Title: What happens when we quantize time?

Speakers: Alexander Smith

Collection: Indefinite Causal Structure

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Abstract: The lesson of general relativity is background independence: a physical theory should not be formulated in terms of external structures. This motivates a relational approach to quantum dynamics, which is necessary for a quantum theory of gravity. Using a covariant POVM to define a time observable, I will introduce the so-called trinity of relational quantum dynamics comprised of three distinct formulations of the same relational quantum theory: evolving constants of motion, the Page-Wootters formalism, and a symmetry reduction procedure. The equivalence between these formulations yields a temporal frame change map that transforms between the dynamics seen by different clocks. This map will be used to illustrate a temporal nonlocality effect that results in a superposition of time evolutions from the perspective of a clock indicating a superposition of different times. Then, a time-nonlocal modification to the SchrÃ¶dinger equation will be shown to manifest when a system is coupled to the clock that is tracking its evolution. Such clock-system interactions should be expected at some scale when the gravitational interaction between them is taken into account. Finally, I will examine relativistic particles with internal degrees of freedom that constitute a clock that tracks their proper time. By evaluating the conditional probability associated with two such clocks reading different proper times, I will show that these clocks exhibit a novel quantum time dilation effect when moving in a superposition of different momenta.

What happens when we quantize time?

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NSERC CRSNG

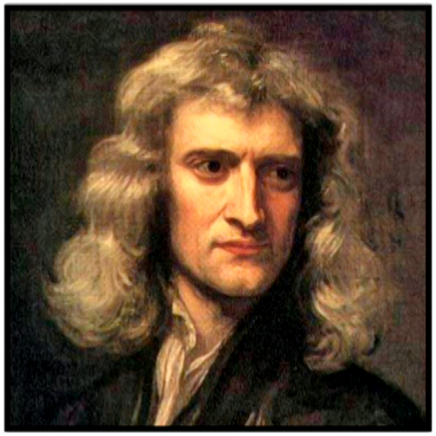
Based on work with Mehdi Ahmadi, Philipp A. Höhn, and Maximilian P. Lock

Quantizing time: Interacting clocks and systems
A. R. H. Smith and M. Ahmadi, Quantum 3, 160 (2019)

Relativistic quantum clocks observe classical and quantum time dilation
A. R. H. Smith and M. Ahmadi, arXiv:1904.12390 [quant-ph] (2019)

The Trinity of Relational Quantum Dynamics
P. A. Höhn, A. R. H. Smith, and M. P. E. Lock, arXiv:1912.00033 [quant-ph] (2019)

Time in Newtonian mechanics



$$F = ma = m\frac{d^2x}{dt^2}$$

"...relative, apparent, and common time, is some sensible and external measure of duration by the means of motion, which is commonly used instead of true time;

Newton, *Philosophiae Naturalis Principia Mathematica* (1687)

Time in quantum theory

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

[2] In the older literature on quantum mechanics, we often find the operator equation

$$H t - t H = \frac{h}{i} I,$$

which arises from (8.6) formally by substituting t for F. It is generally not possible, however, to construct a Hermitian operator (e.g. as function of p and q) which satisfies this equation. This is so because, from the C.R. written above, it follows that H possesses continuously all eigenvalues from $-\infty$ to $+\infty$ (cf. Dirac, Quantum Mechanics, First edition (1930), 34 and 56) whereas on the other hand, discrete eigenvalues of H can be present. *We, therefore, conclude that the introduction of an operator t is basically forbidden and the time t must necessarily be considered as an ordinary number ("c-number")* in Quantum Mechanics* (cf. for this E. Schrödinger, Berl. Ber. (1931) p. 238).**

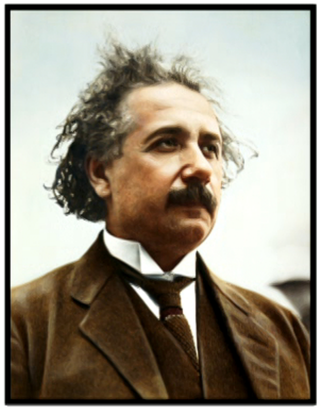
*As opposed to this, operators are usually called "*q-numbers*".

See also P. Carruthers and M.M. Nieto, Rev. Mod. Phys., **40, 411 (1968) for further references.

W. Pauli, *General Principles of Quantum Mechanics* (1958)

Time in general relativity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$



"[Time is] considered measurable by a clock (ideal periodic process) of negligible spatial extent. The time of an event taking place at a point is then defined as the time shown on the clock simultaneous with the event."

A. Einstein, The Special Theory of Relativity (1949)

"Einstein, in seizing on the act of the observer as the essence of the situation, is actually adopting a new point of view as to what the concepts of physics should be, namely, the operational view."

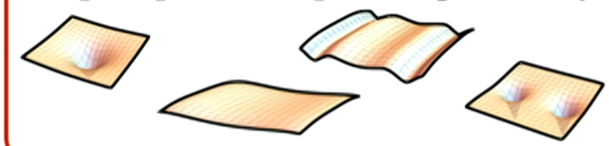
P.W. Bridgman, *The Logic of Modern Physics* (1927)

Quantum gravity and the problem of time

We need to choose canonical variables:

Choose the 3-metric γ_{ab} and its conjugate momentum P^{ab}

Superspace of spatial geometry



Then we can express general relativity in the Hamiltonian form

$$H_{\mathrm{GR}}[\gamma, P] = 16\pi G G_{abcd}[\gamma] P^{ab} P^{cd} + V[\gamma] + \sqrt{\gamma}\rho \approx 0$$

and three momentum constraints

Quantization

1. Constraint

$$H_{GR}\,|\,\Psi\rangle\rangle = 0$$

2. Dynamics

$$i\frac{d}{dt}\,|\,\Psi\rangle\rangle = H_{GR}\,|\,\Psi\rangle\rangle = 0$$

$|\,\Psi\rangle\rangle$ is frozen!

Outline

1. **What is a clock?**
2. **The trinity of relational quantum dynamics**
3. **Changing temporal reference frames**
4. **Interacting clocks and systems**
5. **Relativistic clocks and quantum time dilation**
6. **Causal structure from conditional probabilities**

What is a clock?

A clock (temporal reference frame) is defined by the quadruple

Hilbert space ↘ Fiducial state ↙

$$= \{\mathcal{H}_C,\, H_C,\, \rho_C,\, T_C\}$$

Hamiltonian ↗ Covariant POVM ↖

Elapsed time is unitarily encoded in the state fiducial state

$$\rho_C(\tau) = e^{-iH_C\tau}\rho_C e^{iH_C\tau} \quad \text{for all } \tau \in G \subseteq \mathbb{R}$$

A time observable T_C should satisfy the following properties:

1. On average T_C should estimate the elapsed time τ

2. The variance in a measurement of T_C should be independent of the elapsed time τ

Covariant time observable

T_C is a POVM that best estimates the elapsed time:

$$\left.\begin{aligned} T_C : G &\to \mathcal{E}\left(\mathcal{H}_C\right) \\ \tau &\mapsto E_C(\tau) = |\tau\rangle\langle\tau| \end{aligned}\right\}$$

Clock states

1. $E_C(\tau) \geq 0$
2. $E_C(G_1 + G_2) = E_C(G_1) + E_C(G_2)$
3. $\int_G d\tau \, E(\tau) = \mu \int_G d\tau \; |\tau\rangle\langle\tau| = I_C$

These properties are satisfied if T_C is **covariant** with respect to the group generated by H_C, which is equivalent to the following relation between clocks states

$$|\tau + \tau'\rangle = e^{-iH_C\tau'} \, |\tau\rangle$$

A. S. Holevo, *Probabilistic and Statistical Aspects of quantum Theory*, (North-Holland, 1982)

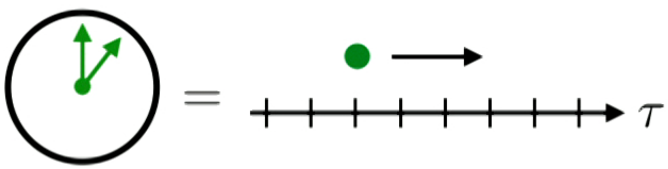
S. L. Braunstein and C. Caves, *Statistical distance and the geometry of quantum states*, Phys. Rev. Lett. 72 3439 (1994)

