Title: On geometry and symmetries in gauge gravity: Cartan connection approach

Speakers: Vadim Belov

Series: Quantum Gravity

Date: December 10, 2019 - 12:00 PM

URL: http://pirsa.org/19120014

Abstract: Our earlier findings indicate the violation of the 'volume simplicity' constraint in the current Spinfoam models (EPRL-FK-KKL). This result, and related problems in LQG, promted to revisit the metric/vielbein degrees of freedom in the classical Einstein-Cartan gravity. Notably, I address in detail what constitutes a 'geometry' and its 'group of motions' in such Poincare gauge theory. In a differential geometric scheme that I put forward the local translations are not broken but exact, and their relation to diffeomorphism transformations is clarified. The refined notion of a tensor takes into account the (relative) localization in spacetime, whereas the key concept of 'development' generalizes parallel transport (of vectors and points) to affine spaces. I advocate for this Cartan connection as the fundamental d.o.f. of the gravitational field, and discuss implications for discretization and quantization. Based on [1905.06931].

On geometry and symmetries in gauge gravity: Cartan connection approach

Vadim Belov

based on the PhD work [1905.06931] done at the University of Hamburg

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December 10, 2019

Emmy Noether-Programm
Deutsche Forschungsgemeinschaft
DFG

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Outline

I Context and motivation

1. Classical gravity: Riemannian metric, fluxes and connection(s)
2. Quantum geometry in LQG. Simplicity in the Spinfoam approach
3. Extended phase space

II Geometry in gauge theory

1. Locally Klein bundle
2. Cartan connection, 'osculation' and development
3. Generalizaed tensors. Universal covariant derivative

III Outlook on quantum kinematics and dynamics

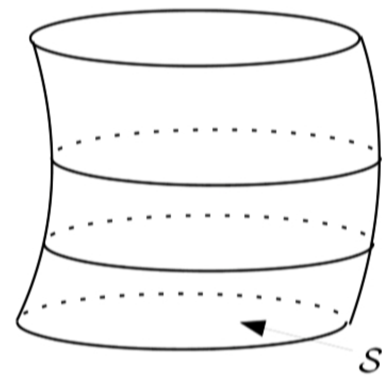
1. Generalized configuration space of connections
2. Geometric significance of Einstein tensor and conservation laws

Classical gravity: metric formulation

- Spacetime picture (covariant)
  - Mathematical model: $(\mathcal{M}, g)/\mathrm{Diff}$ – Riemannian geometry
  - The physical content of General Relativity (derived from $S_{\mathrm{EH}}[g] = \int R(g)$)

$$\underbrace{R_{ij} - \frac{1}{2} R\, g_{ij}}_{E_{ij} \text{ tensor (curvature)}} = \underbrace{\frac{8\pi G}{c^4} T_{ij}}_{\text{energy-momentum (source)}} \qquad \text{(Einstein eq.)}$$

- Hamiltonian dynamics (canonical)



- (ADM) $\rightarrow$ hypersurface (3+1)-foliation picture $\mathcal{M} \cong \mathbb{R} \times \mathcal{S}$, initial value problem with phase space variables (q, K)
- (Dirac) $\rightarrow$ phase space constraints $\mathscr{C}(q, K) \approx 0$

$$\mathscr{C}_a := D_b K^b{}_a - D_a K^b{}_b \approx 0 \qquad \text{(contracted Codazzi eq.)}$$

$$\mathscr{C}_\perp := (\mathrm{tr} K)^2 - \mathrm{tr}(K^2) + {}^{(3)}R \approx 0 \qquad \text{(Gauss' th. egregium)}$$

Classical gravity: connection/flux (re-)formulation

- The idea is to present GR in the form of gauge theory, with well established quantization
  - The curvature is that of *Levi-Civita connection* $\Gamma \sim \partial g$ on the frame bundle $L(\mathcal{M})$
  - The covariant approach is modelled on $(\theta, \omega) : TL(\mathcal{M}) \to (V, \mathfrak{h})$

$$ds^2 = \langle \theta \otimes \theta \rangle = \{\eta_{ij}\theta^i\theta^j\}, \qquad \text{and} \qquad \omega(\cdot) = \langle \Gamma, \theta(\cdot) \rangle = \{{\Gamma^i}_{jk}\theta^k\}$$

  - The action $S[\theta, \omega] = \int \langle (\star + \gamma^{-1})\theta \wedge \theta, \Omega \rangle$, with $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ ($\star$ – Hodge dual)

- The canonical theory is reduced to hypersurface $\mathcal{S}_3 \subset \mathcal{M}$ and $\mathfrak{so}(3) \simeq \mathbb{R}^3$ bundle
  - The variables are **Ashtekar-Barbero** $\mathfrak{su}(2)$-connection and conjugate **momenta-fluxes**

$$A = \Gamma[\vec{\theta}] + \gamma K, \qquad \text{and} \qquad B = \bar{\star}(\vec{\theta} \wedge \vec{\theta}), \qquad \text{s.t.} \qquad \{A, \tilde{B}\} = \gamma \mathbb{1}$$

  - The action $S = \int (\tilde{B}^a_i \dot{A}^i_a - \mathscr{H})$, where $\mathscr{H} = N\mathscr{C}_\perp + N^a\mathscr{C}_a + A^i_0\mathscr{C}_i$

    $\Rightarrow$ GR $\sim$ Yang-Mills with Gauss law $\mathscr{C}_i \equiv \mathcal{D}_a\tilde{B}^a_i \approx 0$

    $+$ additional ADM **constraints** (greatly simplified)

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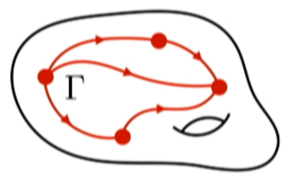
LQG in a nutshell: regularization with holonomies

- The connection representation on $L^2(\bar{\mathcal{A}}, d\mu)$ of wave functionals

$$\hat{A}(\mathbf{x})\Psi[A] = A(\mathbf{x})\Psi[A], \qquad \hat{B}(\mathbf{y})\Psi[A] = \frac{\hbar}{i}\frac{\delta}{\delta A(\mathbf{y})}\Psi[A], \qquad [\hat{A}(\mathbf{x}), \hat{B}(\mathbf{y})] = i\hbar\,\delta^3(\mathbf{x},\mathbf{y})$$

Constraints $\hat{C}\Psi = 0$ express the invariance, e.g. $\Psi[A^g] = \Psi[A]$ for Gauss.

