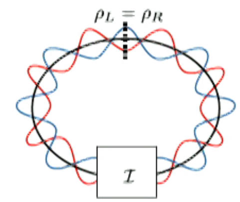


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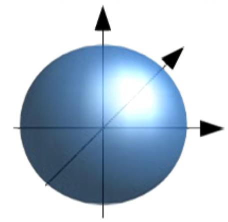

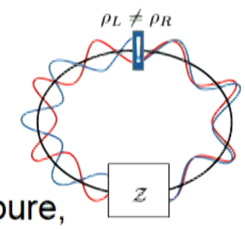


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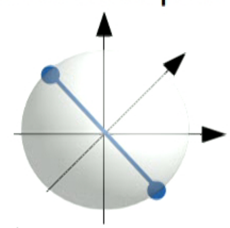

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Case 3: general unitary $U = \sum \exp(\imath \alpha_j) \mid j \rangle \langle j \mid$

- $\hat{H} \mid j \rangle = 0 \mid j \rangle$ only if $\alpha_j \doteq 0$ → arbitrary number of orthogonal ground/consistent states
- $\mathcal{M}(\mid j \rangle \langle j \mid) = \mid j \rangle \langle j \mid \forall j$ → simplex of fixed points
- generally not a valid process: $P(click) = \sum_{j,k} \langle j \mid U \mid j \rangle \langle k \mid U^\dagger \mid k \rangle \neq 1$

# Noisy processes

Case 4: all fixed points are pure states

$\rightarrow$ single fixed point, because of linearity $\rightarrow$ trace and reprepare pure state $|\,\psi\,\rangle$

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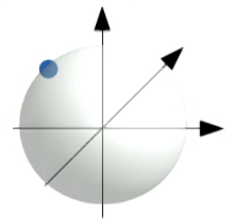
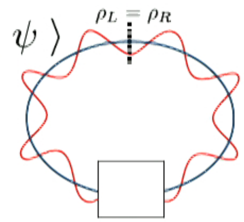
- valid, because the preparation excludes $|\,\psi_\perp\,\rangle$





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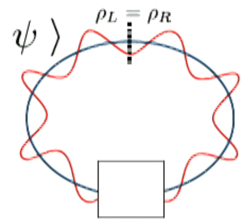


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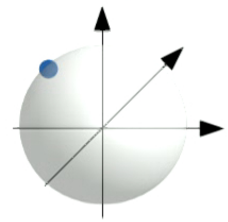

Case 5: fixed points can include mixed states

Element 1: X gate

$\hat{H}_X = E_X\,|\,-\,\rangle\langle\,-\,|_L \otimes \mathbb{I}_A$

allows superposed loop state

$|\,+\,\rangle_L = (|\,0\,\rangle + |\,1\,\rangle)/\sqrt{2}$

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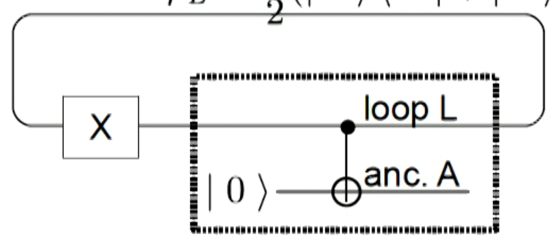

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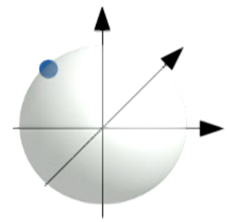

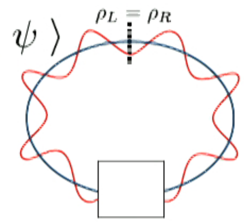


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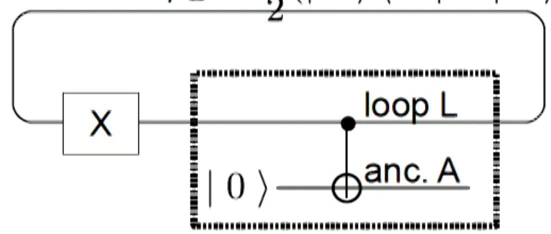



Element 2: Z measurement

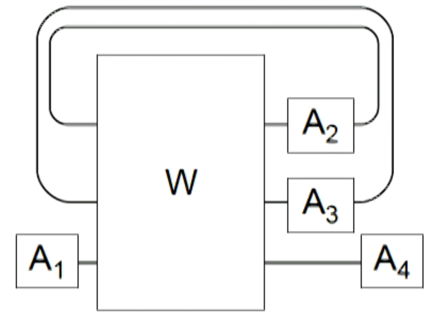
$$\hat{H}_M = 2E_M\,(|\,0\,\rangle\langle\,0\,|_L\,|\,1\,\rangle\langle\,1\,|_A + |\,1\,\rangle\langle\,1\,|_L\,|\,0\,\rangle\langle\,0\,|_A)$$

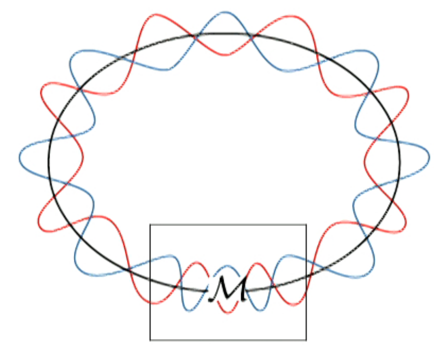
allows entangled states between loop and ancilla

$$\alpha\,|\,0\,\rangle_L\,|\,0\,\rangle_A + \beta\,|\,1\,\rangle_L\,|\,1\,\rangle_A$$

**Summary of proposed construction**

- experiments that use resonators that loop around in space as analogues of the causal loops that appear in a 'rewired' version of processes with indefinite causal structure
- reproduce statistics of instruments applied in local labs
- reveal consistent states on the loops
- allow test of physicality of given map $\mathcal{M}$



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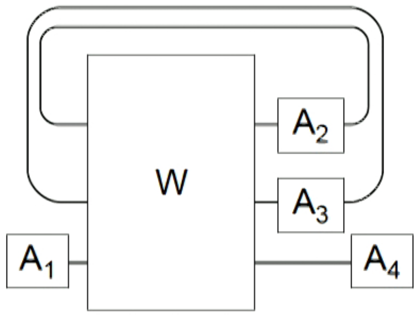


**Observations**

- fixed states are not the same as consistent states
- strange effect of coherence across the map:
  - plays a role in the validity condition
  - excludes mixed states on the loop
  - connection with measurements, entanglement?