Title: Exact bosonization in all dimensions and the duality between supercohomology fermionic SPT and higher-group bosonic SPT phases

Speakers: Yu-An Chen

Series: Condensed Matter

Date: December 03, 2019 - 3:30 PM

URL: http://pirsa.org/19120017

Abstract: The first part of this talk will introduce generalized Jordanâ€“Wigner transformation on arbitrary triangulation of any simply connected manifold in 2d, 3d and general dimensions. This gives a duality between all fermionic systems and a new class of Z2 lattice gauge theories. This map preserves the locality and has an explicit dependence on the second Stiefelâ€“Whitney class and a choice of spin structure on the manifold. In the Euclidean picture, this mapping is exactly equivalent to introducing topological terms (Chern-Simon term in 2d or the Steenrod square term in general) to the Euclidean action. We can increase the code distance of this mapping, such that this mapping can correct all 1-qubit and 2-qubits errors and is useful for the simulation of fermions on the quantum computer. The second part of my talk is about SPT phases. By the boson-fermion duality, we are able to show the equivalent between any supercohomology fermionic SPT and some higher-group bosonic SPT phases. Particularly in (3+1)D, we have constructed a unitary quantum circuit for any supercohomology fermionic SPT state with gapped boundary construction. This fermionic SPT state is derived by gauging higher-form Z2 symmetry in the higher-group bosonic SPT and apply the boson-fermion duality.The first part of this talk will introduce generalized Jordanâ€“Wigner transformation on arbitrary triangulation of any simply connected manifold in 2d, 3d and general dimensions. This gives a duality between all fermionic systems and a new class of Z2 lattice gauge theories. This map preserves the locality and has an explicit dependence on the second Stiefelâ€“Whitney class and a choice of spin structure on the manifold. In the Euclidean picture, this mapping is exactly equivalent to introducing topological terms (Chern-Simon term in 2d or the Steenrod square term in general) to the Euclidean action. We can increase the code distance of this mapping, such that this mapping can correct all 1-qubit and 2-qubits errors and is useful for

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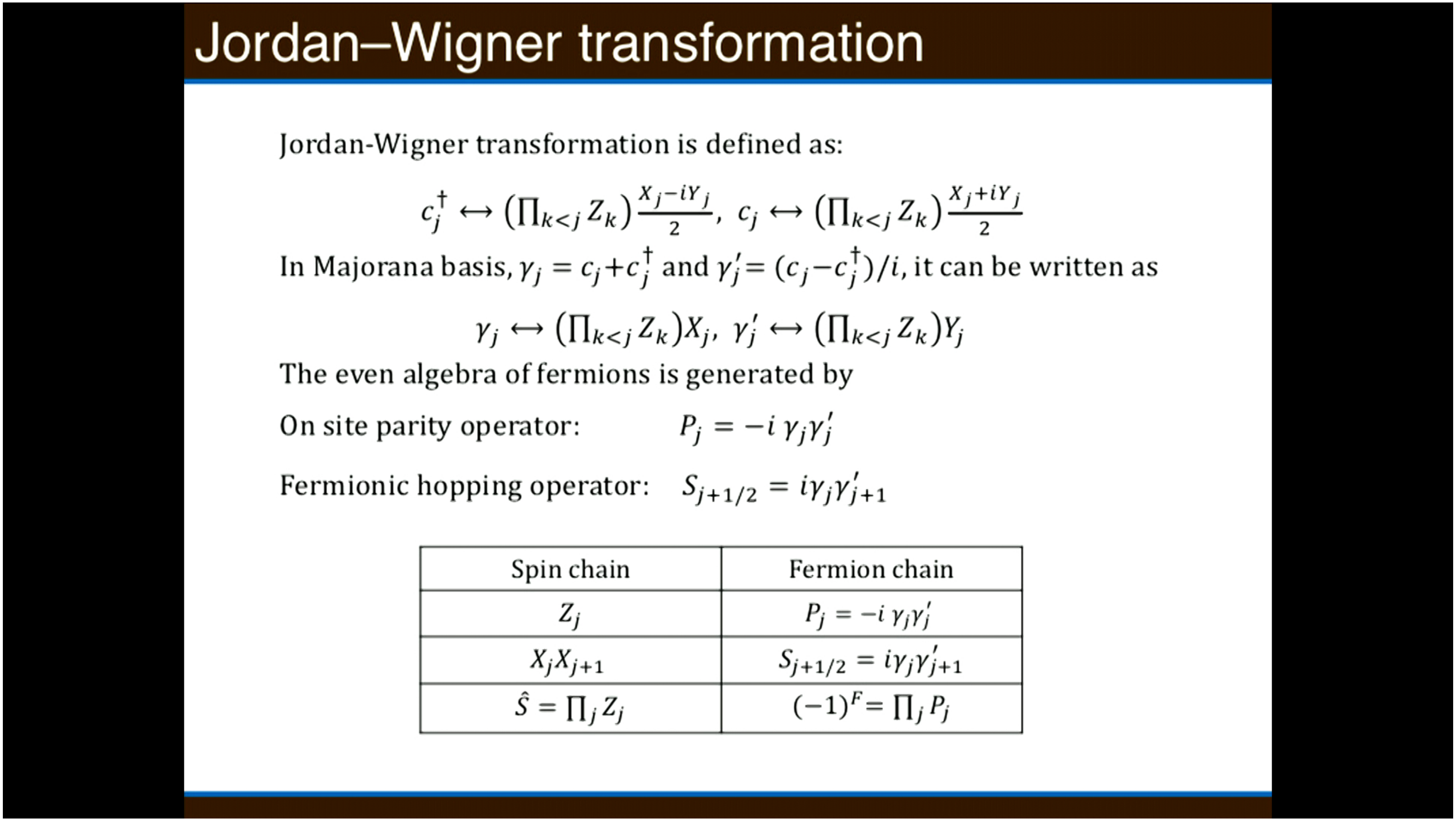
1d spin/fermion chains

	Spin chain	Fermion chain
Hilbert space	$V = \otimes_{j=1}^{N} V_j$, where $V_j \simeq \mathbb{C}^2$	$W = \hat{\otimes} W_j$, where $W_j \simeq \mathbb{C}^2$
Observables	Pauli matrices X_j, Y_j, Z_j	Even algebra formed by $c_j, c_j^\dagger$
Hamiltonian	$H^B = \sum_j H_j^B$, where H_j^B has finite support	$H^F = \sum_j H_j^F$, where H_j^F has finite support
Symmetry	$\mathbb{Z}_2$ symmetry $S = \prod_j Z_j$	$\mathbb{Z}_2$ fermion parity $\prod_j (-1)^{c_j^\dagger c_j}$



Jordan–Wigner transformation

Jordan-Wigner transformation is defined as:

$$c_j^\dagger \leftrightarrow \left(\prod_{k<j} Z_k\right)\frac{X_j - iY_j}{2},\ \ c_j \leftrightarrow \left(\prod_{k<j} Z_k\right)\frac{X_j + iY_j}{2}$$

In Majorana basis, $\gamma_j = c_j + c_j^\dagger$ and $\gamma_j' = (c_j - c_j^\dagger)/i$, it can be written as

$$\gamma_j \leftrightarrow \left(\prod_{k<j} Z_k\right) X_j,\ \ \gamma_j' \leftrightarrow \left(\prod_{k<j} Z_k\right) Y_j$$

The even algebra of fermions is generated by

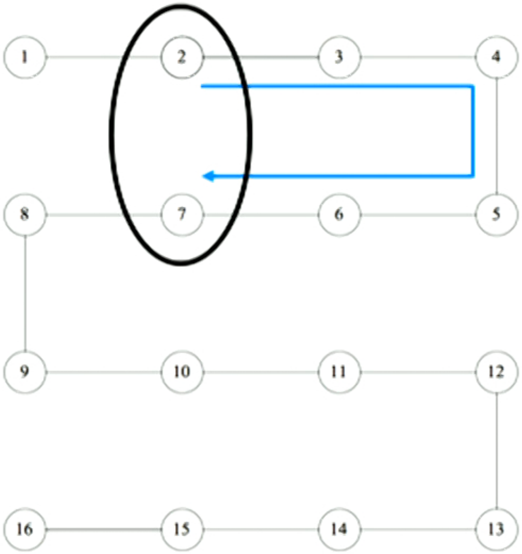
On site parity operator: $P_j = -i\,\gamma_j \gamma_j'$

Fermionic hopping operator: $S_{j+1/2} = i\gamma_j \gamma_{j+1}'$

Spin chain	Fermion chain
Z_j	$P_j = -i\,\gamma_j\gamma_j'$
$X_j X_{j+1}$	$S_{j+1/2} = i\gamma_j\gamma_{j+1}'$
$\hat{S} = \prod_j Z_j$	$(-1)^F = \prod_j P_j$

Jordan–Wigner transformation

$$c_j^\dagger c_i \leftrightarrow \left(\prod_{i<k<j} Z_k\right)\left(\frac{X_j - iY_j}{2}\right)\left(\frac{X_i + iY_i}{2}\right)$$



$$c_7^\dagger c_2 \sim Z_3 Z_4 Z_5 Z_6 \left(\frac{X_2 - iY_2}{2}\right)\left(\frac{X_7 + iY_7}{2}\right)$$

In 2d, local fermionic hopping terms may give nonlocal bosonic operators.

Bosonization map in 2d

Consider a fermionic system on an arbitrary 2d lattice L, with a fermion on each face.

The main result:

Fermionic observables ⟷ $\mathbb{Z}_2$ gauge theory (modified gauge constraints)

- The map only depends on the branching structure of the lattice. Spin structure appear automatically when we require the map to be self-consistent
- The Euclidean path integral can be perform on modified $\mathbb{Z}_2$ gauge theory exactly and Chern-Simons term appears in the action
- The formalism has a direct generalization to higher dimensions

Standard $\mathbb{Z}_2$ gauge theories

A $\mathbb{Z}_2$ gauge theory on lattice L has a spin at each edge e. Let X_e, Y_e, Z_e be Pauli matrices acting on edge.

The Hilbert space of a $\mathbb{Z}_2$ gauge theory is constrained by G_v:

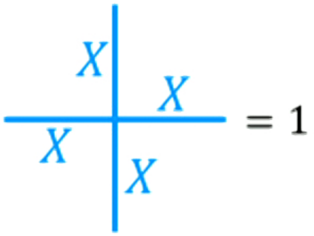
$$\mathcal{H} = \{|\Psi\rangle \mid G_v|\Psi\rangle = |\Psi\rangle \ \forall v\}$$

where v is an vertex and G_v is gauge constraint, satisfying $G_v^2 = 1$ and $G_v G_{v'} = G_{v'} G_v$.

For example, the standard $\mathbb{Z}_2$ gauge theory is

$$G_v = \prod\nolimits_{e \supset v} X_e$$

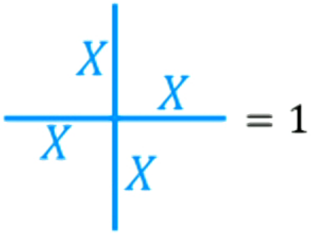
which requires no electric charge excitation at any vertex v.



Kramers-Wannier duality

Under gauge constraints

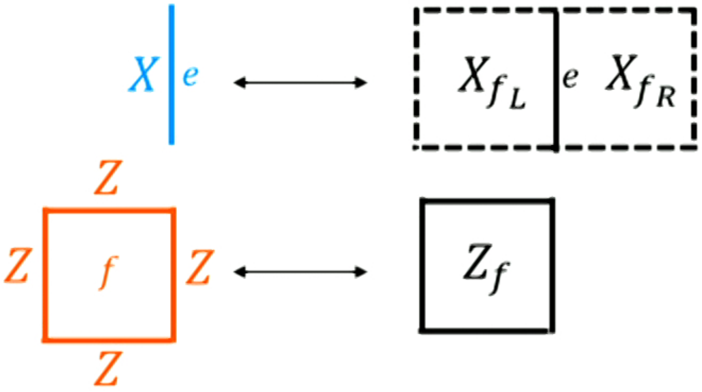
$$G_v = \prod_{e \supset v} X_e \,,$$



The gauge-invariant observables are generated by

$$X_e, W_f \equiv \prod_{e \subset f} Z_e \,.$$

$\mathbb{Z}_2$ gauge theory	Spin model
X_e	$\prod_{f \supset e} \hat{X}_f$
$W_f \equiv \prod_{e \subset f} Z_e$	$\hat{Z}_f$
$G_v = \prod_{e \supset v} X_e$	1
1	$\hat{S} = \prod_f \hat{Z}_f$

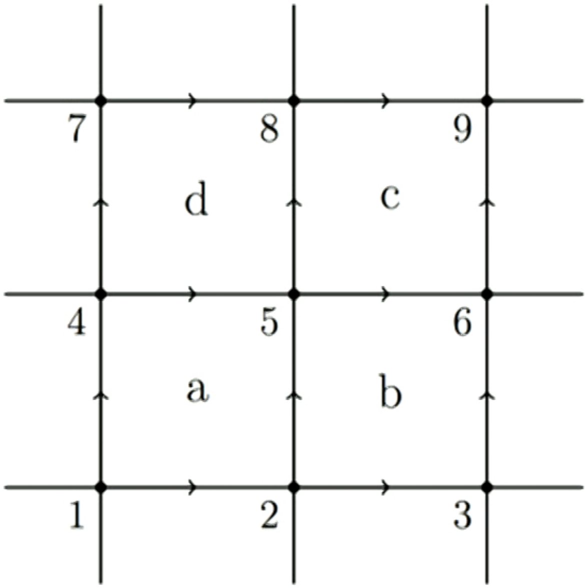


This is known as 2d Kramers-Wannier duality. We can check the dimensions of Hilbert spaces are the same (on genus 0 surface).