- Partial configuration spaces $\mathcal{A}_\Gamma = \mathrm{SU}(2)^{L\subset\Gamma}$. 'Cylindrical' functionals on $\bar{\mathcal{A}} = \lim_{\Gamma\leftarrow} \mathcal{A}_\Gamma$ (w.r.t. poset $\Gamma \le \Gamma'$) project on graphs to $\Psi_\Gamma[A] = \psi_\Gamma\big(\{h_\ell[A]\}_{\ell\in L}\big)$



Gauge-invariant subspace is spanned by ONB of **spin-network** states

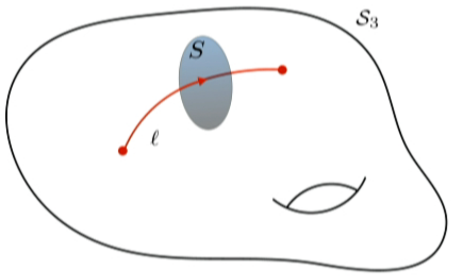
$$\Psi_{\Gamma,j_\ell,\iota_n}[A] = \bigotimes_n \iota_n^* \bigotimes_\ell D^{j_\ell}(h_\ell)$$

Discrete geometry of space

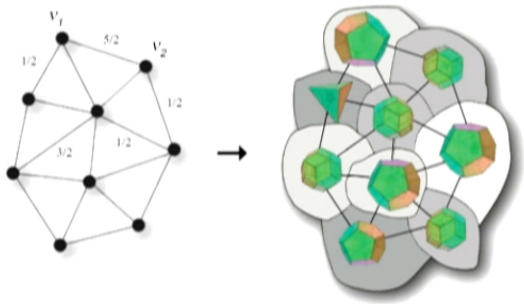
- Conjugate momenta are smeared over complementary 2d surfaces of a dual cellular decomposition

$$B_\ell = \int_{S_\ell} h \triangleright B \quad - \quad \textbf{fluxes}$$

By the action on $\mathcal{A}_\Gamma \Rightarrow$ quantized as angular momentum, determine the eigenvalues of area $\hat{B}_\ell^2 \propto j_\ell(j_\ell + 1)$



- The collection of 'fuzzy' polyhedra, glued along their faces, describe quantized 3d space



Unresolved issues (or raising concern)

- ➔ How to recover spacetime? The dynamics is complicated: $\hat{C}_\perp \Psi = 0$ + ambiguities
- ➔ The significance of discrete configurations: twisted/spinning geometries are discontinuous/torsionful [Freidel,Speziale,Ziprick]. Bug or feature?

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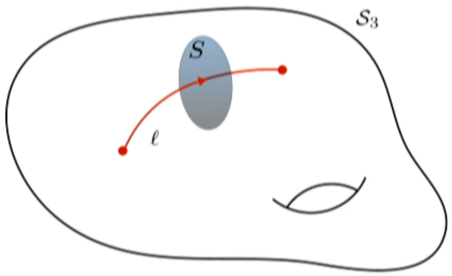

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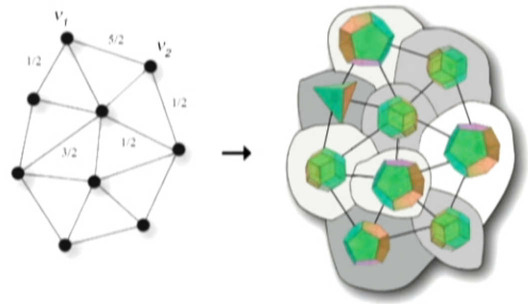

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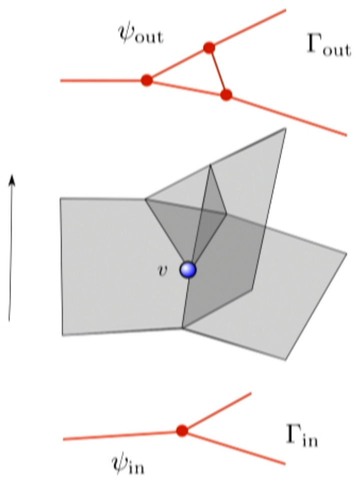
Spin Foams in a nutshell: quantum spacetime

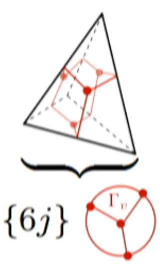


- Imagine the space-slice $\Psi_{\Gamma,j_\ell,\iota_n}$ evolves in time continuously:
  - links sweep faces f, nodes sweep edges e
  - the coloured 2-complex $(\mathcal{K}, j_f, \iota_e)$ is a spinfoam
  - branching vertices correspond to elementary interactions [Iwasaki'95;Reisenberger,Rovelli'97;Baez'98]
- In general, SF amplitude functional $Z_{\mathcal{K}} : \mathcal{H}_{\Gamma=\partial\mathcal{K}} \to \mathbb{C}$

$$Z_{\mathcal{K}} = \sum_{j_f,\iota_e} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \mathcal{A}_v$$

- QG is exact in 3d: $S = \int \langle B \wedge \Omega \rangle$, $B \equiv \star\theta$ – **topological BF theory**

$$\Rightarrow \qquad Z^{\mathcal{K}}_{PR} = \sum_{j_f} \prod_f d_{j_f} \prod_v \underbrace{\text{[tetrahedron]}}_{\{6j\}} \overset{j\to\infty}{\sim} \int d\mu(j) \, \exp\left(iS_R\right)$$



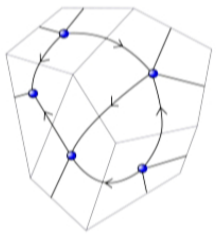
– [Ponzano,Regge'68] state-sum (discrete partition function)

SF from path-integral: "first quantize and then constrain"

- Manifestly 4d (re-)formulation: $S = \int \langle B, \Omega \rangle + \lambda^\alpha C_\alpha[B]$
- $\widehat{C_\alpha[B]} = 0$ restricts summation over $\{\rho_f, \iota_e\}$: only states with $B = \star\theta \wedge \theta$ contribute

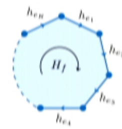
➔ Instead of 'dynamical' constraints – characterization of discrete geometries in terms of B ("reduction to GR" for *some* $E_l = \int_l \theta$)

Expectation is that the shape of a polytope $P \subset \mathbb{R}^4$ is encoded in the boundary