P. Busch, M Grabowski, and P. J. Lahti, *Operational Quantum Physics*, (Springer-Verlag, 1995)

G. Chiribella, *Optimal estimation of quantum signals*, PhD Thesis (2006)

Examples of clocks

The ideal clock



Clock Hilbert space

$$\mathcal{H}_C \simeq L^2(\mathbb{R})$$

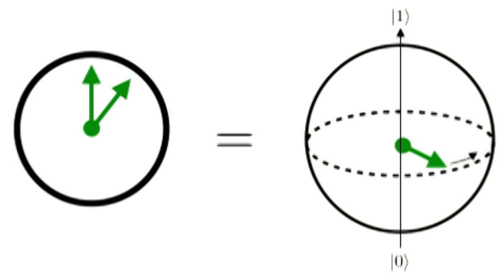
Clock Hamiltonian

$$H_C = P_C$$

Clock states

$$|\tau\rangle = e^{-iP_C\tau}\,|\tau_0\rangle$$

Spin-1/2 clock



Clock Hilbert space

$$\mathcal{H}_C \simeq \mathbb{C}^2$$

Clock Hamiltonian

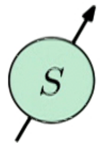
$$H_C = \Omega\sigma_z$$

Clock states

$$|\tau\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\Omega\tau}\,|1\rangle\right)$$

Time as a dynamical variable (i.e. a clock)

System (q, q')



with action $\mathcal{S} = \int_{t_1}^{t_2} dt\, L_S\left(q, q'\right)$

Introducing an arbitrary integration parameter τ, the action is

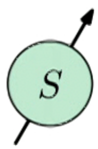
$$\mathcal{S} = \int_{\tau_1}^{\tau_2} d\tau\, \dot{t} L_S\left(q, \dot{q}/\dot{t}\right) = \int_{\tau_1}^{\tau_2} d\tau\, L\left(q, \dot{q}, \dot{t}\right)$$

$\dot{t} = \frac{dt}{d\tau}$

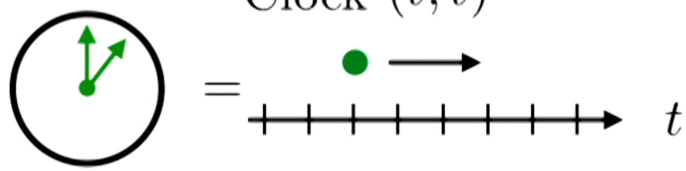
where

$$L\left(q, \dot{q}, \dot{t}\right) := \dot{t} L_S\left(q, \dot{q}/\dot{t}\right)$$

System $(q, \dot{q})$



Clock $(t, \dot{t})$



The total (super) Hamiltonian is constrained to vanish

$$C_H = P_C + H_S \approx 0$$

Is this the most general total Hamiltonian?

$$C_H \,|\, \Psi\rangle\rangle = (P_C \otimes I_S + I_C \otimes H_S)\,|\,\Psi\rangle\rangle = 0$$

1. We can consider a general clock Hamiltonian: $P_C \to H_C$

2. We can include an interaction Hamiltonian: H_{int}

$$H_{\mathrm{GR}}[\gamma, P] = 16\pi G G_{abcd}[\gamma] P^{ab} P^{cd} + V[\gamma] + \underline{\sqrt{\gamma}\rho} \approx 0$$

General total Hamiltonian

$$C_H \,|\, \Psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S + \lambda H_{int})\,|\,\Psi\rangle\rangle = 0$$

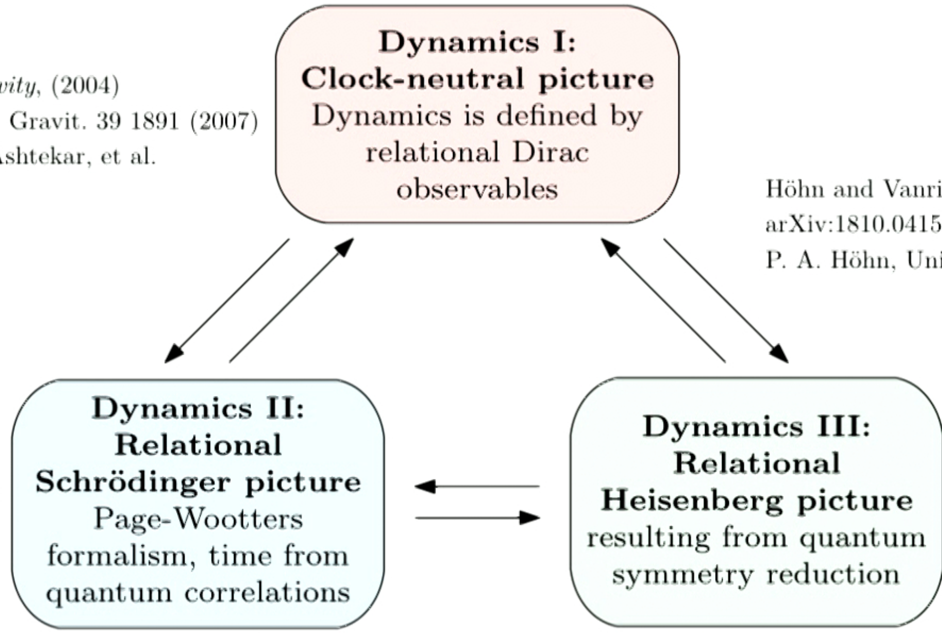
The Trinity of Relational Quantum Dynamics

How do we recover dynamics from a Hamiltonian constraint?

$$C_H \,|\,\Psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S)\,|\,\Psi\rangle\rangle = 0$$

Rovelli, *Quantum Gravity*, (2004)
Dittrich, Gen. Relativ. Gravit. 39 1891 (2007)
+Thiemann, Marolf, Ashtekar, et al.

Höhn and Vanrietvelde,
arXiv:1810.04153 [gr-qc] (2018)
P. A. Höhn, Universe 5, 116 (2019)



Page and Wootters, Phys. Rev. D 27, 2885 (1983)

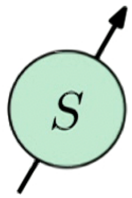
The Trinity of Relational Quantum Dynamics
P. A. Höhn, A. R. H. Smith and M. P. E. Lock, arXiv:1912.00033 [quant-ph] (2019)

Dynamics I: Relational Dirac observables

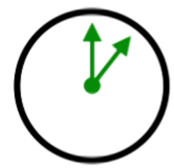
$$C_H \,|\Psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S)\,|\Psi\rangle\rangle = 0$$

A Dirac observable $F_A(\tau)$ is gauge invariant: $[C_H, F_A(\tau)] = 0$

What is the value of A when the clock reads τ ?