$\mathbb{Z}_2$ gauge theory: $\dim \mathcal{H} = 2^{N_e-(N_v-1)} = 2^{N_f-1}$

Spin model: $\dim \mathcal{H} = 2^{N_f-1}$

Bosonization on 2d square lattice



Fermionic operators (at faces):

$$\left\{c_f, c_{f'}^\dagger\right\} = \delta_{ff'}.$$

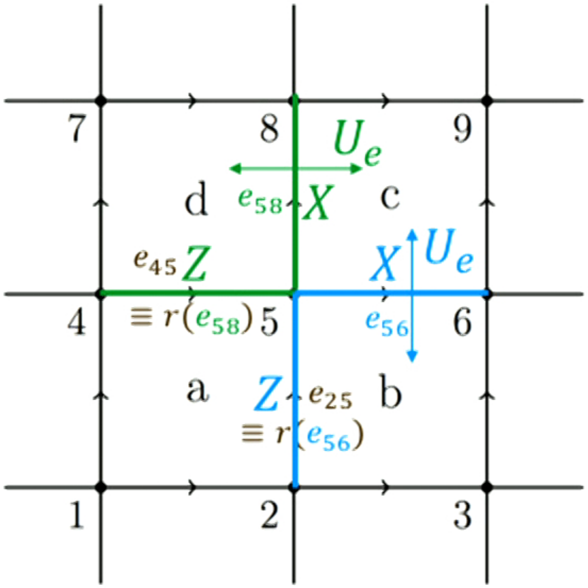
In Majorana basis, the even algebra of fermions is generated by

on site parity: $P_f = -i\,\gamma_f \gamma_f'$

fermionic hopping: $S_e = i\gamma_{L(e)}\gamma_{R(e)}'$

Goal: we want to construct bosonic operators by Pauli matrices X_e, Y_e, Z_e at edges, which have the same algebra as P_f and $S_e = i\gamma_{L(e)}\gamma_{R(e)}'$.

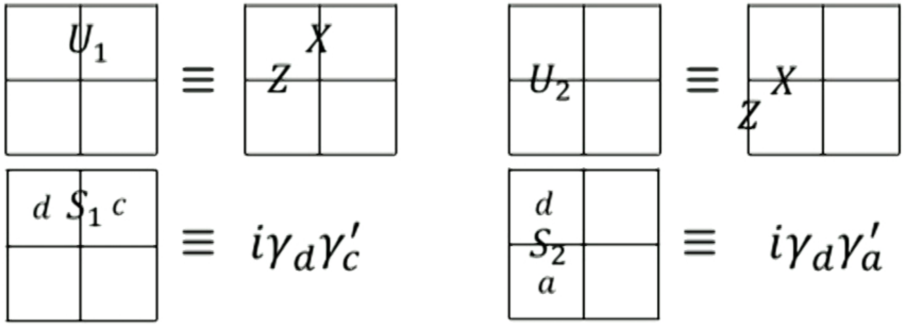
Bosonization on 2d square lattice



Bosonization map:

$\mathbb{Z}_2$ theory	Fermionic theory
$U_e \equiv X_e Z_{r(e)}$	$S_e = i\gamma_{L(e)}\gamma'_{R(e)}$
$W_f \equiv \prod_{e\subset f} Z_e$	$P_f = -i\,\gamma_f\gamma'_f$

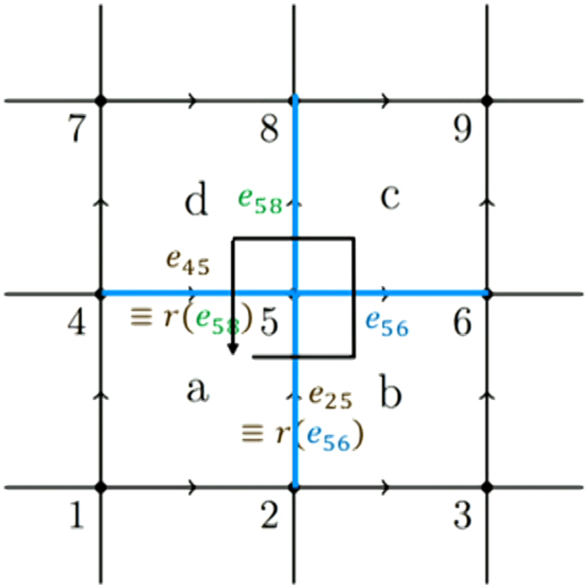
This map preserves the commutation relations.



$$U_1U_2 = -U_2U_1$$

$$S_1S_2 = -S_2S_1$$

Gauge constrains



On fermionic side, the operators satisfy additional conditions:

$$P_a P_c S_{58} S_{56} S_{25} S_{45}$$

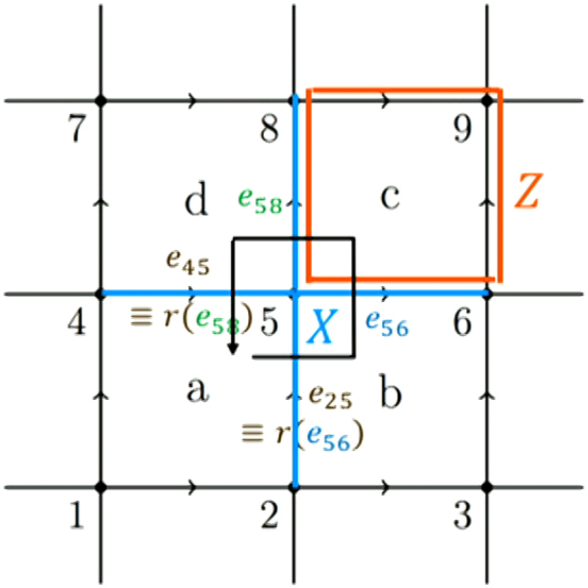
$$= \gamma_a^2 \gamma_a'^2 \gamma_b^2 \gamma_c^2 \gamma_c'^2 \gamma_d^2 = 1.$$

This is mapped to

$$G'_v = W_a W_c U_{58} U_{56} U_{25} U_{45}$$

$$= W_c X_{58} X_{56} X_{25} X_{45}.$$

Gauge constrains



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$$= \gamma_a^2 \gamma_a'^2 \gamma_b^2 \gamma_c^2 \gamma_c'^2 \gamma_d^2 = 1.$$

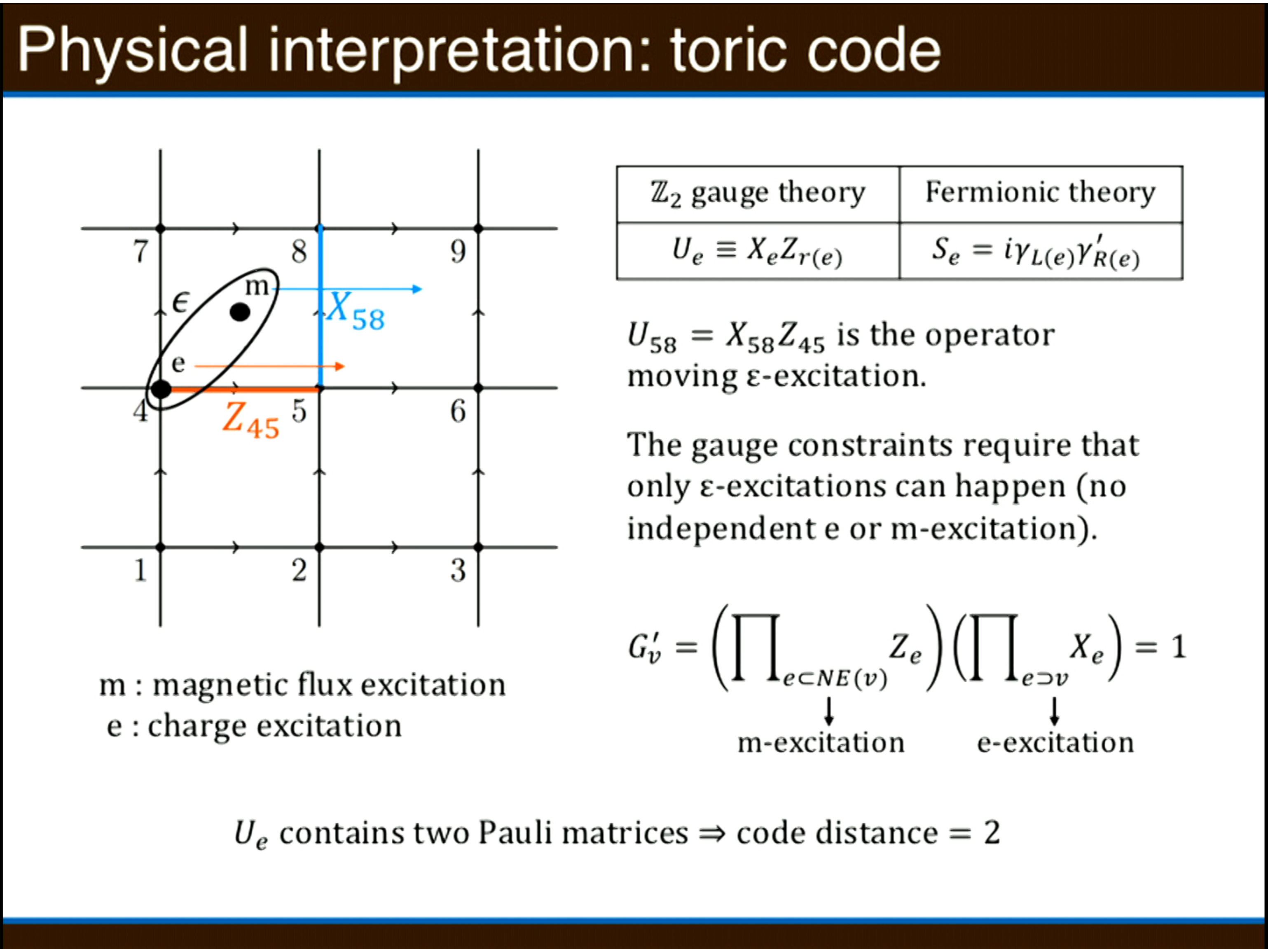
This is mapped to

$$G_v' = W_a W_c U_{58} U_{56} U_{25} U_{45}$$

$$= W_c X_{58} X_{56} X_{25} X_{45}.$$

To illustrate this operator,

$$G_v' = \left(\prod_{e \subset NE(v)} Z_e\right)\left(\prod_{e \supset v} X_e\right)$$



Physical interpretation: toric code

m : magnetic flux excitation
e : charge excitation

$\mathbb{Z}_2$ gauge theory	Fermionic theory
$U_e \equiv X_e Z_{r(e)}$	$S_e = i\gamma_{L(e)}\gamma'_{R(e)}$

$U_{58} = X_{58}Z_{45}$ is the operator moving ε-excitation.