✓ The induced boundary $\mathcal{H}_{\partial\mathcal{K}}$ can be made to match LQG, s.t. $\langle Z|\Psi\rangle = \int_{\mathcal{A}_\Gamma} d\mu_\Gamma \, Z[h]\Psi[h]$

✓ Justification from 4-simplex Regge asymptotics ($j \to \infty$) [Barrett;Conrady,Freidel]

✗ Independent areas $\delta j \;\Rightarrow\;$ flatness problem:  $= 0$ [Magliaro,Perini;Han]

✗ Diffeomorphism invariance broken by discretization [Bahr,Dittrich'09]. Continuum limit via refinement [B,D,Steinhaus], or summation [Group Field Theory]

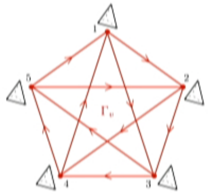
Simplicity constraints: main objectives

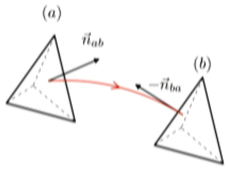
- The meaning: coord. of the (ordered) wedge-product of 2 vectors define a plane element

$$\mathbf{\Sigma} = \mathbf{X} \wedge \mathbf{Y} = \frac{1}{2}\Sigma^{ij}(\mathbf{e}_i \wedge \mathbf{e}_j), \qquad \Sigma^{ij} = X^i Y^j - Y^i X^j = \begin{vmatrix} X^i & Y^i \\ X^j & Y^j \end{vmatrix}$$

NB: *Not every skew-symmetric 2-tensor is simple/planar* (shape is left unspecified)

- [Plebanski'77;BC'98] quadratic constraints for triangulations





✓ 'Diagonal' and 'off-diagonal' (tetrahedral) constraints

$$\star B_f \cdot B_f = 0 \quad \forall f, \qquad \star B_f \cdot B_{f'} = 0 \quad \forall f \cap f' = e$$

✗ 'Volume' constraint $\quad \star B_f \cdot B_{f'} =: V_v(f, f') \quad \forall f \cap f' = v$

- [Gielen,Oriti'10;EPRL'07;FK'08] linear formulation, with *some* 4d normals $\mathcal{V}_e$

✓ 'Cross-simplicity' (orthogonality) $\quad \star B_f \cdot \mathcal{V}_e = 0 \quad \forall f \supset e$

✗ 'Linear volume' constraint $\quad \sum_{\{e',e''\} \not\ni e} \star B_{(ee')} \cdot \mathcal{V}_{e''} = 0 \quad \forall e$

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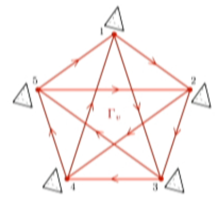

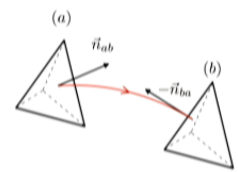

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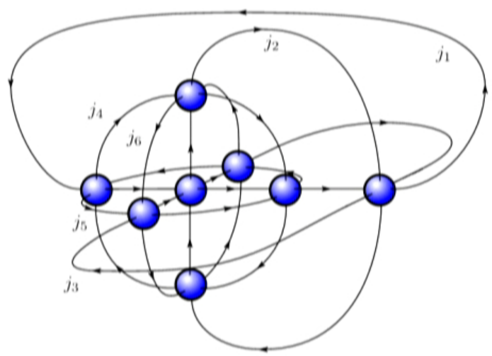
(Semi-classical) bivector geometries

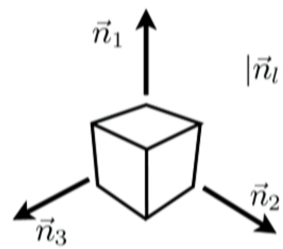
- Induced bivector geometry is given by
  - the set $\{B_\ell\}_{\ell\in\Gamma}$ associated with oriented links of a graph $\Gamma \subset S^3$,
  - satisfying $\sum_{\ell \supset n} [n,\ell] B_\ell = 0$, as well as $\star B_\ell \cdot B_\ell = 0$, and $\star B_\ell \cdot B_{\ell'} = 0$

- Such b.g. appear in the asymptotics of EPRL-FK-KKL model for general 2-complexes $\mathcal{K}$, e.g. for rectangular lattice with hypercuboidal boundary data $\{j_i, \vec{n}_{ab}\}$ [Bahr,Steinhaus'16]



Boundary graph for the vertex amplitude

with [Livine,Speziale'07] $|\iota\rangle$
(semi-classically) $\mathbb{R}^3$-cuboid



$$\mathcal{A}_v^\pm = \int \prod_a dg_a \, e^{S^\pm[g_a]}, \qquad S^\pm[g_a] = \frac{1 \pm \gamma}{2} \sum_l 2j_l \ln\langle -\vec{n}_{ab} | g_a^{-1} g_b | \vec{n}_{ab}\rangle$$

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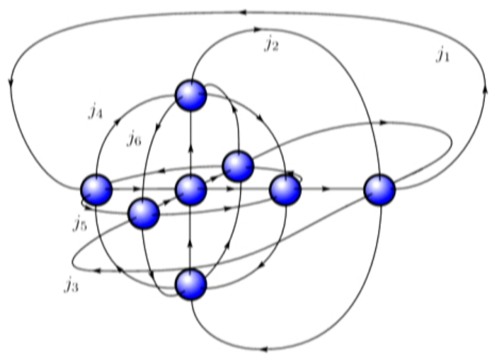

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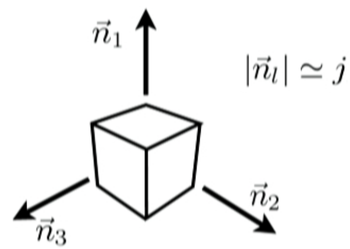

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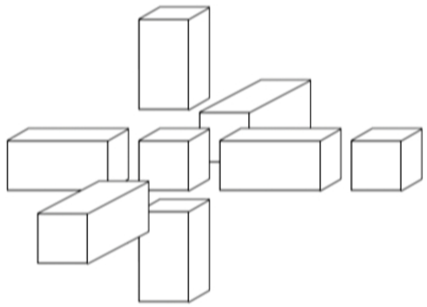
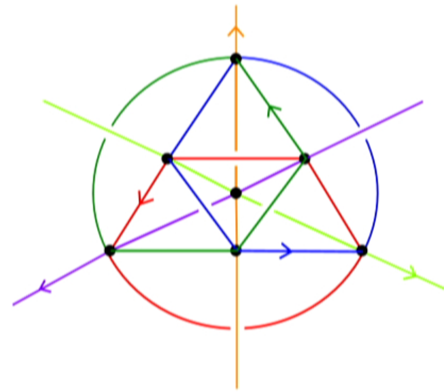
The primer of quboids