$A = \hat{n} \cdot \vec{\sigma}$ WHEN



$= |\tau\rangle$

$$F_A(\tau) = \mathcal{G}\big(\,|\tau\rangle\langle\tau| \otimes A\big) = \int_G dg\, e^{-iC_H g}\big(\,|\tau\rangle\langle\tau| \otimes A\big)e^{iC_H g}$$

C. Rovelli, *Quantum Gravity*, (Cambridge University Press, 2004)

B. Dittrich, *Partial and complete observables for Hamiltonian constrained systems*, Gen. Relativ. Gravit. 39 1891 (2007)

S. D. Bartlett, T. Rudolph, R. W. Spekkens, *Reference frames, superselection rules, and quantum information*, Rev. Mod. Phys. 79, 55 (2007)

S. D. Bartlett, T. Rudolph, R. W. Spekkens, *Quantum communication using a bounded-size quantum reference frame*, New J. Phys, 11 063013 (2009)

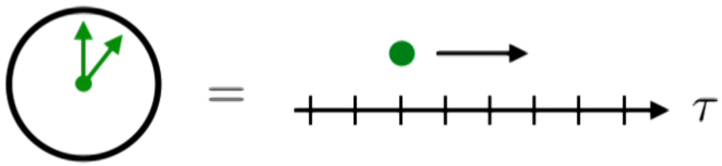
A. R. H. Smith, M. Piani, R. B. Mann, *Quantum reference frames associated with noncompact groups: The case of translations and boosts and the role of mass*, Phys. Rev. A 94 012333 (2016)

A. R. H. Smith, *Communicating without shared reference frames*, Phys. Rev. A 99 052315 (2019)

Dynamics II: The Page-Wootters formalism

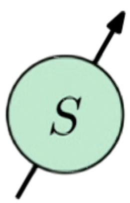
$$C_H \,|\,\Psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S)\,|\,\Psi\rangle\rangle = 0$$

Different clock times are indicated by different clock states



$$|\tau\rangle = e^{-iH_C\tau}\,|\tau_0\rangle$$

The state of the system at the time τ is the joint state of the clock and system $|\,\Psi\rangle\rangle$ conditioned on the clock being in the state $|\tau\rangle$:



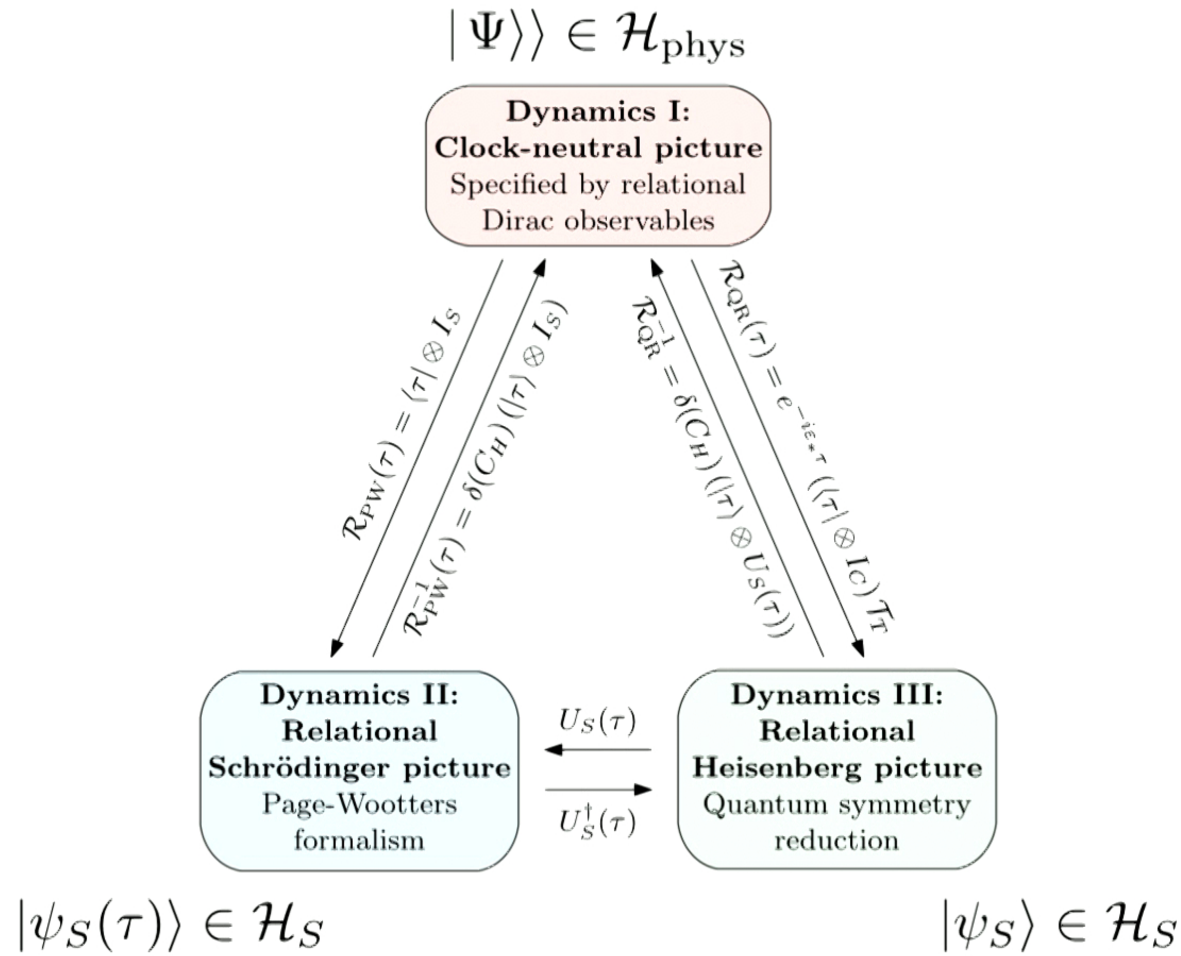
$$|\psi_S(\tau)\rangle := \langle\tau| \otimes I_S\,|\,\Psi\rangle\rangle$$

$$i\frac{d}{d\tau}\,|\psi_S(\tau)\rangle = H_S\,|\psi_S(\tau)\rangle$$

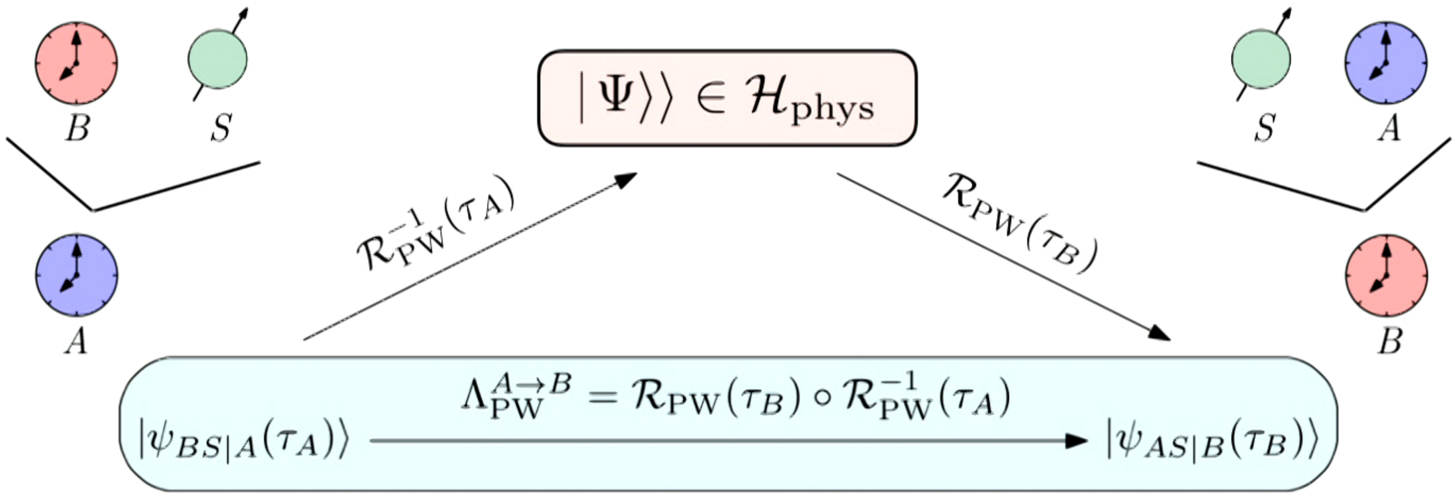
D. N. Page and W. K. Wootters, Phys. Rev. D 27, 2885 (1983)