The gauge constraints require that only ε-excitations can happen (no independent e or m-excitation).

$$G'_v = \Big(\underbrace{\prod\nolimits_{e\subset NE(v)} Z_e}_{\substack{\downarrow \\ \text{m-excitation}}}\Big)\Big(\underbrace{\prod\nolimits_{e\supset v} X_e}_{\substack{\downarrow \\ \text{e-excitation}}}\Big) = 1$$

U_e contains two Pauli matrices ⇒ code distance = 2

Increasing code distance

The bosonization works for any $\tilde{X}_e$ and $\tilde{Z}_e$ satisfying $\tilde{X}_e\tilde{Z}_{e'} = (-1)^{\delta_{ee'}}\tilde{Z}_{e'}\tilde{X}_e$

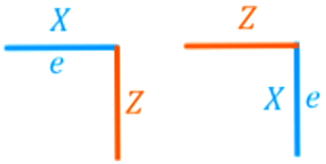
The simplest choice is:

$\tilde{X}_e$:

$\tilde{Z}_e$:

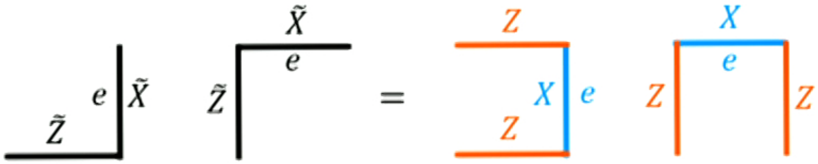
The next choice is

$\tilde{X}_e$:



$\tilde{Z}_e$:

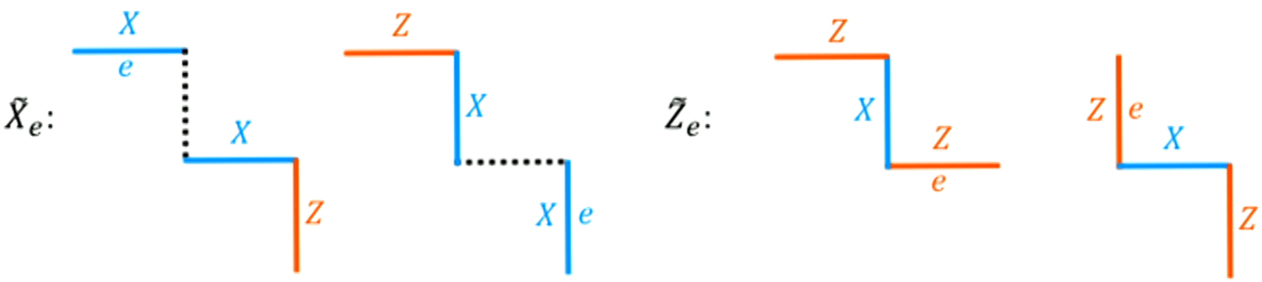
$U_e=\tilde{X}_e\tilde{Z}_{r(e)}$:



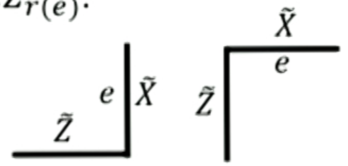
The code distance = 3 ⇒ all 1-qubit errors can be corrected.

Increasing code distance

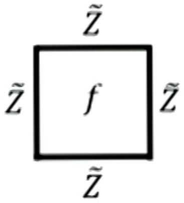
This process can be done recursively:



$U_e = \tilde{X}_e \tilde{Z}_{r(e)}$:



$W_f \equiv \prod_{e \subset f} \tilde{Z}_e$:

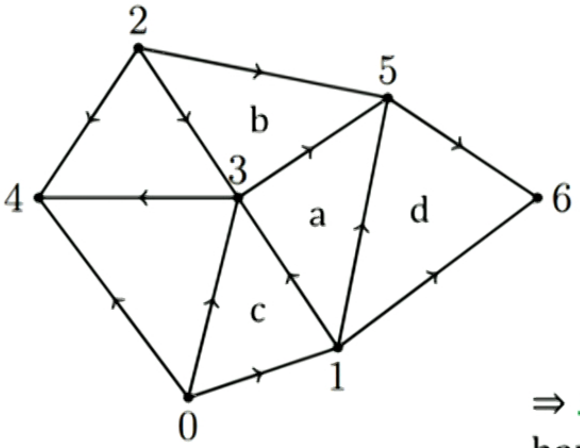


$\mathbb{Z}_2$ gauge theory	Fermionic theory
$U_e \equiv \hat{X}_e \hat{Z}_{r(e)}$	$S_e = i\gamma_{L(e)}\gamma'_{R(e)}$
$W_f \equiv \prod_{e \subset f} \hat{Z}_e$	$P_f = -i\,\gamma_f \gamma'_f$
G'_v	1
1	$(-1)^F = \prod_f P_f$

The code distance ≥ 5 (checked by computer) ⇒ all 2-qubit errors can be corrected.

Commutation relations on any triangulation

Generators: $P_f = -i\,\gamma_f \gamma'_f, \quad S_e = i\gamma_{L(e)}\gamma'_{R(e)}$



$[P_{f_1}, P_{f_2}] = 0$

$P_f S_e = (-1)^{\delta_{L(e),f} + \delta_{R(e),f}}\, S_e P_f$

$S_{e_1} S_{e_2} = s(e_1, e_2)\, S_{e_2} S_{e_1}$

What is the sign $s(e_1, e_2)$?

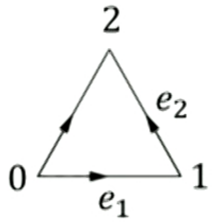
$S_{35} = i\gamma_b \gamma'_a \quad S_{13} = i\gamma_c \gamma'_a \quad S_{15} = i\gamma_a \gamma'_d$

$\{S_{35}, S_{13}\} = [S_{35}, S_{15}] = [S_{13}, S_{15}] = 0$

$\Rightarrow s(e_1, e_2) = -1$ iff e_1 and e_2 share a face and have the same orientation.

Cup product: $s(e_1, e_2) = (-1)^{\int e_1 \cup e_2 + e_2 \cup e_1}$

$e_1 \cup e_2 + e_2 \cup e_1\,(012) = e_1(01) e_2(12) + e_1(12) e_2(01)$



(Even) algebra of fermions

Generators: $P_f = -i\,\gamma_f \gamma_f'$, $\quad S_e = i\gamma_{L(e)}\gamma_{R(e)}'$

- P_f and S_e are idempotent (square to 1)
- $[P_{f_1}, P_{f_2}] = 0$
- $P_f S_e = (-1)^{\delta_{L(e),f} + \delta_{R(e),f}}\, S_e P_f$
- $S_{e_1} S_{e_2} = (-1)^{\int e_1 \cup e_2 + e_2 \cup e_1} S_{e_2} S_{e_1}$
- $(\prod'_{e \supset v} S_e)\left(\prod_f P_f^{\int v \cup f + f \cup v}\right) = (-1)^{\int_{w_2} v}$

$\downarrow$

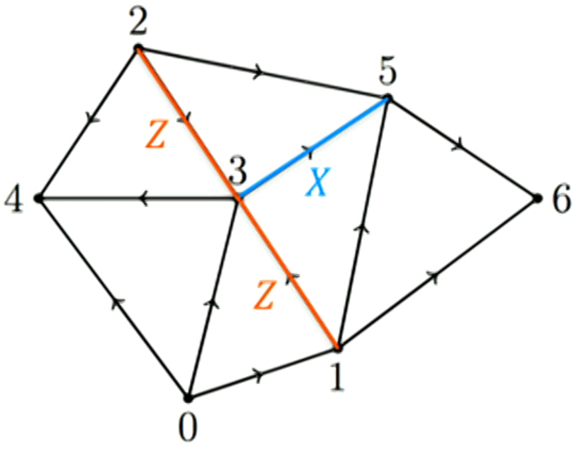
second Stiefel-Whitney class

Construction for bosonic operators

$$S_{e_1}S_{e_2} = (-1)^{\int e_1\cup e_2 + e_2\cup e_1} S_{e_2}S_{e_1}$$

We can construct $U_e \equiv X_e \prod_{e'} Z_{e'}^{\int e'\cup e}$, which satisfies the same commutation relations:

$$U_{e_1}U_{e_2} = (-1)^{\int e_1\cup e_2 + e_2\cup e_1} U_{e_2}U_{e_1}$$



For $e = e_{35}$, only nontrivial $e' \cup e$ is $e' = e_{12}$ and $e' = e_{13}$:

$$U_{35} = X_{35}Z_{23}Z_{13}$$

The fermion parity P_f correspond to

$$W_f \equiv \prod_{e\subset f} Z_e$$

$\{S_e, P_f\}$ and $\{U_e, W_f\}$ satisfy the same commutation relations.

Bosonization on any triangulation T

$\mathbb{Z}_2$ gauge theory　　　　Fermionic theory

$$W_f = \prod_{e \subset f} Z_e \longleftrightarrow P_f = -i\gamma_f \gamma'_f,$$

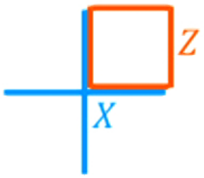
spin structure E:
$\partial E = w_2$

$$U_e = X_e(\prod_{e'} Z_{e'}^{\int \boldsymbol{e'} \cup \boldsymbol{e}}) \longleftrightarrow (-1)^{\int_E \boldsymbol{e}} S_e = (-1)^{\int_E \boldsymbol{e}} i\gamma_{L(e)}\gamma'_{R(e)},$$

$$G_v = \prod_{e \supset v} X_e(\prod_{e'} Z_{e'}^{\int \delta \boldsymbol{v} \cup \boldsymbol{e'}}) \longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{v}} S_{\delta \boldsymbol{v}} \prod_f P_f^{\int \boldsymbol{v} \cup \boldsymbol{f} + \boldsymbol{f} \cup \boldsymbol{v}} = 1,$$

$$\prod_f W_f = 1 \longleftrightarrow \prod_f P_f$$

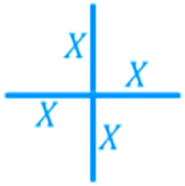
second Stiefel-Whitney class



Euclidean Action

For a Hamiltonian H in d-dimension, we can perform Euclidean path integral to get classical action in (d+1)-dimension. For standard $\mathbb{Z}_2$ lattice gauge theory in 2d

$$H = g^2 \sum_e X_e + \frac{1}{g^2} \sum_f W_f$$



$= 1$

with gauge constraints $G_v \equiv \prod_{e \supset v} X_e = 1$, the Euclidean action S is defined by:

$$Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, T^M = \int e^{-s}$$

where $T = \left(\prod_v \delta_{G_v,1}\right) e^{-\delta\tau H}$ is the transfer matrix. The corresponding classical action is 3D Ising gauge theories:

$$S[a] = \frac{1}{g'^2} \sum_f |\delta a(f)|$$

$$\big\downarrow$$

$a(e) \in \{0,1\}$ and $S_e \in \{\pm 1\}$
$a(e) = (1 - S_e)/2$

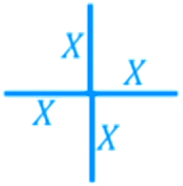
$$|\nabla \times a| = \left(1 - \prod_{e \subset f} S_e\right)/2$$

Kogut. Rev. Mod. Phys. 51, 659 (1979)

New lattice gauge theory

Standard $\mathbb{Z}_2$ lattice gauge theory

Hamiltonian: $H = g^2 \sum_e X_e + \frac{1}{g^2}\sum_f W_f$



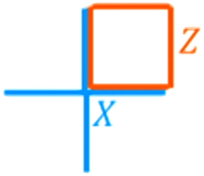
$= 1$

Gauge constraints: $G_v \equiv (\prod_{e \supset v} X_e)=1$

Action: $S[a] = \frac{1}{g'^2}\sum_f |\delta a(f)|$ $\quad (Z = \mathrm{Tr}\, e^{-\beta H} = \int e^{-S})$

New $\mathbb{Z}_2$ lattice gauge theory

Hamiltonian: $H' = g^2 \sum_e U_e + \frac{1}{g^2}\sum_f W_f$



$= 1$

Gauge constraints: $G'_v \equiv (\prod_{e \supset v} X_e)\left(\prod_{e'} Z_{e'}^{\int \delta v \cup e'}\right)=1$

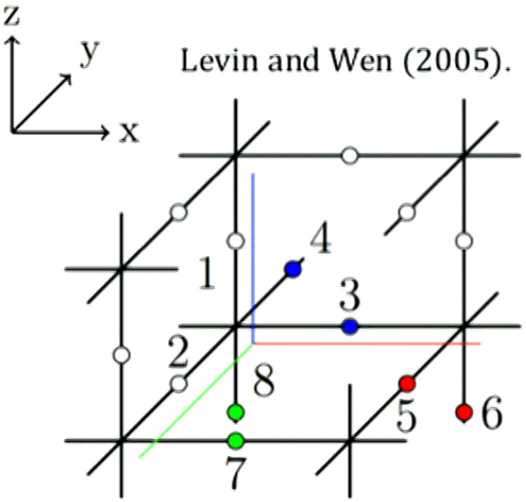
Action: $S'[a] = \frac{1}{g'^2}\sum_f |\delta a(f)| + i\pi \sum a \cup \delta a$ → Discrete version of Chern-Simons term $\int A \wedge dA$

H' is dual to $H^f = g^2 \sum_e i\gamma_{L(e)}\gamma'_{R(e)} + \frac{1}{g^2}\sum_f(-i\,\gamma_f\gamma'_f)$

This exact bosonization map is analogous to "particle-vortex duality"

$$Z_{\text{scalar+flux}}[A]\, e^{\frac{i}{2}S_{CS}[A]} = Z_{\text{fermion}}[A]$$

Bosonization on 3d cubic lattice



Fermionic operators:

$$P_c = -i\,\gamma_c\gamma_c', \quad S_f = i\gamma_{L(f)}\gamma'_{R(f)}$$

Bosonic operators:

$$W_c = \prod_{f\subset c} Z_f, \quad U_f$$

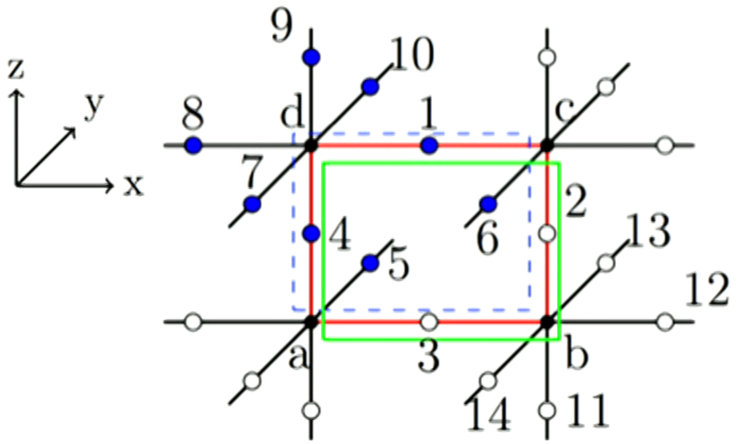
where

$$U_1 = X_1 Z_3 Z_4$$
$$U_2 = X_2 Z_7 Z_8$$
$$U_3 = X_1 Z_5 Z_6$$

$\{P_c, S_f\}$ and $\{W_c, U_f\}$ satisfy the same commutation relations.