- Consider $\{B_\ell\}$ forming 3d cuboid at each node: coincide on 6 independent great circles

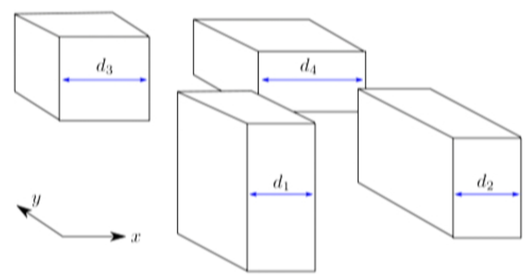




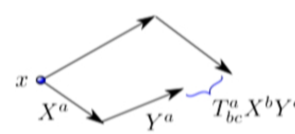
$$B_1 = a_1\, e_y \wedge e_z, \qquad B_2 = a_2\, e_z \wedge e_x, \qquad B_3 = a_3\, e_x \wedge e_y$$
$$B_4 = a_4\, e_z \wedge e_t, \qquad B_5 = a_5\, e_t \wedge e_y, \qquad B_6 = a_6\, e_x \wedge e_t.$$

- 3 ways to form volume (if $a_1a_6 \neq a_2a_5 \neq a_3a_4$):

$$V_{\mathcal{H}_1} = \frac{1}{3}a_1a_6, \quad V_{\mathcal{H}_2} = \frac{1}{3}a_2a_5, \quad V_{\mathcal{H}_3} = \frac{1}{3}a_3a_4$$



- Shape-mismatched configurations *do contribute* to KKL-asymptotics [Donà,Fanizza,Sarno,Speziale'18], and the physical states may possess *torsion* $||\psi_T||^2_{\text{phys}} \sim e^{-C_\alpha |T|^2}$, $T^x_{xy} = \frac{1}{2}(d_1 + d_2 - d_3 - d_4)$



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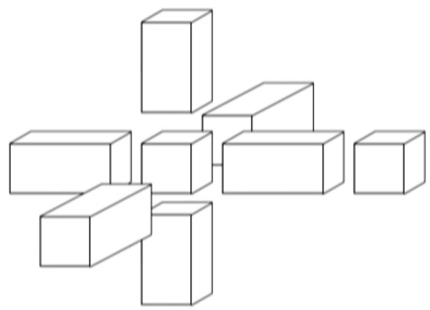

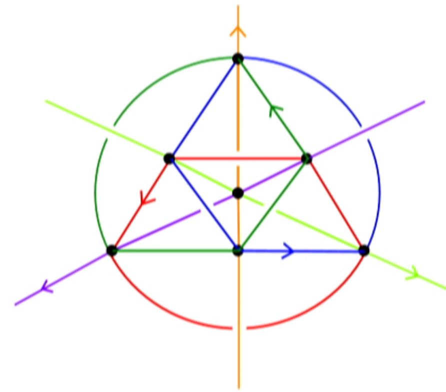

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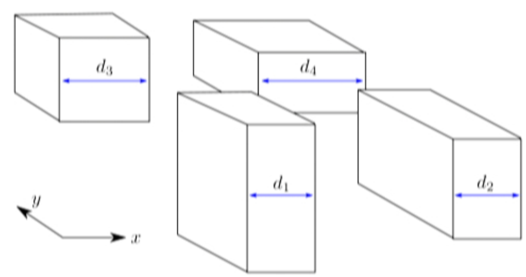
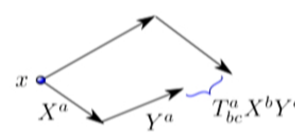


- Shape-mismatched configurations *do contribute* to KKL-asymptotics [Donà,Fanizza,Sarno,Speziale'18], and the physical states may possess *torsion* $||\psi_T||^2_{\text{phys}} \sim e^{-C_\alpha |T|^2}$, $T^x_{xy} = \frac{1}{2}(d_1 + d_2 - d_3 - d_4)$

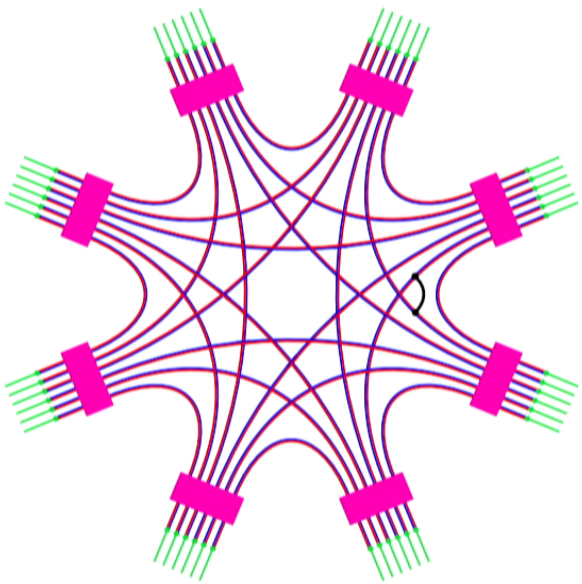


Problem with the EPRL-model [VB'18]

- The case of 4-simplex: closure is sufficient $\sum_{f \supset e} B_f = 0 \qquad \forall\, e = 1, ..., 5$

  → $\star B(\triangle_{12}) \cdot B(\triangle_{45}) + \star B(\triangle_{13}) \cdot B(\triangle_{45}) = - \star B(\triangle_{14}) \cdot B(\triangle_{45}) - \star B(\triangle_{15}) \cdot B(\triangle_{45}) = 0$

  → Reconstruction theorem [Barrett et al.'98'09] in terms of face bivectors



- Not applicable to arbitrary polytopes (hypercuboidal counter-example leads to tautology $\star B_{ij} \cdot B_{kt} = - \star B_{ji} \cdot B_{kt}$, $i, j, k = x, y, z$)

Failure to define 'volume'

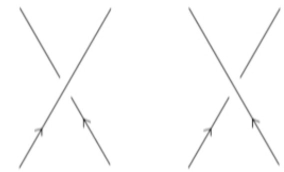
→ Part of constraints is not implemented properly