W. K. Wootters, *"Time" Replaced by Quantum Correlations* (1984)

The Trinity of Relational Quantum Dynamics



Changing temporal reference frames



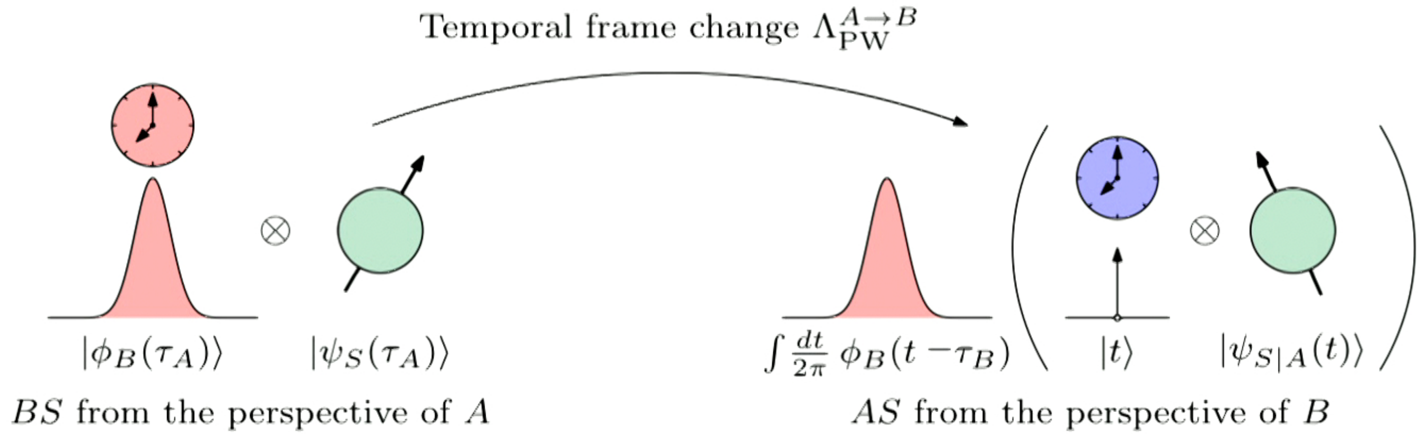
The temporal frame change map

$$\Lambda^{A\to B}_{\text{PW}} : \mathcal{H}_B \otimes \mathcal{H}_S \to \mathcal{H}_A \otimes \mathcal{H}_S,$$

$$|\psi_{BS|A}(\tau_A)\rangle \mapsto |\psi_{AS|B}(\tau_B)\rangle = \Lambda^{A\to B}_{\text{PW}} \, |\psi_{BS|A}(\tau_A)\rangle$$

$$\Lambda^{A\to B}_{\text{PW}} = \langle \tau_B | \, \delta(C_H) \, |\tau_A\rangle$$

Temporal locality and nonlocality



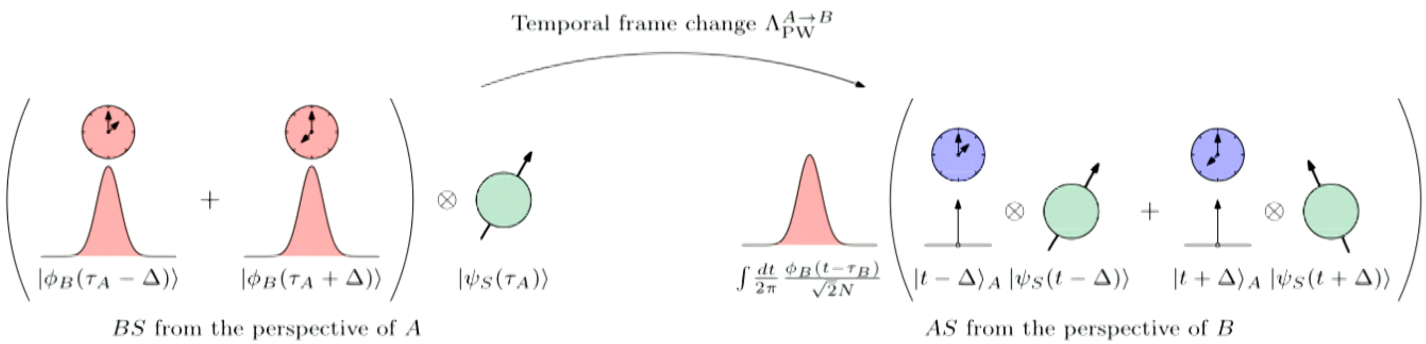
Temporally local evolution of AS from the perspective of B

$$|\psi_{AS|B}(\tau_B)\rangle = U_{AS}(\tau_B) \int_{\mathbb{R}} \frac{dt}{2\pi}\, \phi_B(t)\, |t\rangle_A \, |\psi_S(t)\rangle$$

The reduced state of S is localized in clock B time

$$\rho_{S|B}(t) \approx |\psi_S(\tau_B)\rangle\langle\psi_S(\tau_B)|$$

Temporal locality and nonlocality



Temporally nonlocal evolution of AS from the perspective of B

$$|\psi_{AS|B}(\tau_B)\rangle = \frac{1}{\sqrt{2N}} \left[U_{AS}(\tau_B - \Delta) + U_{AS}(\tau_B + \Delta) \right] \int_{\mathbb{R}} \frac{dt}{2\pi}\, \phi_B(t)\, |t\rangle_A \, |\psi_S(t)\rangle$$

The localization of S in clock B time is indefinite

$$\rho_{S|B}(t) \approx \frac{1}{2} \Big(|\psi_S(\tau_B - \Delta)\rangle\langle\psi_S(\tau_B - \Delta)| + |\psi_S(\tau_B + \Delta)\rangle\langle\psi_S(\tau_B + \Delta)| \Big)$$

Temporal analog of frame dependent superposition/entanglement

Quantum mechanics and the covariance of physical laws in quantum reference frames
F. Giacomini, E. Castro-Ruiz, and Č. Brukner, Nat. Comm. 10, 494 (2019)

Time reference frames and gravitating quantum clocks
E. Castro-Ruiz, F. Giacomini, A. Belenchia and Č. Brukner, arXiv:1908.10165 [quant-ph] (2019)

Interacting clocks and systems

$$C_H \,|\,\Psi\rangle\rangle = (P_C \otimes I_S + I_C \otimes H_S)\,|\,\Psi\rangle\rangle = 0$$

1. We can consider a general clock Hamiltonian: $P_C \to H_C$

2. We can include an interaction Hamiltonian: H_{int}

$$H_{\mathrm{GR}}[\gamma, P] = 16\pi G G_{abcd}[\gamma] P^{ab} P^{cd} + V[\gamma] + \underline{\sqrt{\gamma}\rho} \approx 0$$

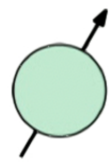
General total Hamiltonian

$$C_H \,|\,\Psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S + \lambda H_{int})\,|\,\Psi\rangle\rangle = 0$$

The modified Schrödinger equation

$$C_H \,|\Psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S + \lambda H_{int})\,|\Psi\rangle\rangle = 0$$

$$|t\rangle := e^{-iH_C t}\,|t_0\rangle$$



$$|\psi_S(t)\rangle := \big(\langle t| \otimes I_S\big)\,|\Psi\rangle\rangle$$

$$i\frac{d}{dt}\,|\psi_S(t)\rangle = H_S\,|\psi_S(t)\rangle + \lambda \int dt'\, K(t,t')\,|\psi_S(t')\rangle$$

$$K(t,t') := \langle t|H_{int}|t'\rangle$$

The modified Schrödinger is time-nonlocal

Quantizing time: Interacting clocks and systems
A. R. H. Smith and M. Ahmadi, Quantum 3, 160 (2019)