$\mathbb{Z}_2$ theory	Fermionic theory
U_f	$S_f = i\gamma_{L(f)}\gamma'_{R(f)}$
$W_c = \prod_{f\subset c} Z_f$	$P_c = -i\,\gamma_c\gamma_c'$

Gauge constraints



On fermionic side, the operators satisfy additional conditions:

$$-S_1 S_2 S_3 S_4 P_b P_d = 1.$$

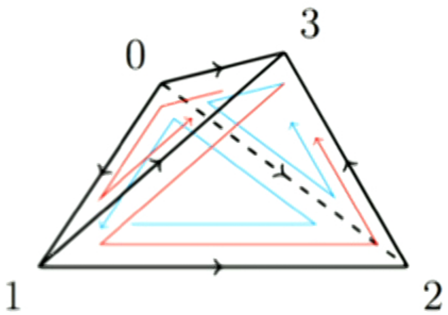
We need to impose the constraints on bosonic operators:

$$G'_e = X_1 X_2 X_3 X_4 Z_1 Z_4 Z_5 Z_6 Z_7 Z_8 Z_9 Z_{10} = 1$$

$\mathbb{Z}_2$ gauge theory	Fermionic theory
U_f	$S_f = i\gamma_{L(f)}\gamma'_{R(f)}$
$W_c = \prod_{f\subset c} Z_f$	$P_c = -i\ \gamma_c\gamma'_c$
G'_e	1
1	$(-1)^F = \prod_c P_c$

Commutation relations of fermions

Generators: $P_t = -i\,\gamma_t \gamma'_t, \quad S_f = i\gamma_{L(f)}\gamma'_{R(f)}$



$$[P_{t_1}, P_{t_2}] = 0$$

$$P_t S_f = (-1)^{\delta_{L(f),t}+\delta_{R(f),t}} S_f P_t$$

$$S_{f_1} S_{f_2} = s(f_1, f_2)\, S_{f_2} S_{f_1}$$

$s(f_1, f_2) = -1$ iff f_1 and f_2 are both inward or outward faces of a tetrahedron.

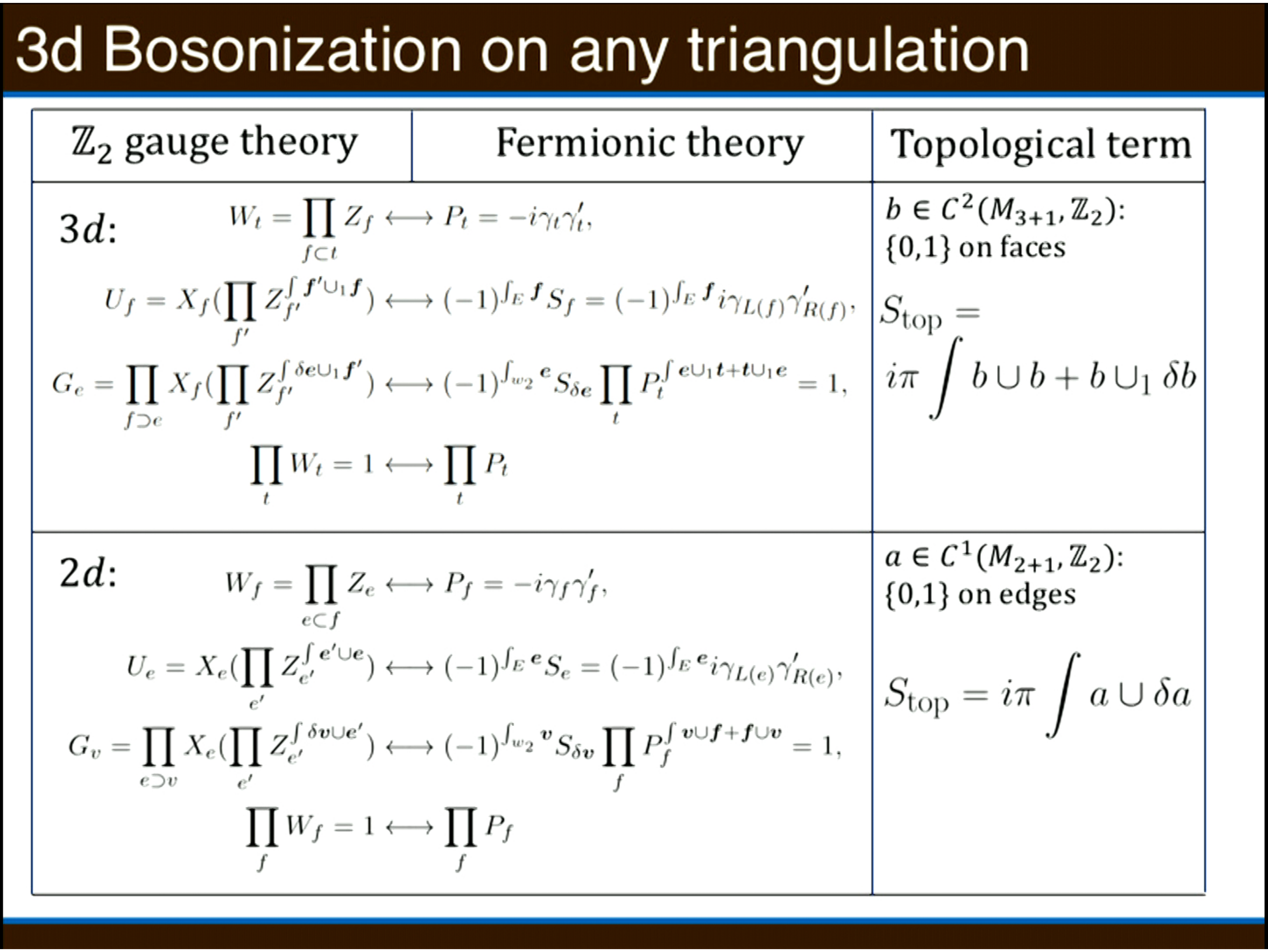
The higher cup product $\cup_1$ of two 2-cochains f_1 and f_2 is defined by

$$f_1 \cup_1 f_2(0123) = f_1(023) f_2(012) + f_1(013) f_2(123)$$

We can write the commutation relation as:

$$s(f_1, f_2) = (-1)^{\int f_1 \cup_1 f_2 + f_2 \cup_1 f_1}$$

We can construct $U_f \equiv X_f \prod_{f'} Z_{f'}^{\int f' \cup_1 f}$, satisfying the same commutation relations.



3d Bosonization on any triangulation

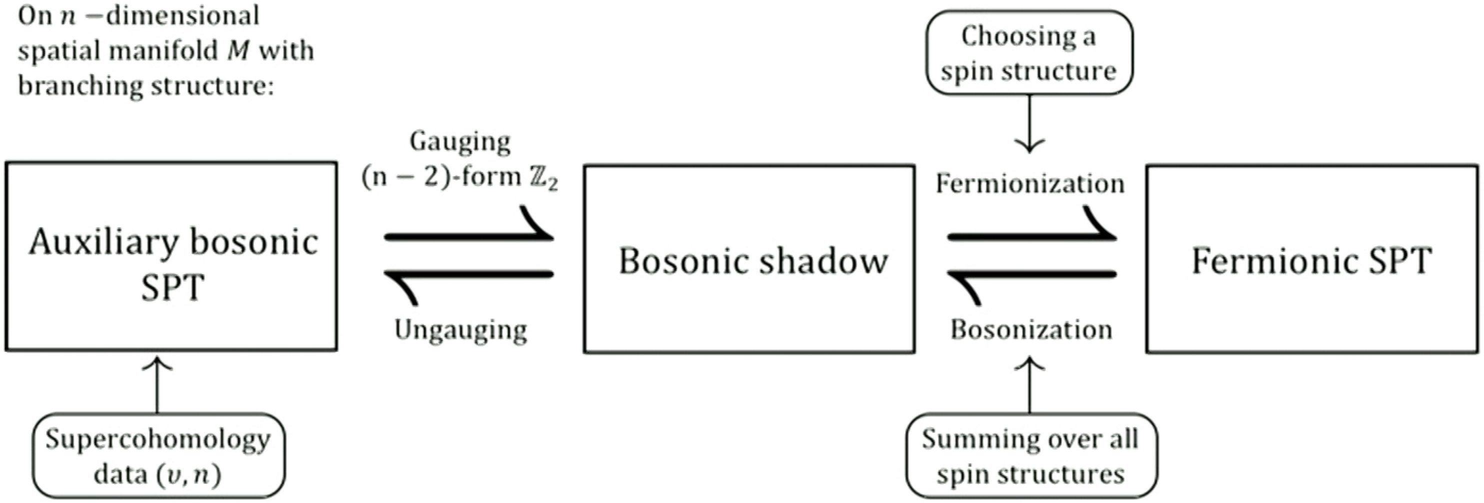
<table>
<tr><th>$\mathbb{Z}_2$ gauge theory</th><th>Fermionic theory</th><th>Topological term</th></tr>
<tr><td colspan="2">3d:
$W_t = \prod_{f \subset t} Z_f \longleftrightarrow P_t = -i\gamma_t\gamma'_t,$
$U_f = X_f(\prod_{f'} Z_{f'}^{\int \boldsymbol{f'} \cup_1 \boldsymbol{f}}) \longleftrightarrow (-1)^{\int_E \boldsymbol{f}} S_f = (-1)^{\int_E \boldsymbol{f}} i\gamma_{L(f)}\gamma'_{R(f)},$
$G_e = \prod_{f \supset e} X_f(\prod_{f'} Z_{f'}^{\int \delta \boldsymbol{e} \cup_1 \boldsymbol{f'}}) \longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{e}} S_{\delta \boldsymbol{e}} \prod_t P_t^{\int \boldsymbol{e} \cup_1 \boldsymbol{t} + \boldsymbol{t} \cup_1 \boldsymbol{e}} = 1,$
$\prod_t W_t = 1 \longleftrightarrow \prod_t P_t$</td><td>$b \in C^2(M_{3+1}, \mathbb{Z}_2)$:
{0,1} on faces
$S_{\text{top}} =$
$i\pi \int b \cup b + b \cup_1 \delta b$</td></tr>
<tr><td colspan="2">2d:
$W_f = \prod_{e \subset f} Z_e \longleftrightarrow P_f = -i\gamma_f\gamma'_f,$
$U_e = X_e(\prod_{e'} Z_{e'}^{\int \boldsymbol{e'} \cup \boldsymbol{e}}) \longleftrightarrow (-1)^{\int_E \boldsymbol{e}} S_e = (-1)^{\int_E \boldsymbol{e}} i\gamma_{L(e)}\gamma'_{R(e)},$
$G_v = \prod_{e \supset v} X_e(\prod_{e'} Z_{e'}^{\int \delta \boldsymbol{v} \cup \boldsymbol{e'}}) \longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{v}} S_{\delta \boldsymbol{v}} \prod_f P_f^{\int \boldsymbol{v} \cup \boldsymbol{f} + \boldsymbol{f} \cup \boldsymbol{v}} = 1,$
$\prod_f W_f = 1 \longleftrightarrow \prod_f P_f$</td><td>$a \in C^1(M_{2+1}, \mathbb{Z}_2)$:
{0,1} on edges
$S_{\text{top}} = i\pi \int a \cup \delta a$</td></tr>
</table>