→ No reduction to GR for higher valence graphs

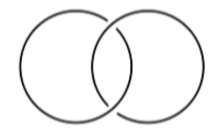
- The source of a breakdown: *'cycles'*
  Crossings carry information

Towards quantum 4-volume [Bahr,VB'18]

- Trace the origin of missing condition(s) to the **embedding** of $\Gamma \subset S^3$
  - Γ stereographically projects to $\mathbb{R}^3$, and then to $\mathbb{R}^2$ with crossings $\mathcal{C}$
  - Crossing number $\sigma(\mathcal{C}) = \pm 1$ depends on orientations of ℓ



- Define the Hopf-link volume $\quad V_{\mathcal{C}} := \frac{1}{6}\sigma(\mathcal{C}) \star (B_1 \wedge B_2), \quad V_{\mathcal{H}} := \sum_{\mathcal{C} \between \mathcal{H}} V_{\mathcal{C}}$



$\mathcal{H}$ – subset of edges which form two linked (non-intersecting) cycles when embedded in S^3

✓ The 4-polytope total volume in terms of 2d faces $V_P = \sum_{\mathcal{C}} V_{\mathcal{C}}$ and $V_{\mathcal{H}}$ are **independent of the projection** $\Rightarrow$ properties of b.g. only (and choice of $\mathcal{H}$) [Bahr'18]

Hopf link volume-simplicity constraint (conjecture)

$V_{\mathcal{H}}$ is independent of the choice of Hopf link $\mathcal{H}$ in Γ

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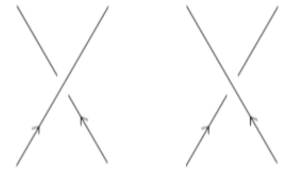

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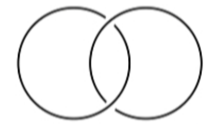

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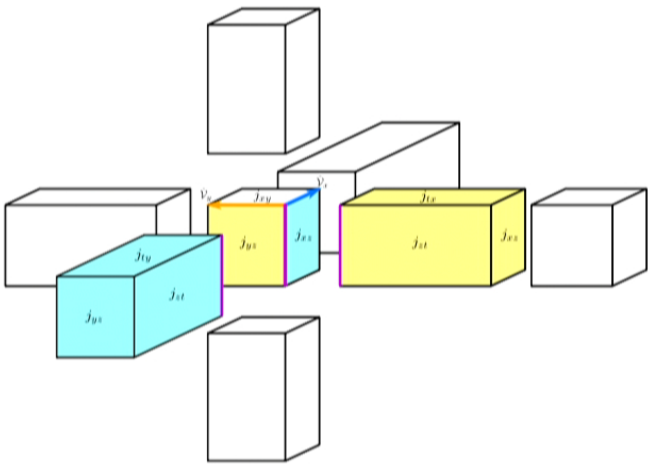
Fully linear treatment [VB'18]

- 4d normal $\mathcal{V}_e = \int_{\tau_e} \vartheta$ is dual to 3-form $\vartheta = \star\theta \wedge \theta \wedge \theta$
  - ➔ Norm $|\mathcal{V}_e|$ has the meaning of 3-volume
  - ➔ Linearized volume constr. requires closure $\sum_{e \supset v} \mathcal{V}_e^A \;=\; 0 \quad \forall\, v$ for 4-simplex



- We applied [Gielen,Oriti'10] beyond triangulation

$$\sum_{\{i,j\} \neq k} \star B_{ij} \cdot \mathcal{V}_k \;=\; 0 \qquad \forall\, k$$

Results in proportionality

- ➔ $|E_l| = |\mathcal{V}_i| / |B_{il}|$ independent of i (heights)
- ➔ $|E_i||B_{il}| = |E_j||B_{jl}| = |E_k||B_{kl}| \quad \forall\, i,j,k \neq l$ (3-volume)

- Suggestive to pass to $E_l = \int_l \theta$ (by duality)
  Vanishing torsion $D\theta = 0 \Rightarrow$ closure (no e.o.m.)

Poincaré-Plebański formulation [VB'18]

Extend the phase space by including θ-variables (and corresponding momenta)

- The relevant TQFT ansatz: $S_0 = \int B_{ij} \wedge \Omega^{ij} + \beta_k \wedge D^\omega \theta^k$
- Add simplicity constraints (e.g. with Lagrange multipliers λ)

$$B = \star\theta \wedge \theta \quad \Leftrightarrow \quad \mathrm{Sym}\left(B_{ij} \otimes \theta^j\right) = 0$$ (dual linear version)

- Geometric meaning: existence of 3-volume $B_{ij} \wedge \theta^j = (\star\theta \wedge \theta \wedge \theta)_i =: \mathcal{V}_i$ ("pyramid")

- We investigated this Poincaré-BF with $\mathfrak{so}(3,1) \ltimes \mathfrak{p}^{3,1} \ni \varpi := \omega + \theta$ and $\mathcal{B} := B + \beta$
  - EOM: $\Omega^{ij} = 0, \quad D^\omega \theta^k = 0, \quad DB^{ij} - \theta^{[i} \wedge \beta^{j]} = 0, \quad D\beta^k = 0$
  - gauge symmetries from Noether id-s: $\varpi \to g^{-1}(\varpi + d)g, \quad \mathcal{B} \to \mathrm{Ad}^*(g)\mathcal{B}$
  - generalized topological symmetry: $\delta\varpi = 0, \quad \delta\mathcal{B} = \left(D\Xi^{ij} - \theta^i \wedge \xi^j\right)\mathcal{J}_{ij} + D\xi^k \mathcal{P}_k$
  - reproduced in canonical formalism with **gauge generator(s)** $\mathcal{G}$ and 1st class algebra

Aftermath

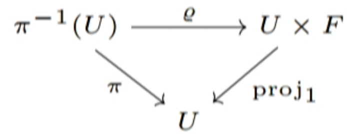
- Complementary (in continuum) to 'higher gauge theory/BFCG' with 2-connection $\omega + \beta$ [Girelli,Pfeiffer,Popescu'08;Baratin,Freidel'15;Miković,Oliveira,Vojinović'16;Dittrich et al'19]
- Complications seem to be rooted in the primacy of area-variables: embracing the θ brings us closer to metric GR
- **Klein geometry** $(M, x) \simeq G/H_x$, associated to any *'group of motion'* (acts transitively), naturally corresponds to homogeneous Minkowski spacetime, thereby elevating the earlier 'flux formulation' based on $|\Omega = 0\rangle$-state with just H-invariance [Dittrich,Geiller'15]

We need to come to terms about the "gauge potential of translations" θ:

Is not the local symmetry broken down to H by curvature in GR? How diffeomorphisms arise in gauge description? What is torsion $\Theta \equiv D^\omega \theta$ geometrically, and how to discretize it?