Examples of different clock-system interactions

Clocks coupled through Newtonian gravity

$$H_{int} = -\frac{G m_C \, m_S}{d} = -\frac{G}{d}\frac{H_C}{c^2} \otimes \frac{H_S}{c^2}$$

The modified Schrödinger equation simplifies to

$$i\frac{d}{dt}\left|\psi_S(t)\right\rangle = \frac{H_S}{I_S + \frac{G}{c^4 d} H_S}\left|\psi_S(t)\right\rangle$$

$$= \left[H_S + \frac{G}{c^4 d} H_S^2 + \mathcal{O}\left(\left[\frac{G}{c^4 d}\right]^2\right)\right]\left|\psi_S(t)\right\rangle$$

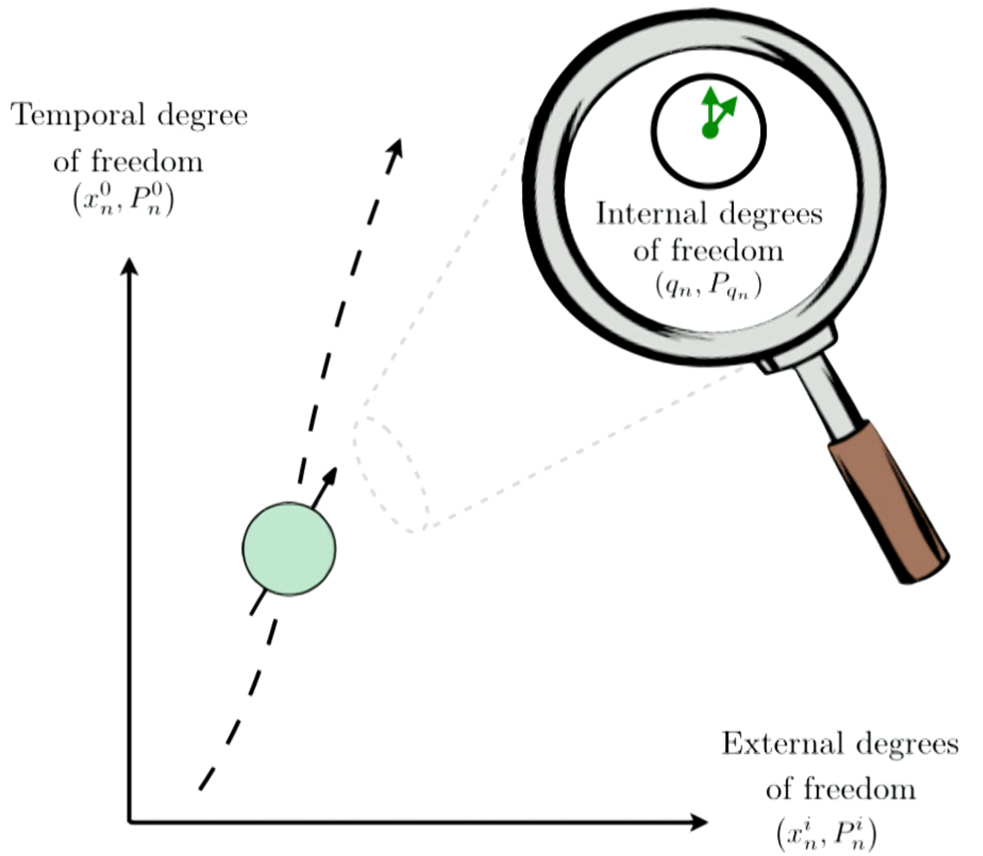
Time-dependent system Hamiltonian

$$H_{int} = f(T) \otimes S \qquad T := \int dt \, t \, |t\rangle\langle t|$$

The modified Schrödinger equation simplifies to

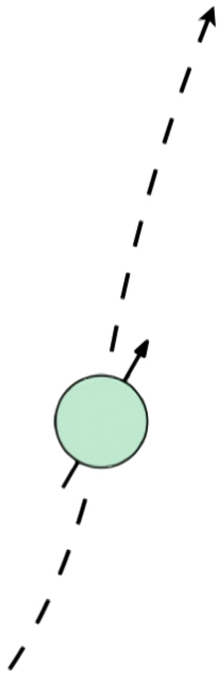
$$i\frac{d}{dt}\left|\psi_S(t)\right\rangle = \left[H_S + \lambda f(t)\, S\right]\left|\psi_S(t)\right\rangle$$

Classical and quantum time dilation



Relativistic quantum clocks observe classical and quantum time dilation
A. R. H. Smith and M. Ahmadi, arXiv:1904.12390 [quant-ph] (2019)

A particle moving in curved spacetime



$$\mathcal{S} = \int d\tau \left[-mc^2 + P_q \frac{dq}{d\tau} - H^{\text{clock}} \right]$$

$$= \int dt \sqrt{-\dot{x}^2} \left(-mc^2 + \frac{P_q \dot{q}}{\sqrt{-\dot{x}^2}} - H^{\text{clock}} \right)$$

$$\dot{x}^2 := g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

Construct the associated Hamiltonian

$$H = g_{\mu\nu} P^\mu P^\nu + \left(mc^2 + H^{\text{clock}} \right)^2 \approx 0$$

The Hamiltonian may be factorized as

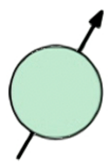
$$H = C^+ C^- \quad \text{where} \;\; C^\pm := P^0 \pm H_S \approx 0$$

$$H_S := \frac{1}{\sqrt{g_{00}}} \sqrt{g_{ij} P^i P^j + (mc^2 + H^{\text{clock}})^2}$$

Quantization and recovery of relativistic QM

$$C^{+} := P^{0} + H_S \approx 0 \quad \xrightarrow{\text{Quantization}} \quad C^{+}\,|\Psi\rangle\rangle = \left(P^{0} + H_S\right)|\Psi\rangle\rangle = 0$$

Define the conditional state in the same way as before



$$|\psi_S(t)\rangle := (\langle t| \otimes I_S)\,|\Psi\rangle\rangle \in \mathcal{H}^{\text{ext}} \otimes \mathcal{H}^{\text{clock}}$$

which satisfies the relativistic Schrödinger equation

$$i\frac{d}{dt}\,|\psi_S(t) = H_S\,|\psi_S(t)\rangle\rangle$$

$$H_S := \frac{1}{\sqrt{g_{00}}}\sqrt{g_{ij}P^iP^j + (mc^2 + H^{\text{clock}})^2}$$

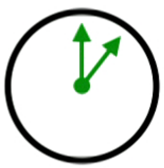
A clock measuring proper time

$$C^{+}\,|\Psi\rangle\rangle = \left(P^0 + H_S\right)|\Psi\rangle\rangle = 0$$

$$H_S := \frac{1}{\sqrt{g_{00}}}\sqrt{g_{ij}P^iP^j + (mc^2 + H^{\text{clock}})^2}$$

$$= H^{\text{clock}} + H^{\text{ext}} + H^{\text{int}} + \mathcal{O}\left(\frac{1}{m^2c^4}\right)$$

The internal/clock degrees of freedom evolve in the particles proper time



$$|\tau\rangle = e^{-iH^{\text{clock}}\tau}\,|\tau_0\rangle \longrightarrow E_C(\tau) = |\tau\rangle\langle\tau|$$

Clock effect operators

$$H^{\text{int}} := -\frac{1}{mc^2}\left(H^{\text{ext}} \otimes H^{\text{clock}} + \left(H^{\text{ext}}\right)^2\right)$$

This coupling is responsible for time dilation

Proper time and mass uncertainty relational

The Helstrom-Holevo lower bound implies a generalized uncertainty relation between proper time and mass:

$$\langle \Delta T_C^2 \rangle_{\rho_C} \geq \frac{1}{4\left\langle (\Delta H^{\text{clock}})^2 \right\rangle_{\rho_C}} = \frac{\hbar}{4c^4} \frac{1}{\left\langle (\Delta M)^2 \right\rangle_{\rho_C}}$$