Bosonization in arbitrary dimensions

<table>
<tr><th>$\mathbb{Z}_2$ gauge theory</th><th>Fermionic theory</th><th>Topological term</th></tr>
<tr><td colspan="2">n spatial dimensions:
$W_{\Delta_n} \equiv \prod_{\Delta_{n-1} \subset \Delta_n} Z_{\Delta_{n-1}} \longleftrightarrow P_t = -i\gamma_{\Delta_n}\gamma'_{\Delta_n},$
$U_{\Delta_{n-1}} \equiv X_{\Delta_{n-1}} \left(\prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \mathbf{\Delta}_{n-1}' \cup_{n-2} \mathbf{\Delta}_{n-1}} \right)$
$\longleftrightarrow (-1)^{\int_E \mathbf{\Delta}_{n-1}} S_{\Delta_{n-1}} = (-1)^{\int_E \mathbf{\Delta}_{n-1}} i\gamma_{L(\Delta_{n-1})}\gamma'_{R(\Delta_{n-1})},$
$G_{\Delta_{n-2}} \equiv \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} \left(\prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta\mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_{n-1}'} \right)$
$\longleftrightarrow (-1)^{\int_{w_2} \mathbf{\Delta}_{n-2}} S_{\delta\mathbf{\Delta}_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_n + \mathbf{\Delta}_n \cup_{n-2} \mathbf{\Delta}_{n-2}} = 1,$
$\prod_{\Delta_n} W_{\Delta_n} = 1 \longleftrightarrow \prod_{\Delta_n} P_{\Delta_n}$</td><td>$(n+1)$-dimensional spacetime action
$A_{n-1} \in C^{n-1}(M_{n+1}, \mathbb{Z}_2)$: $\{0,1\}$ on every $(n-1)$-simplex
$S_{\text{top}} = i\pi \times \int_{M_{n+1}} (A_{n-1} \cup_{n-3} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1})$
which is Steenrod square topological term.</td></tr>
</table>

Duality between SPT phases

On n −dimensional spatial manifold M with branching structure:



Auxiliary bosonic SPT

Gauging (n − 2)-form $\mathbb{Z}_2$

Ungauging

Bosonic shadow

Fermionization

Bosonization

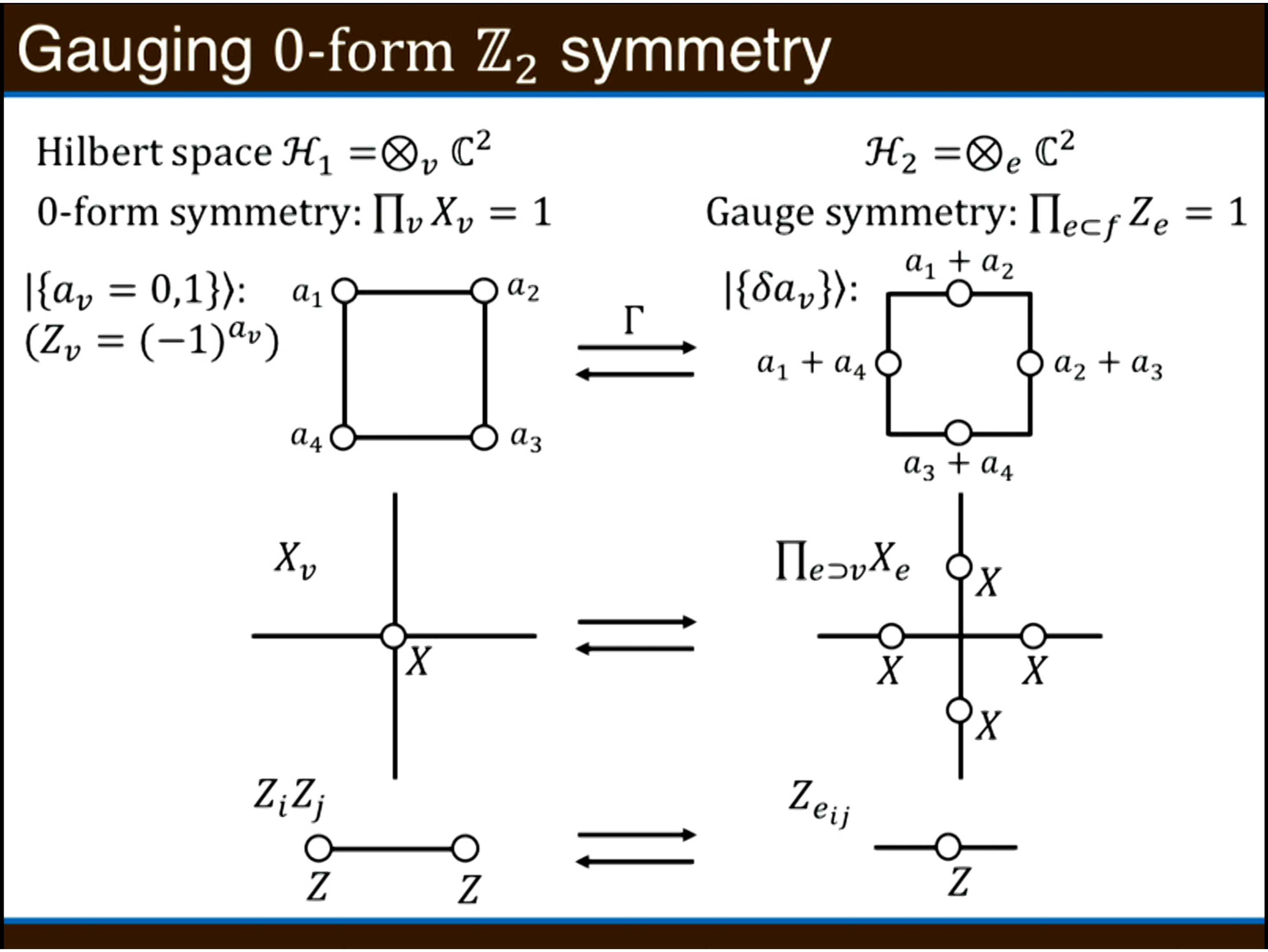
Fermionic SPT

Supercohomology data (ν, n)

Choosing a spin structure

Summing over all spin structures

Bhardwaj, L., Gaiotto, D. & Kapustin, A. J. High Energ. Phys. (2017).
Ellison and Fidkowski. Phys. Rev. X 9, 011016 (2019).



Gauging 0-form $\mathbb{Z}_2$ symmetry

Hilbert space $\mathcal{H}_1 = \otimes_v \mathbb{C}^2$

0-form symmetry: $\prod_v X_v = 1$

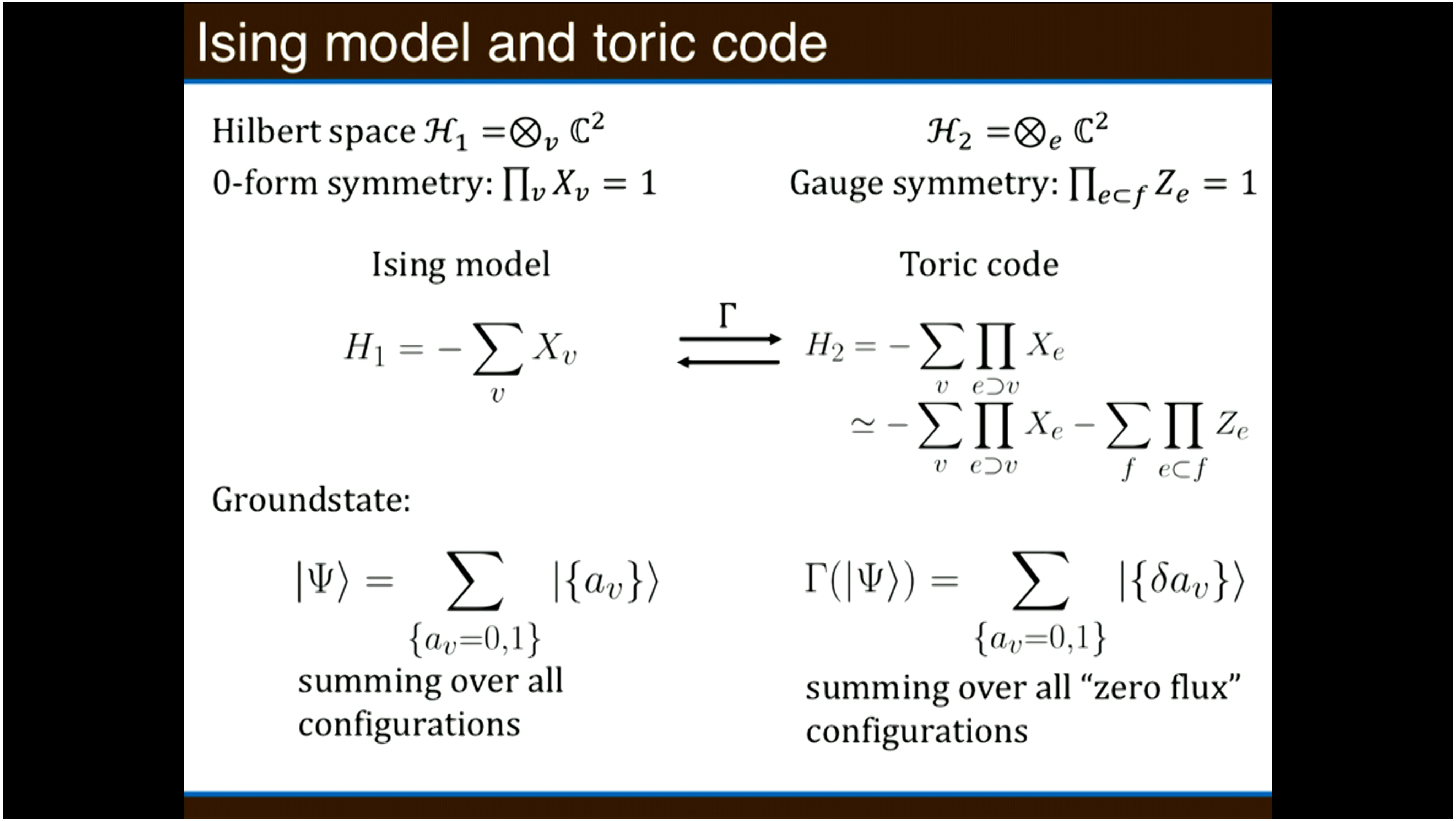
$|\{a_v = 0,1\}\rangle$:

$(Z_v = (-1)^{a_v})$

$\mathcal{H}_2 = \otimes_e \mathbb{C}^2$

Gauge symmetry: $\prod_{e \subset f} Z_e = 1$

$|\{\delta a_v\}\rangle$:



Ising model and toric code

Hilbert space $\mathcal{H}_1 = \bigotimes_v \mathbb{C}^2$

0-form symmetry: $\prod_v X_v = 1$

$\mathcal{H}_2 = \bigotimes_e \mathbb{C}^2$

Gauge symmetry: $\prod_{e \subset f} Z_e = 1$

Ising model

Toric code

$$H_1 = -\sum_v X_v \quad \overset{\Gamma}{\rightleftarrows} \quad H_2 = -\sum_v \prod_{e \supset v} X_e$$

$$\simeq -\sum_v \prod_{e \supset v} X_e - \sum_f \prod_{e \subset f} Z_e$$

Groundstate:

$$|\Psi\rangle = \sum_{\{a_v = 0,1\}} |\{a_v\}\rangle$$

summing over all configurations

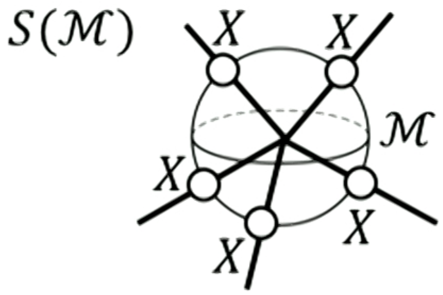
$$\Gamma(|\Psi\rangle) = \sum_{\{a_v = 0,1\}} |\{\delta a_v\}\rangle$$

summing over all “zero flux” configurations

Gauging 1-form $\mathbb{Z}_2$ symmetry

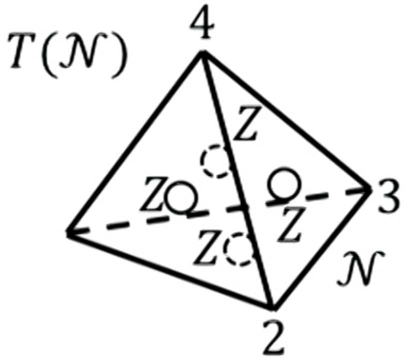
Hilbert space $\mathcal{H}_1 = \otimes_e \mathbb{C}^2$