Recap: gauge theory basics

- The arena for fields

  - Bundle $\pi : E \to \mathcal{M}$ with standard fiber F is locally trivial  .

  - F is a G-module on which Lie group acts smoothly. Then transition functions

$$(U_\alpha \cap U_\beta) \times F \ \ni \ \varrho_\alpha \varrho_\beta^{-1}(x, f) = (x, s_{\alpha\beta}(x) f), \quad s(x) \in G$$

    determine the G-bundle by "gluing" $E = \bigcup_\alpha (U_\alpha \times F)/\sim$.

  - Principal bundle with $P_x = \pi^{-1}(x) \simeq G$ is determined by right R_g-action. Then

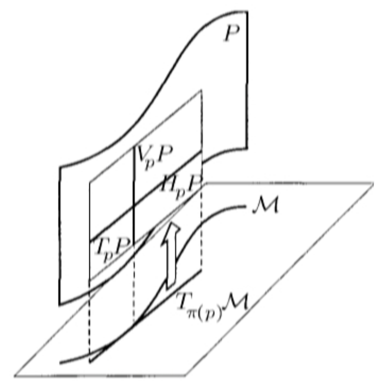
$$(U, \varrho) \quad \leftrightarrow \quad \tilde{\sigma}(x) = \varrho^{-1}(x, e) \in P_U \quad \leftrightarrow \quad \sigma(pg) = \sigma(p) g \in G$$

- (Ehresmann) connection on p.f.b. lifts curves $\gamma \in \mathcal{M}$ to P *horizontally*

(i) by specifying smooth distribution of tangent subspaces complementary to (canonical) vertical fields

$$TP = VP \oplus HP, \qquad VP := \ker \pi_*$$

(ii) satisfying right G invariance $R_{g*}(H_p P) = H_{pg} P$



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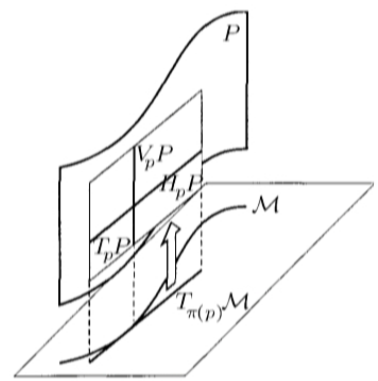

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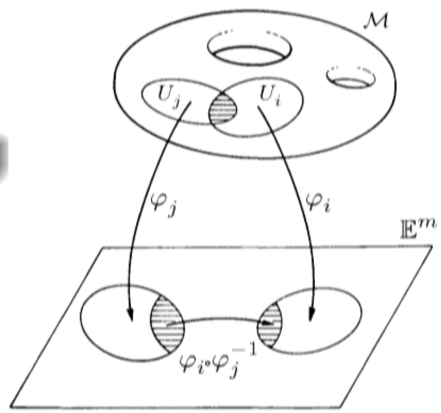
General knowledge: manifolds

- Smooth $\mathcal{M}$ "looks like" Euclidean space in the local chart

$$\varphi : U \to \mathbb{E}^m, \quad \text{s.t.} \quad \varphi_i \circ \varphi_j^{-1} \in C^\infty$$



Note: *affine* structure $\mathbb{E} + V \to \mathbb{E}$ allows differentiation

- ("External") linear frame bundle $L\mathcal{M} = \bigcup_{x \in \mathcal{M}} \{u_x\}$ at $x = \varphi^{-1}$

$$\begin{aligned} u \ : \quad T\mathbb{E} \cong \mathbb{E} \times V \quad &\to \quad T\mathcal{M} \\ (a, \mathbf{v}) \quad &\mapsto \quad X_{x(a)} \in T_{x(a)}\mathcal{M}. \end{aligned}$$

➔ $P_x \simeq \mathrm{GL}(V)$ (or its subgroup H) by $R_A(u) = u \circ A$, $A \in H$

Commonly, the gauge group is "H+diffeos", where Diff $\ni \phi : \mathcal{M} \to \mathcal{M}$ is bijective C^∞-map.
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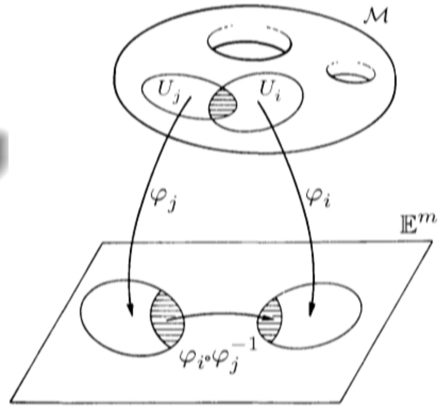

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Affine spaces and bases

Basic idea: the points-positions are made relative ("radius vector" w.r.t. any other point)

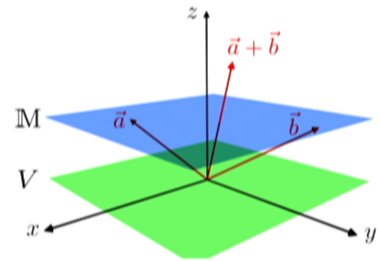
- Group $G = V \rtimes H$ acts freely and transitively on affine bases

$$(o', e') = (o, e)\, g \;=\; (o + \mathbf{a}, eA), \qquad g = (\mathbf{a}, A)$$

– principal homogeneous space (torsor) $P \times G \to P$. Factors down to $\mathbb{M} \cong G/H$.