C. W. Helstrom, *Quantum detection and estimation theory*, (1976)
A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, (1982)

Mass operator

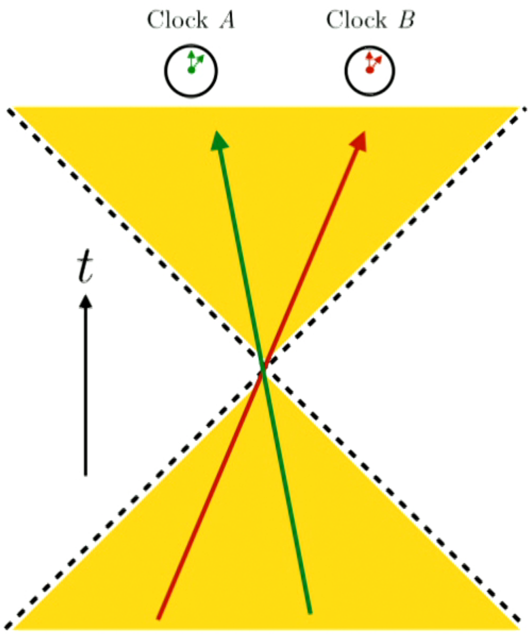
$$M := mI^{\text{clock}} + \frac{1}{c^2} H^{\text{clock}}$$

M. Zych and Č. Brukner, *Quantum formulation of the Einstein equivalence principle*, Nat. Phys. 14, 1027 (2018)
I. Pikovski, M. Zych, F. Costa, and Č. Brukner, *Universal decoherence due to gravitational time dilation*, Nat. Phys. 11, 668 (2015)

Mass and proper time are quantum observables

Daniel M. Greenberger, *Conceptual Problems Related to Time and Mass in Quantum Theory*, (2010), (RQI-N, Vienna, 2017)

Probabilistic time dilation



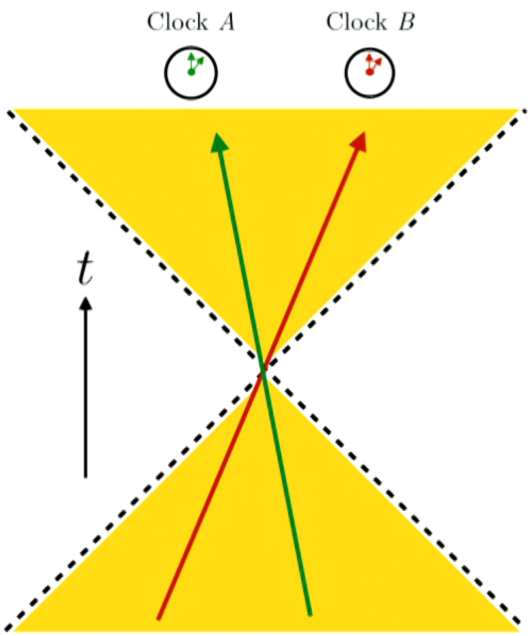
$$\text{Prob}\left[T_A = \tau_A \text{ when } T_B = \tau_B\right]$$

$$= \frac{\langle\langle\Psi|E_A(\tau_A)E_B(\tau_B)|\Psi\rangle\rangle}{\langle\langle\Psi|E_B(\tau_B)|\Psi\rangle\rangle}$$

To leading order in Minkowski space the average proper time read by clock A is

$$\langle T_A \rangle = \left(1 - \frac{\langle H_A^{\text{ext}}\rangle - \langle H_B^{\text{ext}}\rangle}{mc^2}\right)\tau_B + \mathcal{O}\left(\left(\tfrac{1}{mc^2}\right)^2\right)$$

Recovering classical time dilation



Supposing that the conditional state of external degree of freedom of both clocks at $t = 0$ is Gaussian

$$|\psi_n^{\text{ext}}\rangle = \frac{1}{\pi^{1/4}\sqrt{\Delta}} \int d\mathbf{p}\, e^{-\frac{(\mathbf{p}-\bar{\mathbf{p}}_n)^2}{2\Delta^2}} |\mathbf{p}_n\rangle$$

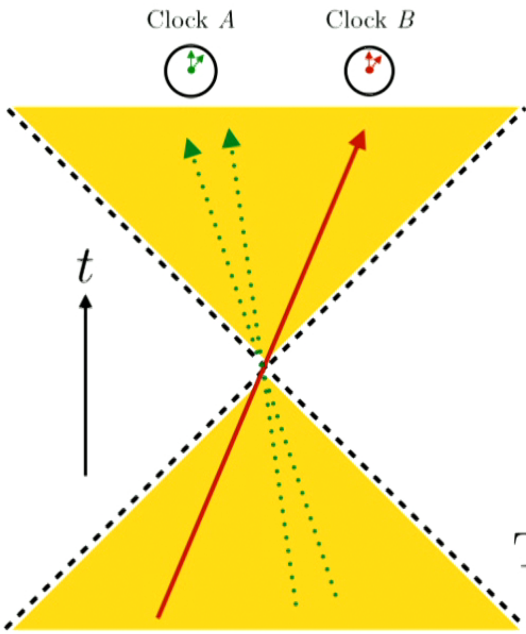
Then the average time read by clock A is

$$\langle T_A \rangle = \left[1 - \frac{\bar{\mathbf{p}}_A^2 - \bar{\mathbf{p}}_B^2}{2m^2c^2}\right] \tau_B + \cdots$$

Agrees with the proper time dilation observed by classical clocks

$$\tau_A = \frac{\gamma_B}{\gamma_A}\tau_B = \left[1 - \frac{\bar{\mathbf{p}}_A^2 - \bar{\mathbf{p}}_B^2}{2m^2c^2}\right] \tau_B + \cdots$$

Classical and quantum time dilation effects



Consider a momentum superposition

$$|\psi_A^{\text{ext}}\rangle = \frac{1}{\sqrt{N}}\Big(\cos\theta\,|\bar{\mathbf{p}}_A\rangle + \sin\theta e^{i\phi}\,|\bar{\mathbf{p}}'_A\rangle\Big)$$

Then the average time read by clock A is

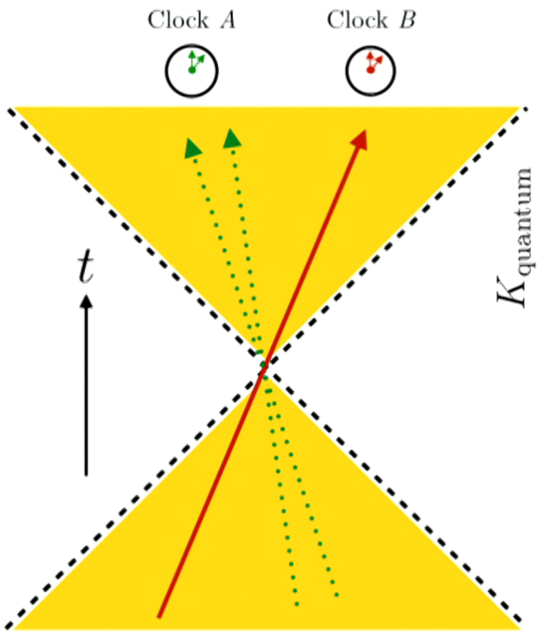
$$\langle T_A\rangle = \Bigg(1 - \underbrace{\frac{\langle H_A^{\text{ext}}\rangle - \langle H_B^{\text{ext}}\rangle}{mc^2}}_{K_{\text{classical}} + K_{\text{quantum}}}\Bigg)\tau_B + \cdots$$