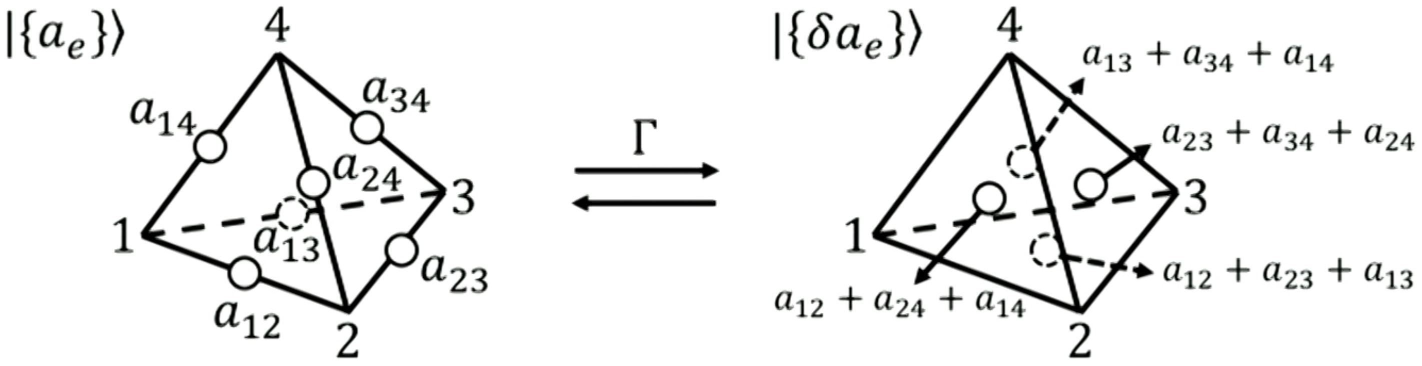
1-form symmetry: $S(\mathcal{M}) = 1$



$\mathcal{H}_2 = \otimes_f \mathbb{C}^2$

Gauge symmetry: $T(\mathcal{N}) = 1$





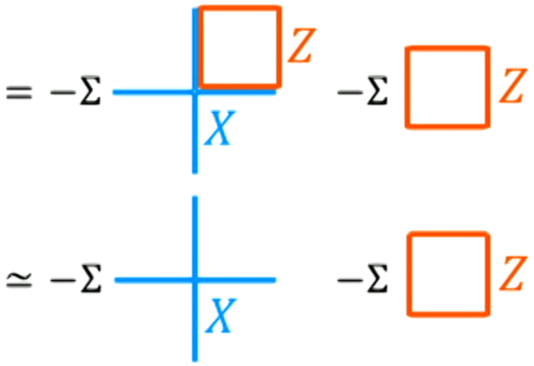
2d duality: warm-up

$$W_f = \prod_{e \subset f} Z_e \longleftrightarrow P_f = -i\gamma_f\gamma'_f,$$

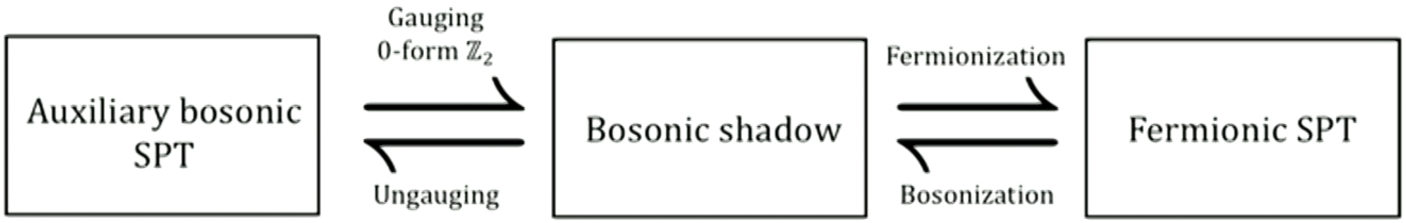
$$U_e = X_e(\prod_{e'} Z_{e'}^{\int \boldsymbol{e'} \cup \boldsymbol{e}}) \longleftrightarrow (-1)^{\int_E \boldsymbol{e}} S_e = (-1)^{\int_E \boldsymbol{e}} i\gamma_{L(e)}\gamma'_{R(e)}$$

$$G_v = \prod_{e \supset v} X_e(\prod_{e'} Z_{e'}^{\int \delta\boldsymbol{v} \cup \boldsymbol{e'}}) \longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{v}} S_{\delta \boldsymbol{v}} \prod_f P_f^{\int \boldsymbol{v} \cup \boldsymbol{f} + \boldsymbol{f} \cup \boldsymbol{v}} = 1$$

bosonic shadow		trivial fermion
$H_s = -\sum_v G_v - \sum_f W_f$	$\longleftrightarrow$	$H_f = -\sum_f \gamma_f\gamma'_f$



2d duality: warm-up



$$H_b = -\sum_v X_v \qquad H_s = -\sum_v G_v - \sum_f W_f \qquad H_f = -\sum_f i\gamma_f\gamma'_f$$

$$|\Psi_b\rangle = \sum_{\{a_v\}} |a_v\rangle \qquad |\Psi_s\rangle = \sum_{\{a_v\}} |\delta a_v\rangle \qquad |\Psi_f\rangle = |\mathrm{vac}\rangle$$

3d duality

$$W_t = \prod_{f \subset t} Z_f \longleftrightarrow P_t = -i\gamma_t\gamma'_t,$$

$$U_f = X_f(\prod_{f'} Z_{f'}^{\int \boldsymbol{f'} \cup_1 \boldsymbol{f}}) \longleftrightarrow (-1)^{\int_E \boldsymbol{f}} S_f = (-1)^{\int_E \boldsymbol{f}} i\gamma_{L(f)}\gamma'_{R(f)}$$

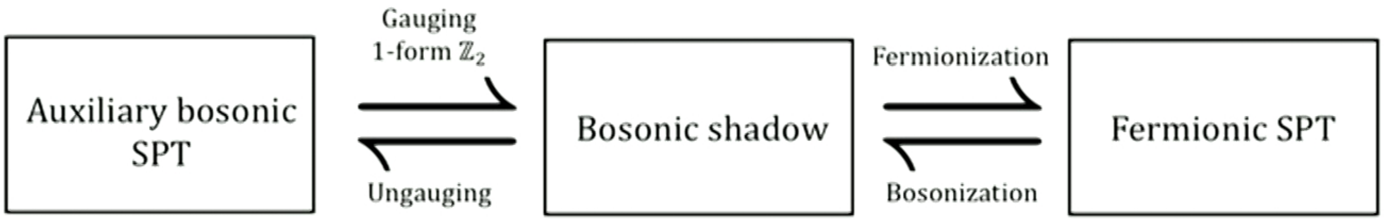
$$G_e = \prod_{f \supset e} X_f(\prod_{f'} Z_{f'}^{\int \delta \boldsymbol{e} \cup_1 \boldsymbol{f'}}) \longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{e}} S_{\delta \boldsymbol{e}} \prod_t P_t^{\int \boldsymbol{e} \cup_1 \boldsymbol{t} + \boldsymbol{t} \cup_1 \boldsymbol{e}} = 1,$$

bosonic shadow **trivial fermion**

$$H_s = -\sum_e G_e - \sum_t W_t \longleftrightarrow H_t = -\sum_t i\gamma_t\gamma'_t$$

$$|\Psi_s\rangle = \sum_{\{a_e\}} (-1)^{\int a_e \cup \delta a_e} |\{\delta a_e\}\rangle \qquad |\Psi_f\rangle = |\text{vac}\rangle$$

3d duality



Auxiliary bosonic SPT

Gauging 1-form $\mathbb{Z}_2$

Ungauging

Bosonic shadow

Fermionization

Bosonization

Fermionic SPT

$$H_b = U_{CZ}(-\sum X_e)U_{CZ}^\dagger$$

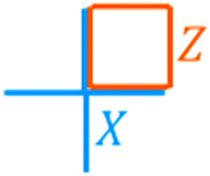
$$U_{CZ} \equiv \prod_{t=\langle 0123\rangle} CZ(e_{01}, e_{12})CZ(e_{01}, e_{13})CZ(e_{01}, e_{23})$$

$$H_s = -\sum_e G_e - \sum_t W_t$$

$$H_f = -\sum_t i\gamma_t\gamma_t'$$

$$|\Psi_b\rangle = \sum_{\{a_e\}}(-1)^{\int a_e \cup \delta a_e}|\{a_e\}\rangle \quad |\Psi_s\rangle = \sum_{\{a_e\}}(-1)^{\int a_e \cup \delta a_e}|\{\delta a_e\}\rangle \quad |\Psi_f\rangle = |\mathrm{vac}\rangle$$

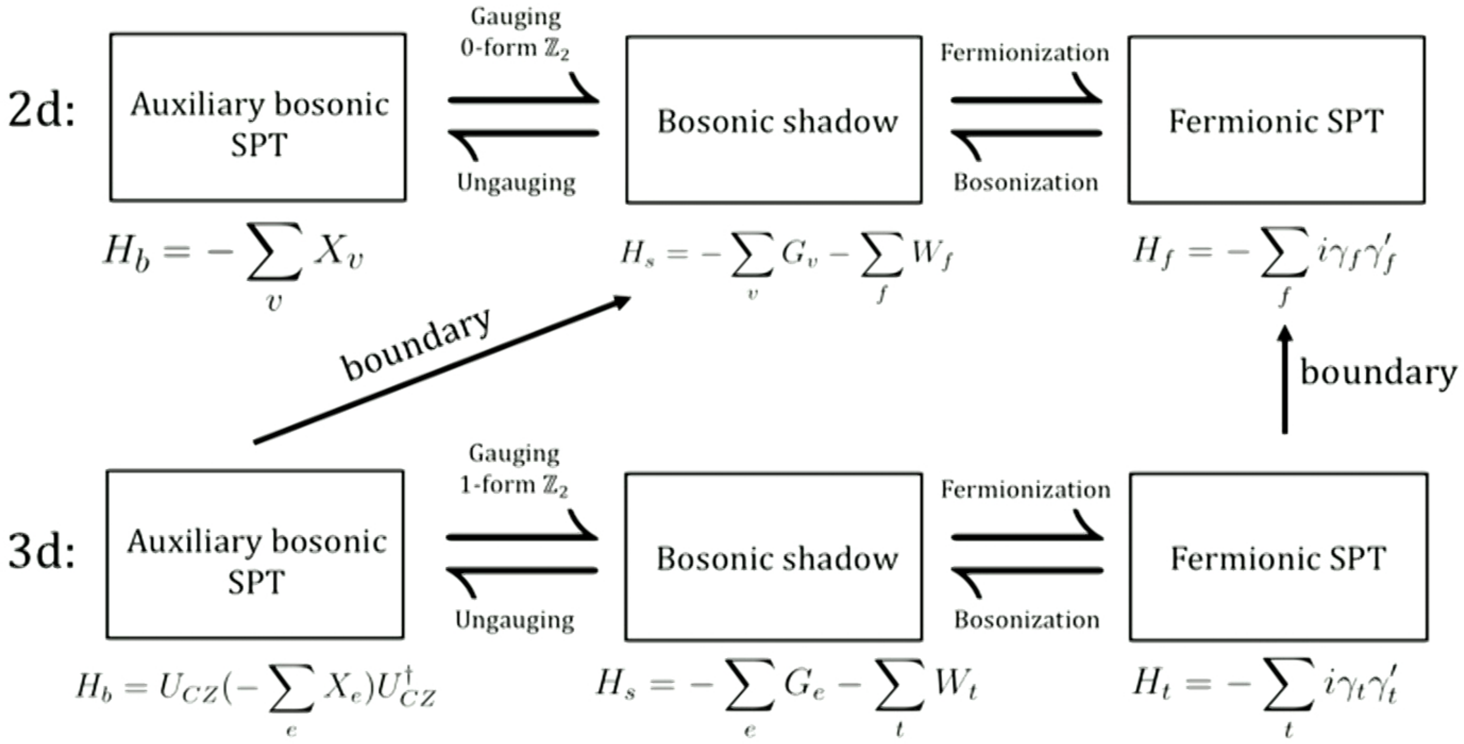
$|\Psi_b\rangle$ is a bosonic 1-form $\mathbb{Z}_2$ SPT. On its boundary, the anomalous symmetry action is $G_v = 1$ (boundary anomaly of $a_e \to a_e + \delta\lambda$). The following is an allowed boundary:



Toric code: $H_s = -\sum_v G_v - \sum_f W_f$

Yoshida. Phys. Rev. B 93, 155131 (2016).
Tsui and Wen. arXiv:1908.02613 (2019).