- Customary model: $(x^0 = 1)$-hyperplane in $\mathbb{R}^{m+1}$, with linear matrix representation

$$G = \begin{pmatrix} 1 & 0 \\ V & H \end{pmatrix} \subset \mathrm{GL}(m+1, \mathbb{R})$$



Points: $\mathbf{a} + \mathbf{b} \notin \mathbb{M}$, but $(\mathbf{b} - \mathbf{a}) \in V$ – **free vector**

- 'Bound' or 'sliding' multivectors: $\mathbf{X}_0 = \mathbf{e}_0 + \mathbf{x} \quad \Rightarrow \quad \mathbf{\Sigma}_0^{(k)} := \mathbf{X}_0 \wedge \mathbf{X}_1 \wedge \cdots \wedge \mathbf{X}_k$

Particle at $\mathbf{x}$ with velocity $\mathbf{X}$ has $\mathbf{\Sigma}_0^{(1)} = \underbrace{X^i(\mathbf{e}_0 \wedge \mathbf{e}_i)}_{\text{linear momentum}} + \underbrace{\frac{1}{2}(x^i X^j - x^j X^i)\, \mathbf{e}_i \wedge \mathbf{e}_j}_{\text{angular momentum}}$

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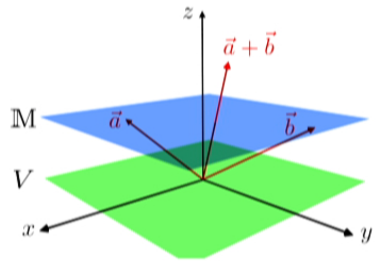

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Locally Klein bundle

- We want to actively use affine $\mathbb{M}$. Consider $\kappa := (\bar{\varphi}, \mathrm{Id}) \circ \varrho = (\varphi, \sigma)$, where $\varphi = \pi^* \bar{\varphi}$

$$\begin{array}{ccccc} U \equiv P|_{\bar{U}} = \pi^{-1}(\bar{U}) & \xrightarrow{\kappa} & \mathscr{O} \times H & \longleftrightarrow & G \\ \pi \downarrow & & \downarrow \mathrm{proj}_1 & & \downarrow \pi_G \\ \bar{U} & \xrightarrow{\bar{\varphi}} & \mathscr{O} \subset \mathbb{E}^m & \longleftrightarrow & G/H \end{array}$$

Klein G structure (def.)

Let $\pi : P \to P/H$. Specify an atlas $\mathcal{K} = \{U_\alpha, \kappa_\alpha\}$ of Klein charts $\kappa : U \to \kappa(U) \subset G$

(i) satisfying right G equivariance $\kappa(pg) = \kappa(p)g$

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Connection as parallelization

Underlying logic: "borrow" all kinematics from the model space $\mathbb{M}$, including admissible frames

- Parallelization of $\mathcal{N}$ = trivialization of $T\mathcal{N} \simeq \mathcal{N} \times W$

  smooth basis $\{X_1(p), ..., X_n(p)\} \subset T_p\mathcal{N}$ $\Leftrightarrow$ $\beta_p : T_p\mathcal{N} \xrightarrow{\sim} W$ – linear isomorphism

- Every Lie group is naturally parallelizable by (left) G action

  $L_{g^{-1}*} : T_gG \to T_eG$ $\Rightarrow$ $\mathfrak{g}$-valued (canonical) **Maurer-Cartan form** $\omega_G \equiv g^{-1}dg$

  Induced parallelism on $(P, \mathcal{K})$ via pullback $(\kappa^*\omega_G) : TP \xrightarrow{\kappa_*} TG \xrightarrow{\omega_G} \mathfrak{g}$

We call **fundamental vector fields** on $(P, \mathcal{K})$

$$X_p^{\mathbf{W}} := \frac{d}{ds}(p \exp s\mathbf{W})\Big|_{s=0} \in T_pP \qquad \text{generated by } \mathbf{W} \in \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$$

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Cartan connection

Definition

$\mathfrak{g}$-valued 1-form on $(P, \mathcal{K})$ determines infinitesimal parallelization $\varpi : TU \to \mathfrak{g}$ in local charts:

(i) $\varpi(X^{\mathbf{W}}) = \mathbf{W} \in \mathfrak{g}$ (point-wise Maurer-Cartan for each $p \in P$)

(ii) $(R_g^*)\varpi = \mathrm{Ad}(g^{-1})\varpi$ for all $g \in G$

- ➔ Unlike Ehresmann conn. $\ker \varpi = 0$. Unlike [Sharpe'97] covariance (ii) w.r.t. full G (not H).
- ➔ Transforms under $f \in \mathcal{G}(P, \mathcal{K})$ as $f^*\varpi = \tau^{-1}(\varpi + d)\tau$ for $\tau \in C(P, G)$.
- ➔ Introducing the split $\varpi = \omega + \theta \in \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ $\to$ *horizontal* and *vertical* distributions

$$T_pP = H_pP \oplus V_pP, \qquad HP = \ker \omega, \qquad VP = \ker \theta$$

Should not the (finite) connection be about lifting curves horizontally?
And then Ω measures non-involutivity?

Cartan connection

Definition

$\mathfrak{g}$-valued 1-form on $(P, \mathcal{K})$ determines infinitesimal parallelization $\varpi : TU \to \mathfrak{g}$ in local charts:

(i) $\varpi(X^{\mathbf{W}}) = \mathbf{W} \in \mathfrak{g}$ (point-wise Maurer-Cartan for each $p \in P$)

(ii) $(R_g^*)\varpi = \mathrm{Ad}(g^{-1})\varpi$ for all $g \in G$

- Unlike Ehresmann conn. $\ker \varpi = 0$. Unlike [Sharpe'97] covariance (ii) w.r.t. full G (not H).
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Development along paths

For a finite separation, choose intermediate frames and perform matching step-by-step

- Pull back the osculation eq. to I by a piecewise smooth path $\gamma : (I, s_i, s_j) \to (P, p_i, p_j)$ with $\dot\gamma(s) = \gamma_*(\partial_s) \in T_{\gamma(s)}P$. The (unique) smooth map

$$\begin{aligned}(\kappa\gamma):\ (I, s_0) &\to (G, g_0),\\ s &\mapsto (\mathbf{a}, A)(s),\end{aligned}$$

satisfying ordinary differential equations

$$\gamma^*\varpi(\partial_s) \;=\; (\kappa\gamma)^*\omega_G(\partial_s) \qquad \Leftrightarrow \qquad \begin{cases} -A^{-1}\dfrac{d\mathbf{a}}{ds} &= \theta(\dot\gamma(s)),\\ A^{-1}\dfrac{dA}{ds} &= \omega(\dot\gamma(s)),\end{cases}$$

is called the **development of** ϖ **on** G **along** γ starting at g_0.

➔ Non-homogeneous V-part generalizes H-holonomy to points and bound multivectors.