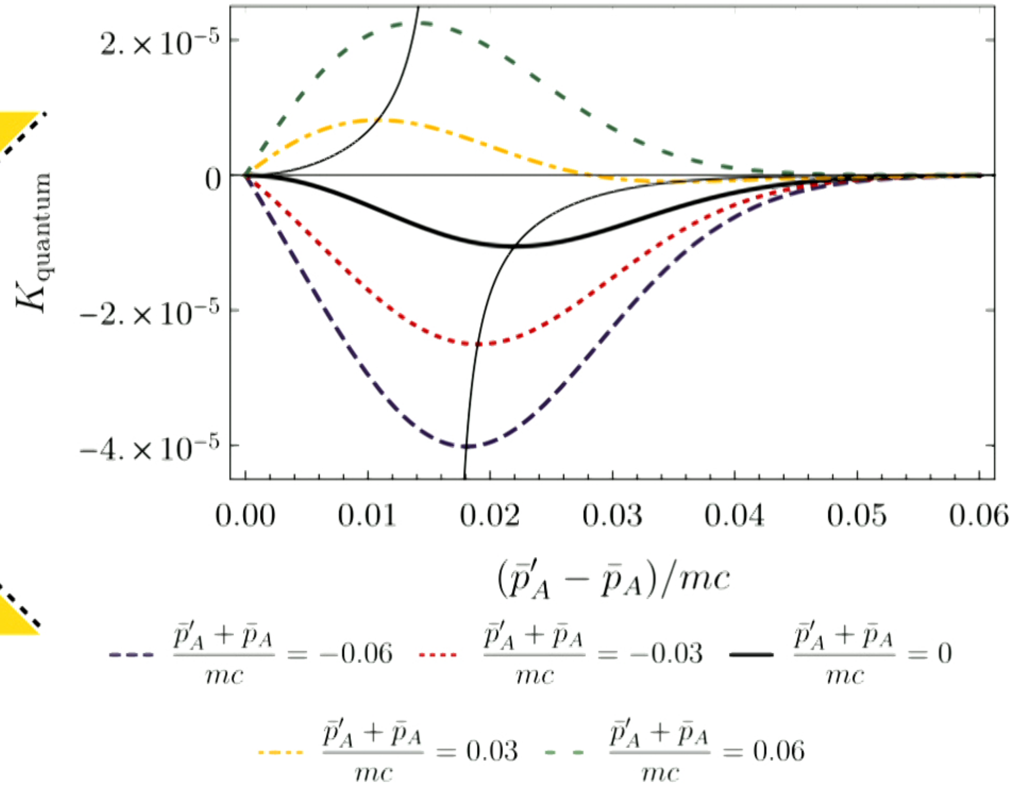
The classical and quantum contributions are

$$K_{\text{classical}} = \frac{\bar{\mathbf{p}}_A^2\cos^2\theta + \bar{\mathbf{p}}_A'^2\sin^2\theta - \bar{\mathbf{p}}_B^2}{2m^2c^2} + \frac{\Delta_A^2 - \Delta_B^2}{4m^2c^2}$$

$$K_{\text{quantum}} = \frac{\sin 2\theta\cos\phi}{8m^2c^2N}e^{-\frac{(\bar{\mathbf{p}}'_A - \bar{\mathbf{p}}_A)^2}{4\Delta_A^2}}\left[2\left(\bar{\mathbf{p}}_A'^2 - \bar{\mathbf{p}}_A^2\right)\cos 2\theta - \left(\bar{\mathbf{p}}'_A - \bar{\mathbf{p}}_A\right)^2\right]$$

Quantum time dilation effects





$$K_{\text{quantum}} = \frac{\sin 2\theta \cos\phi}{8m^2c^2N} e^{-\frac{(\bar{\mathbf{p}}'_A - \bar{\mathbf{p}}_A)^2}{4\Delta_A^2}} \left[2\left(\bar{\mathbf{p}}'^2_A - \bar{\mathbf{p}}^2_A\right)\cos 2\theta - \left(\bar{\mathbf{p}}'_A - \bar{\mathbf{p}}_A\right)^2\right]$$

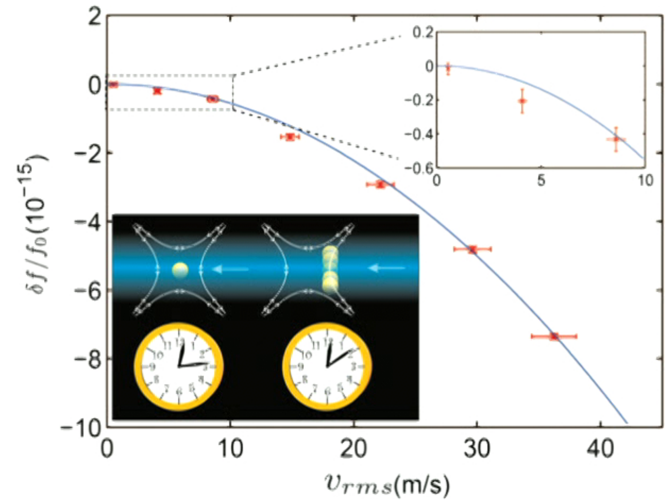
Is quantum time dilation observable?

1. Time dilation observed for clocks moving several meters per second

C. W. Chou, D. B. Hume, T. Rosenband, D. J. Wineland, *Optical Clocks and Relativity*, Science 329, 1630 (2010)

2. Superpositions of atoms moving at speeds that differ by several meters per second has been realize

P. R. Berman, Atom Interferometry (Academic Press, 1997)
P. Cladé, S. Guellati-Khélifa, F. Nez, and F. Biraben, Phys. Rev. Lett.102, 240402 (2009)



3. An estimate of the quantum contribution is $K_{\text{quantum}} \approx 10^{-15}$

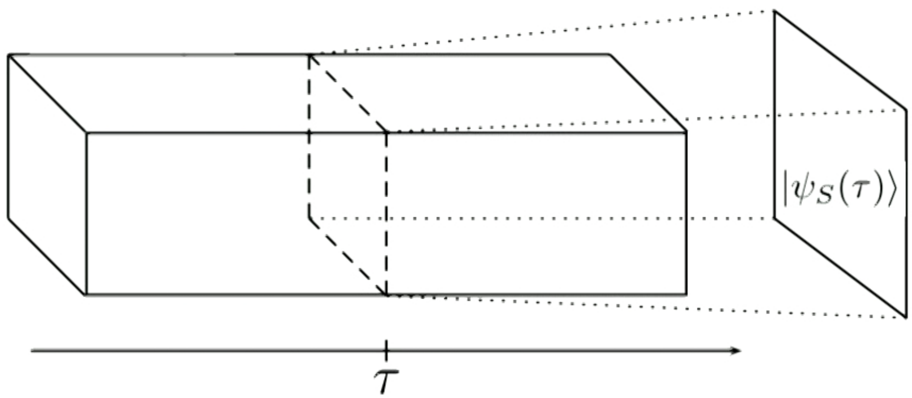
4. Given the resolution of atomic clocks, the coherence time of the superposition must be ~10 seconds to observe quantum time dilation

T. Kovachy, P. Asenbaum, C. Overstreet, C. A. Donnelly, S. M. Dickerson, A. Sugarbaker, J. M. Hogan and M. A. Kasevich, *Quantum superposition at the half-metre scale*, Nature 528, 530–533 (2015)

It appears that quantum time dilation may be observable with present day technology

Conditional probabilities and causality

$$|\,\Psi\rangle\rangle = \left(\int d\tau\ |\tau\rangle\langle\tau| \otimes I_S\right)|\,\Psi\rangle\rangle = \int d\tau\ |\tau\rangle\,|\psi_S(\tau)\rangle$$



V. Giovannetti, S. Lloyd, and L. Maccone, *Quantum time*, Phys. Rev. D 92, 045022 (2015)

Why formulate quantum theory this way?

'One motivation for considering such a "condensation" of history is the desire for economy as regards the number of basic elements of the theory: quantum correlations are an integral part of quantum theory already; so one is not adding a new element to the theory. And yet an old element, time, is being eliminated, becoming a secondary and even approximate concept.'