SPT boundary



2d:

Auxiliary bosonic SPT

Gauging 0-form $\mathbb{Z}_2$

Ungauging

Bosonic shadow

Fermionization

Bosonization

Fermionic SPT

$H_b = -\sum_v X_v$

$H_s = -\sum_v G_v - \sum_f W_f$

$H_f = -\sum_f i\gamma_f\gamma'_f$

boundary

boundary

3d:

Auxiliary bosonic SPT

Gauging 1-form $\mathbb{Z}_2$

Ungauging

Bosonic shadow

Fermionization

Bosonization

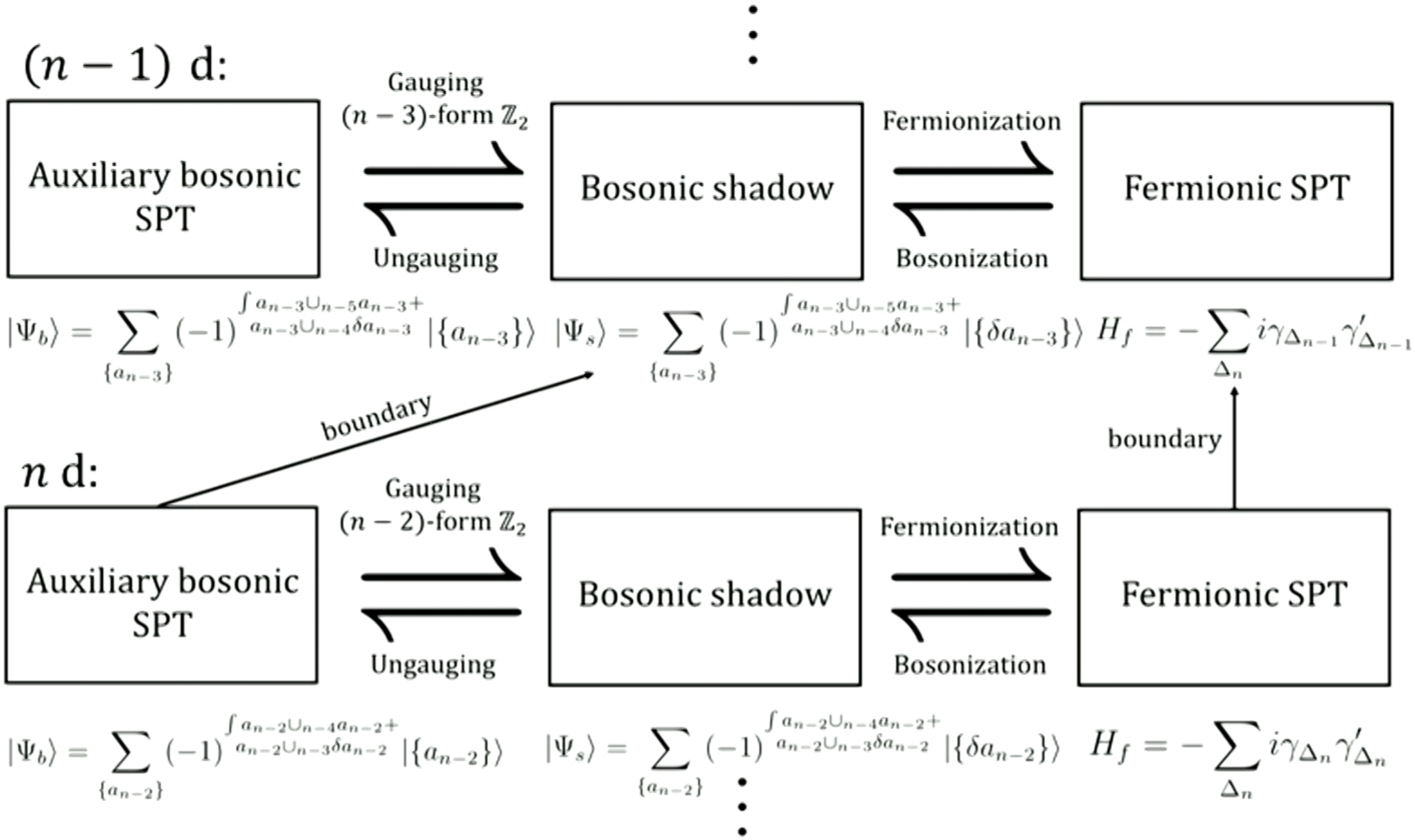
Fermionic SPT

$H_b = U_{CZ}(-\sum_e X_e)U_{CZ}^\dagger$

$H_s = -\sum_e G_e - \sum_t W_t$

$H_t = -\sum_t i\gamma_t\gamma'_t$

Hierarchy of higher-form $\mathbb{Z}_2$ SPTs



$(n-1)$ d:

Auxiliary bosonic SPT — Gauging $(n-3)$-form $\mathbb{Z}_2$ / Ungauging — **Bosonic shadow** — Fermionization / Bosonization — **Fermionic SPT**

$$|\Psi_b\rangle = \sum_{\{a_{n-3}\}} (-1)^{\substack{\int a_{n-3}\cup_{n-5}a_{n-3}+ \\ a_{n-3}\cup_{n-4}\delta a_{n-3}}} |\{a_{n-3}\}\rangle \quad |\Psi_s\rangle = \sum_{\{a_{n-3}\}} (-1)^{\substack{\int a_{n-3}\cup_{n-5}a_{n-3}+ \\ a_{n-3}\cup_{n-4}\delta a_{n-3}}} |\{\delta a_{n-3}\}\rangle \quad H_f = -\sum_{\Delta_n} i\gamma_{\Delta_{n-1}}\gamma'_{\Delta_{n-1}}$$

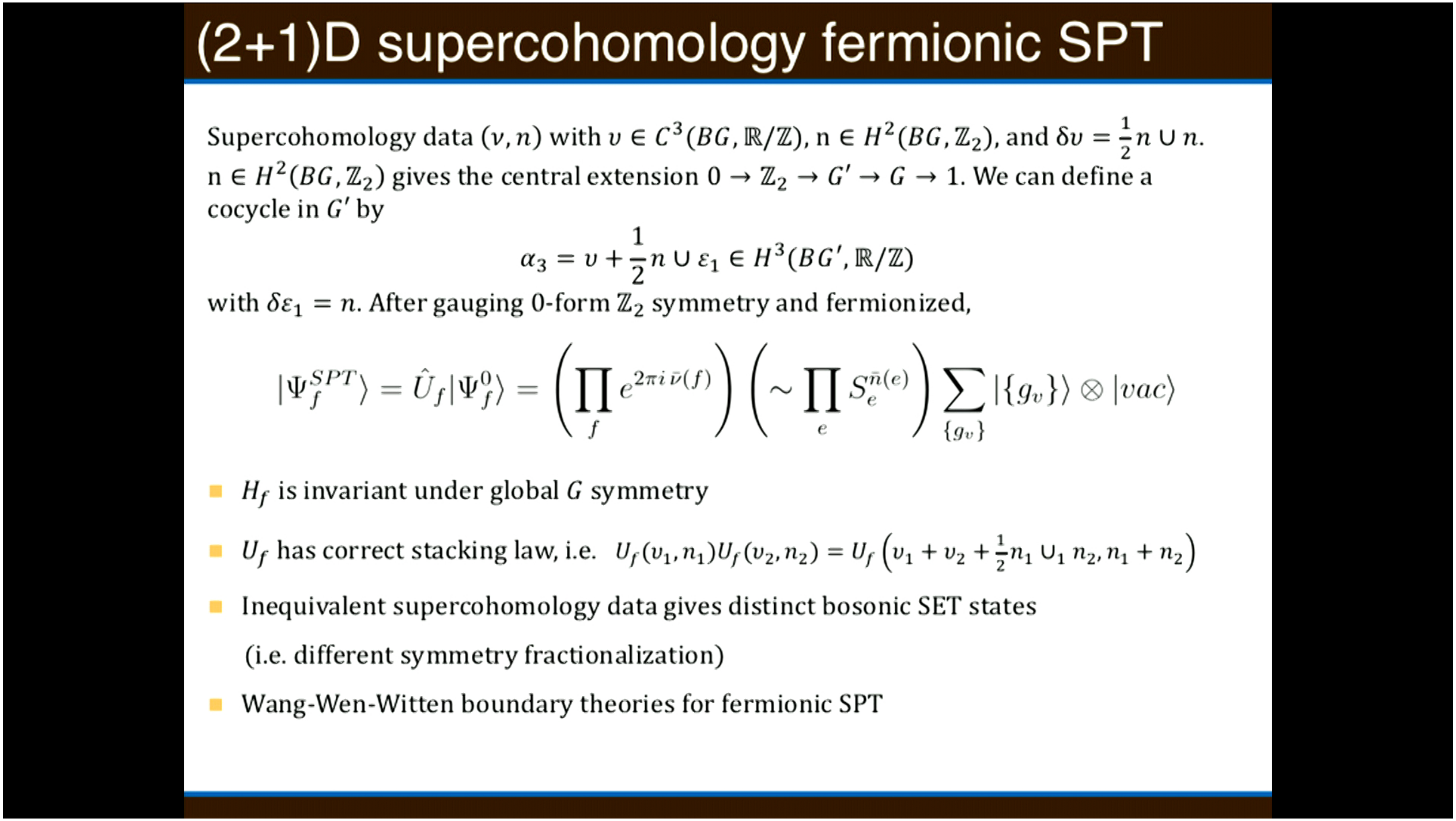
boundary

boundary

n d:

Auxiliary bosonic SPT — Gauging $(n-2)$-form $\mathbb{Z}_2$ / Ungauging — **Bosonic shadow** — Fermionization / Bosonization — **Fermionic SPT**

$$|\Psi_b\rangle = \sum_{\{a_{n-2}\}} (-1)^{\substack{\int a_{n-2}\cup_{n-4}a_{n-2}+ \\ a_{n-2}\cup_{n-3}\delta a_{n-2}}} |\{a_{n-2}\}\rangle \quad |\Psi_s\rangle = \sum_{\{a_{n-2}\}} (-1)^{\substack{\int a_{n-2}\cup_{n-4}a_{n-2}+ \\ a_{n-2}\cup_{n-3}\delta a_{n-2}}} |\{\delta a_{n-2}\}\rangle \quad H_f = -\sum_{\Delta_n} i\gamma_{\Delta_n}\gamma'_{\Delta_n}$$



(2+1)D supercohomology fermionic SPT

Supercohomology data (ν, n) with $\upsilon \in C^3(BG, \mathbb{R}/\mathbb{Z})$, n $\in H^2(BG, \mathbb{Z}_2)$, and $\delta\upsilon = \frac{1}{2} n \cup n$. n $\in H^2(BG, \mathbb{Z}_2)$ gives the central extension $0 \to \mathbb{Z}_2 \to G' \to G \to 1$. We can define a cocycle in G' by

$$\alpha_3 = \upsilon + \frac{1}{2} n \cup \varepsilon_1 \in H^3(BG', \mathbb{R}/\mathbb{Z})$$

with $\delta\varepsilon_1 = n$. After gauging 0-form $\mathbb{Z}_2$ symmetry and fermionized,

$$|\Psi_f^{SPT}\rangle = \hat{U}_f |\Psi_f^0\rangle = \left(\prod_f e^{2\pi i \, \bar{\nu}(f)}\right)\left(\sim \prod_e S_e^{\bar{n}(e)}\right) \sum_{\{g_v\}} |\{g_v\}\rangle \otimes |vac\rangle$$

- H_f is invariant under global G symmetry
- U_f has correct stacking law, i.e. $U_f(\upsilon_1, n_1) U_f(\upsilon_2, n_2) = U_f\left(\upsilon_1 + \upsilon_2 + \frac{1}{2} n_1 \cup_1 n_2, n_1 + n_2\right)$
- Inequivalent supercohomology data gives distinct bosonic SET states

  (i.e. different symmetry fractionalization)
- Wang-Wen-Witten boundary theories for fermionic SPT

(3+1)D gapped boundary construction

Supercohomology data (υ, n) with $\upsilon \in C^4(BG, \mathbb{R}/\mathbb{Z})$, $\mathrm{n} \in H^3(BG, \mathbb{Z}_2)$, and $\delta\upsilon = \frac{1}{2} n \cup_1 n$. On the boundary, we extend G to G':

$$0 \to K \to G' \to G \to 1$$

Kobayashi, Ohmori, Tachikawa.
arXiv:1905.05391 (2019).

such that in G'

$$\upsilon = \delta\eta + \frac{1}{2}\beta \cup_1 \delta\beta + \frac{1}{2}\beta \cup \beta, \qquad n = \delta\beta.$$

Extra degrees of freedom on the boundary:

1) Group elements $k_v \in K$ at boundary vertices
2) Fermions at boundary faces

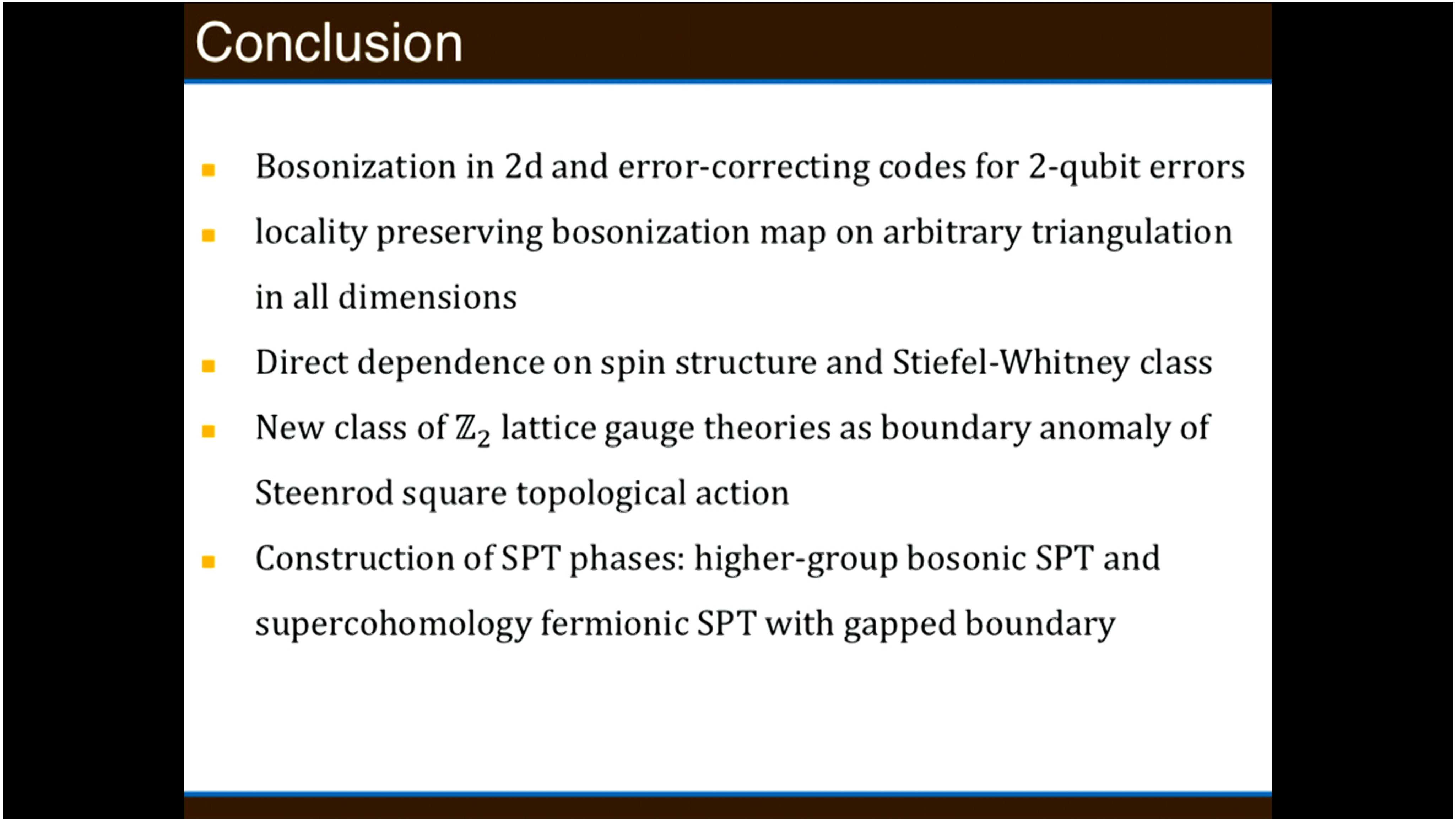
(3+1)D femionic SPT states with gapped boundary (from bosonic Wang-Wen-Witten):

$$|\Psi_f\rangle = \left(\prod_{f \in bdy} \exp(2\pi i\, \bar{\eta}(f))\right)\left(\sim \prod_{e \in bdy} S_e^{\bar{\beta}(e)}\right)\left(\prod_{t \in bulk} \exp(2\pi i\, \bar{\nu}(t))\right)\left(\sim \prod_{f \in bulk} S_f^{\bar{n}(f)}\right)$$

$$\sum_{\{g_v\},\{k_v\}} |\{g_v\}_{bulk}, \{\delta k_v\}_{boundary}\rangle \otimes |vac\rangle_{bulk} \otimes |vac\rangle_{boundary}$$

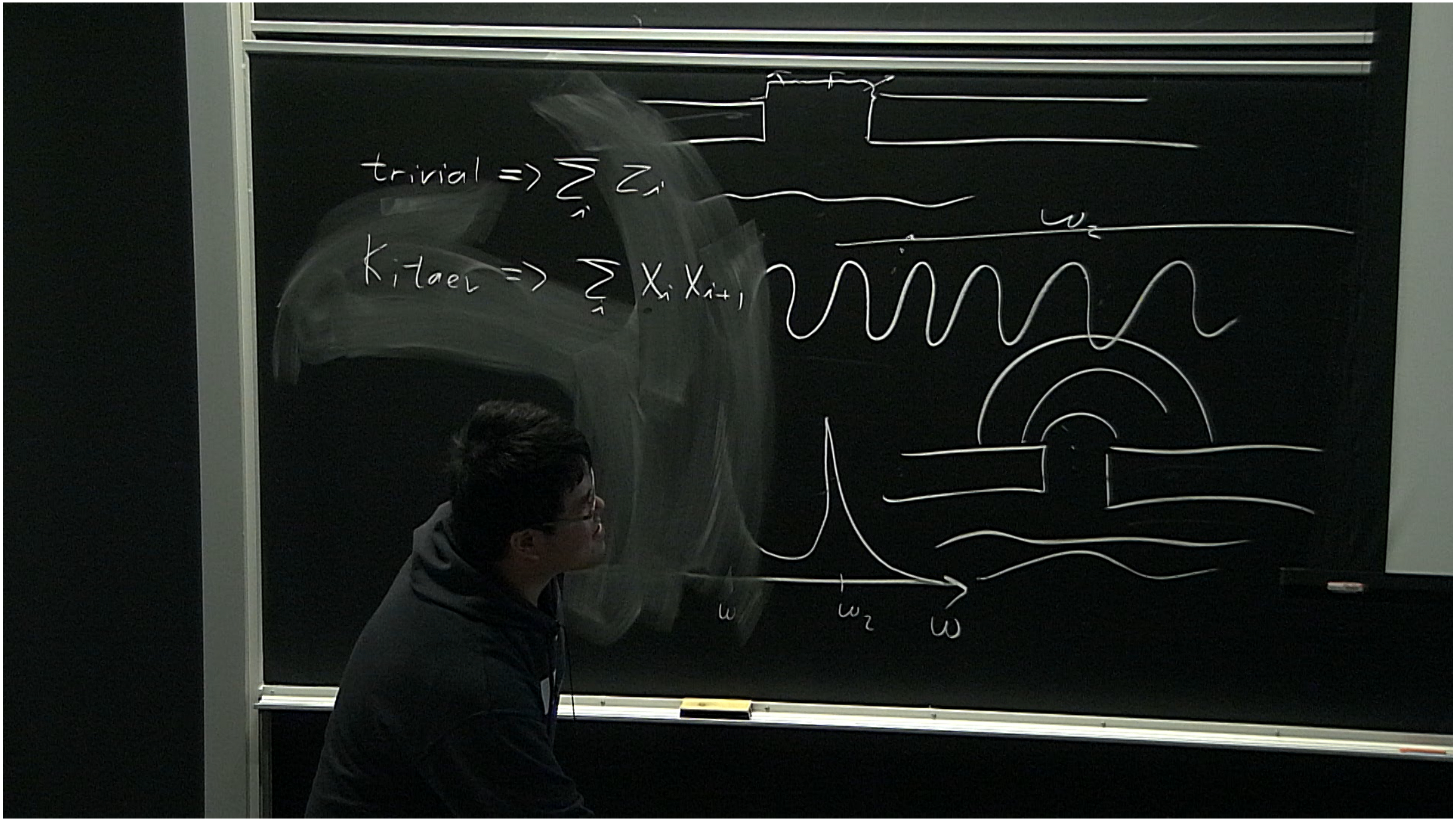
$G = \mathbb{Z}_4 \times \mathbb{Z}_2$ $\quad$ $K = \mathbb{Z}_4$ $\quad$ ⇒ boundary $\mathbb{Z}_4$ gauge theory

Fidkowski, Vishwanath, and Metlitski. arXiv:1804.08628 (2018).



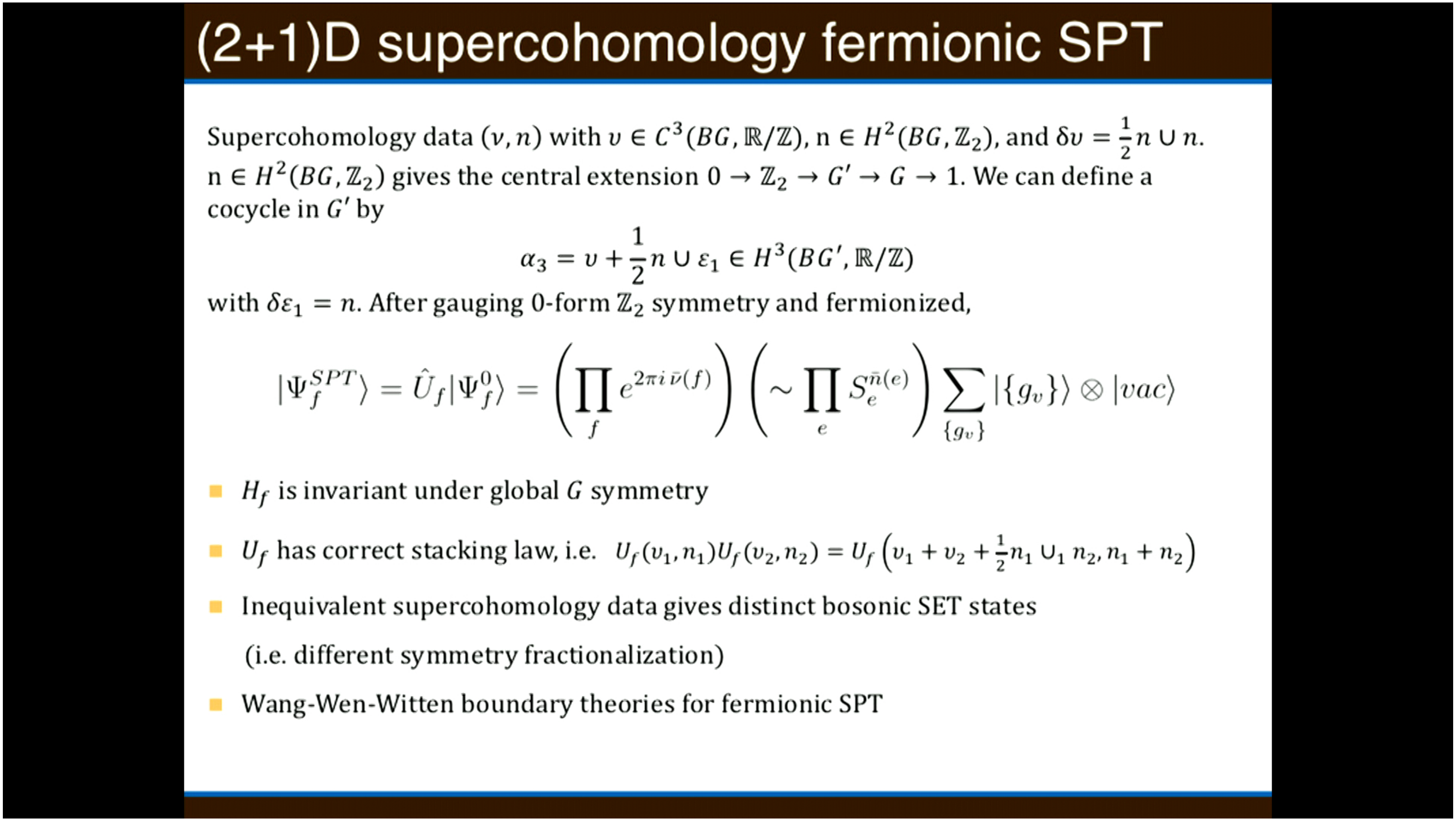
Conclusion

- Bosonization in 2d and error-correcting codes for 2-qubit errors
- locality preserving bosonization map on arbitrary triangulation in all dimensions
- Direct dependence on spin structure and Stiefel-Whitney class
- New class of $\mathbb{Z}_2$ lattice gauge theories as boundary anomaly of Steenrod square topological action
- Construction of SPT phases: higher-group bosonic SPT and supercohomology fermionic SPT with gapped boundary



trivial $\Rightarrow \sum_i Z_i$

Kitaev $\Rightarrow \sum_i X_i X_{i+1}$



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