- From the properties of development $\Rightarrow$ κ behaves as covariant functor from the path-groupoid to G (w.r.t. $\circ$ and $^{-1}$)
- The right covariance $(\kappa\gamma h) = (\kappa\gamma)h$ w.r.t. $h : I \to H$ allows the projected $\overline{\kappa\gamma} \equiv \pi_G(\kappa\gamma)$ **development of the path** $\bar{\gamma} : I \to \mathcal{M} \cong P/H$ on $\mathbb{M} \cong G/H$

→ Curves that develop to π_G-projections of left G-translates generalize *autoparallels* to arbitrary groups of motion (straightest paths, or 'circles')

→ They coincide with shortest *geodesics* $\delta \int_I \theta = 0$ of Riemannian geometry (if 'vector torsion' is vanishing)

We propose this notion of development as a natural regularization tool for QG

Outline

How is the gravitational field encoded in the language of Cartan connections?

- The momentum of μ-particle $\gamma : I \to \mathbb{M}$ remains constant in the absence of interactions

$$\mathbf{P} \equiv \mu\, d\mathbf{m}(\dot\gamma) = \mu\, \frac{dx^i}{ds}\, \mathbf{e}_i, \qquad \dot{\mathbf{P}} \equiv \mu\, \frac{d^2x^i}{ds^2}\, \mathbf{e}_i = 0 \qquad \text{('principle of inertia')}$$

or is changed by the action of (grav.) force $\quad \dot{\mathbf{P}} = \mathbf{g} \quad$ ('mechanical law')

- In concordance with 'relativity postulate', write in the covariant form

$$dx \to d\mathbf{m} = \theta, \qquad d/ds \to \nabla_{\dot\gamma} \qquad \text{(with integrable connection of } \mathbb{M}\text{)}$$

Invariant content: curve will develop into straight line iff $\quad \dot\theta^i(\dot\gamma) + \omega^i{}_j(\dot\gamma)\,\theta^j(\dot\gamma) = \lambda\,\theta^i(\dot\gamma)$

'Equivalence principle': the parallelism of freely-falling observers is modified by $\mathbf{g}$ (on the level of geodesics), s.t. new coefficients $\varpi = \theta + \omega$ are *non-integrable*. Including torque and 'momentum law', the modified dynamics expresses the preservation of the sliding vector

$$d(\mathbf{m} \wedge \mathbf{P}) = (\mathbf{m} \wedge \mathbf{e}_i)\,[dP^i + \omega^i{}_j \wedge P^j] + (\mathbf{e}_i \wedge \mathbf{e}_j)[\theta^i \wedge P^j] = 0$$

Quantum connections?

Our main assertion: the fundamental local d.o.f. of the gravitational field are provided by (non-integrable) Cartan connections. We advance the quantum theory based on these variables. What could it look like?

The configuration space $\bar{\mathcal{A}}$ of generalized (distr.) connections in LQG arise as the spectrum of holonomy C^*-algebra $\overline{\mathrm{Cyl}}(\mathcal{A}) \cong C(\bar{\mathcal{A}})$, and characterized algebraically as $\mathrm{Hom}(\Upsilon, \mathrm{SU}(2))$

- Our constructed mapping κ from paths in $(P, \mathcal{K})$ to G extends it in a natural way
- Lacking is the description in terms of projective construction. What are the partial conf. spaces and their discrete geometry? Our spacetime connection being Lorentz and translation covariant, it is suggestive to consider the graph regularization in 4d.
- (Non-binding) particle picture: straight developed autoparallels correspond to "free motion", the "interactions" happening at the vertices ("spacetime coincidences").

Note: one could gain better control over diffeomorphisms in discrete, which correspond now to regular shifts (of "internal" space) $df(\mathbf{m} + s\mathbf{q}, e)/ds = -\rho_*(\mathbf{q})f(\mathbf{m}, e)$, $\rho_*(\mathbf{e}_i) = -\partial_i$

- **'Cartan moment of rotation'**

$$\star(\boldsymbol{\theta}\wedge\boldsymbol{\Omega}) = \hat{\mathbf{e}}^j\, E^i{}_j\, d^3\Sigma_i, \qquad E_{ij} \equiv R_{ij} - \frac{1}{2} R\, g_{ij}$$

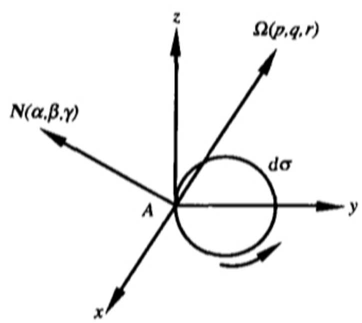
– *'geometric representation of the physical vector of energy-momentum'*

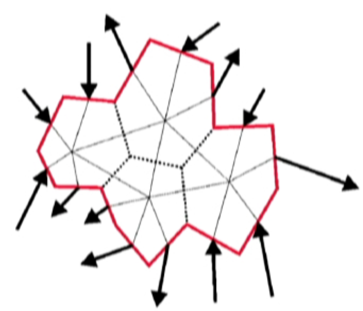
Construction: take (bivector of) rotation associated with elements (at $\mathbf{m}$) of $S_2 = \partial S_3$, enclosing small 3-volume around $\mathbf{a}$; take a sum of projections onto hyperplane $\perp(\mathbf{m}-\mathbf{a})$ (of the duals, essentially $\mathbf{J}$ of $\mathrm{SO}(3)$), multiplied by $|\mathbf{m}-\mathbf{a}|$.

- Has the structure of the Pauli-Lubanski vector, if $\boldsymbol{\theta}\mapsto\mathcal{P}\in\mathfrak{p}$, $\boldsymbol{\Omega}\mapsto\mathcal{J}\in\mathfrak{h}$

$$\star(\boldsymbol{\Omega}\wedge\boldsymbol{\theta}) \;\mapsto\; \mathcal{W} = \star(\mathcal{J}\wedge\mathcal{P}), \qquad \mathcal{W}_i \;:=\; \frac{1}{2}\varepsilon_{ijkl}\mathcal{J}^{jk}\mathcal{P}^l$$

- (Non-binding) mechanical analogy in 3d: writing $d\boldsymbol{\sigma} = \star\boldsymbol{\Sigma} = (\alpha,\beta,\gamma)d\sigma$ for the cycle, $\star\boldsymbol{\Omega} = (p,q,r)d\sigma$ for associated rotation and $K_{ij} = \frac{1}{4}\varepsilon_{ikl}\,\Omega^{kl}{}_{mn}\,\varepsilon_j{}^{mn}$ for the "double dual" of the curvature tensor, then relations are identical to the formulas of elasticity. Conservation laws then express the medium in equilibrium.





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