W. K. Wootters, *"Time" Replaced by Quantum Correlations* (1984)

Conditional probabilities

Dynamics from the Born rule

The probability of the event of $B = b$ when $T = \tau_2$ conditioned on $A = a$ when $T = \tau_1$ is

$$\text{Prob}\left(B = b \text{ when } T_C = \tau_2 \;\middle|\; A = a \text{ when } T_C = \tau_1\right)$$

$$= \frac{\langle\langle\Psi|\, F_{\Pi_a,T}(\tau_1) \cdot F_{\Pi_b,T}(\tau_2) \cdot F_{\Pi_a,T}(\tau_1)\; |\Psi\rangle\rangle_{\text{phys}}}{\langle\langle\Psi|\, F_{\Pi_a,T}(\tau_1)\, |\Psi\rangle\rangle_{\text{phys}}}$$

$$= \left|\, \langle\psi_S^b(\tau_2)|\; U_S(\tau_2 - \tau_1)\; |\psi_S^a(\tau_1)\rangle \,\right|^2$$

Time and interpretations of quantum gravity
K. Kuchar, Int. J. Mod. Phys. D 20, 3 (2011)

The Trinity of Relational Quantum Dynamics
P. A. Höhn, A. R. H. Smith and M. P. E. Lock, arXiv:1912.00033 [quant-ph] (2019)

Generalized probability rules from a timeless formulation of Wigner's friend scenarios
V. Baumann, F. Del Santo, A. R. H. Smith, F. Giacomini, E. Castro-Ruiz, and Č. Brukner arXiv:1911.09696 [quant-ph] (2019)

Causal structure from conditional probabilities

The **cause** ($\textsc{Event}_1$) is likely to manifest the **effect** ($\textsc{Event}_2$) if the cause happening **raises the probability** of the effect compared to the cause not happening:

$$\mathrm{Prob}\left[\textsc{Event}_2 \,\middle|\, \textsc{Event}_1\right] > \mathrm{Prob}\left[\textsc{Event}_2 \,\middle|\, \neg\textsc{Event}_1\right]$$

Hitchcock, Christopher, *Probabilistic Causation*, The Stanford Encyclopedia of Philosophy (Fall 2018 Edition)

Computed from physical state and relational Dirac observables:

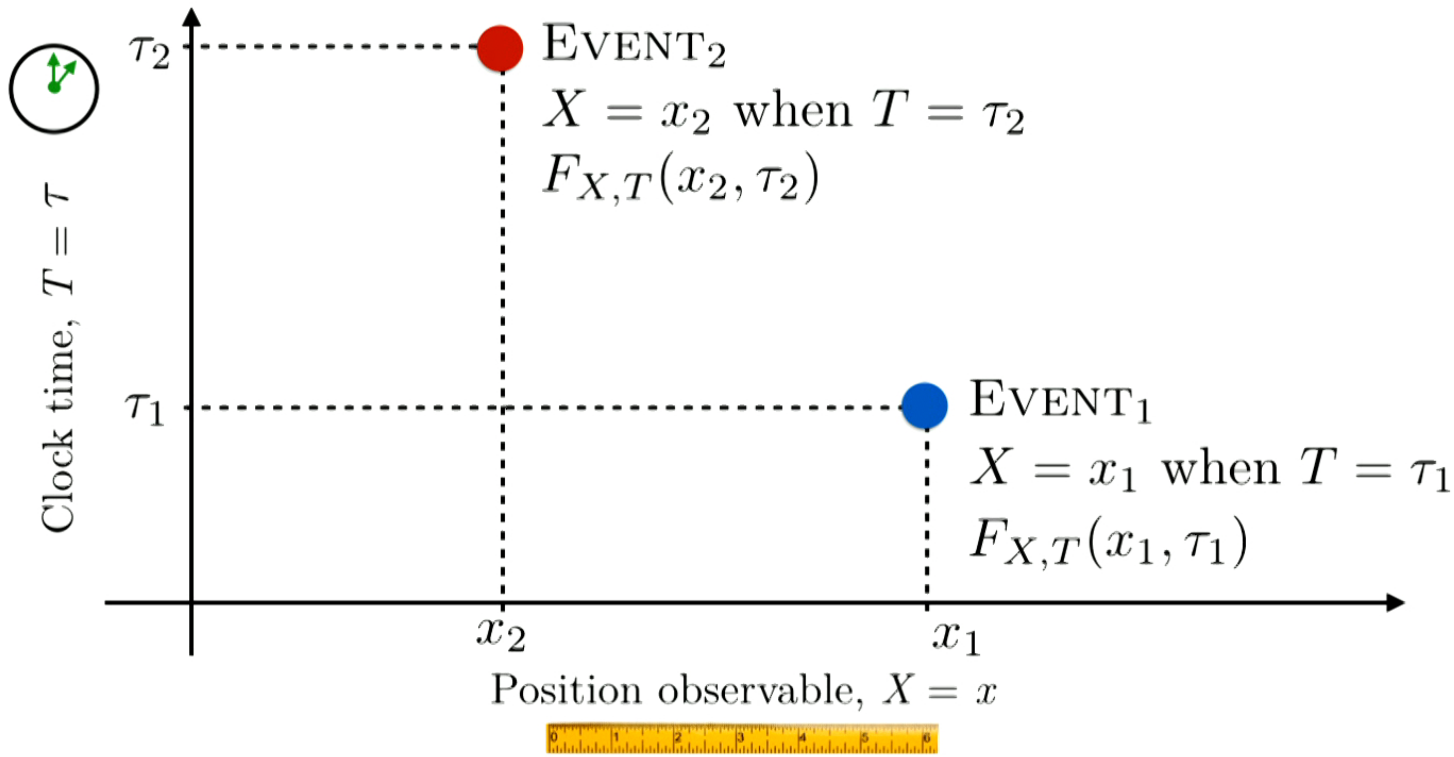
$$\mathrm{Prob}\left(B = b \text{ when } T_C = \tau_2 \,\middle|\, A = a \text{ when } T_C = \tau_1\right)$$
$$= \frac{\langle\langle\Psi|\, F_{\Pi_a,T}(\tau_1) \cdot F_{\Pi_b,T}(\tau_2) \cdot F_{\Pi_a,T}(\tau_1)\, |\Psi\rangle\rangle_{\text{phys}}}{\langle\langle\Psi|\, F_{\Pi_a,T}(\tau_1)\, |\Psi\rangle\rangle_{\text{phys}}}$$

$$C_H\, |\Psi\rangle\rangle = 0$$

Physics is here!

Causal structure from conditional probabilities

$$\text{Prob}\left[\textsc{Event}_2 \,\middle|\, \textsc{Event}_1\right] > \text{Prob}\left[\textsc{Event}_2 \,\middle|\, \neg\textsc{Event}_1\right]$$



Emergent causal structure from conditional probabilities
A. R. H. Smith, in preparation (2020)

Summary

1. **What is a clock?**
   A POVM that is covariant with respect to the clock Hamiltonian
2. **The trinity of relational quantum dynamics**
   Evolving constants of motion = PW = symmetry reduction
3. **Changing temporal reference frames**
   Temporal superposition / superposition of time evolutions
4. **Interacting clocks and systems**
   Time nonlocal modified Schrödinger equation
5. **Relativistic clocks and quantum time dilation**
   Relativistic quantum time dilation
6. **Causal structure from conditional probabilities**
   Probability raising
7. **Outlook and next steps**
   - Experimental proposal for quantum time dilation
   - Causal structure
   - Quantum field theory and quantum gravity

Smith and Ahmadi, Quantum 3, 160 (2019)
Smith and Ahmadi, arXiv:1904.12390 [quant-ph] (2019)
Baumann, Del Santo, Smith, Giacomini, Castro-Ruiz, and Brukner arXiv:1911.09696 [quant-ph] (2019)
Höhn, Smith, and Lock, arXiv:1912.00033 [quant-ph] (2019)