

Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 14

Speakers: Francois David

Collection: PSI 2019/2020 - Quantum Field Theory II (David)

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Gauge Fixing: Continued... $G = SU(2)$

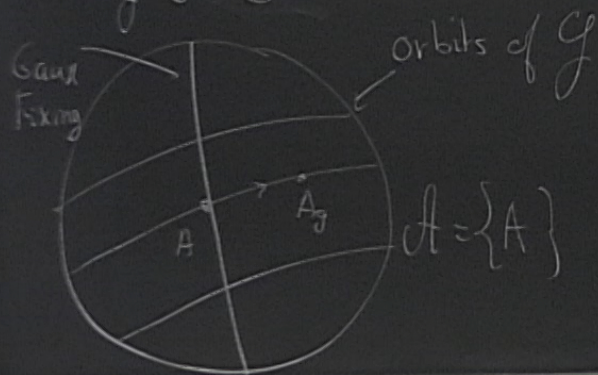
A Gauge Potentials

$A = (A_\mu) \in \text{Lie Algebra of } G$

$g \in G$

$$A \rightarrow A_g = g(A + \partial_\mu)g^{-1}$$

gauge transformation



Gauge Fixing Choice

$$F[A] = 0$$

$F: A \rightarrow \text{Lie Algebra}$
function $M_4 \rightarrow \text{Lie Algebra}$

$$A_\mu^a(x)$$

$$F^a(x) = F[A_\mu^b, \partial A_\mu^b, \dots]$$

$\uparrow$ Functional of $A_\mu^a(x) \rightarrow$ Function

$$S = \int \mathcal{L}[A_\mu^b, \partial A_\mu^b, \dots]$$

functional of $A_\mu^b(x) \rightarrow$ Function

Choose a $\mathcal{F}$

$$\int_{\mathcal{A}} \mathcal{D}[A] \rightarrow \int_{\mathcal{A}} \mathcal{D}[A] \cdot \delta[\mathcal{F}[A]]$$

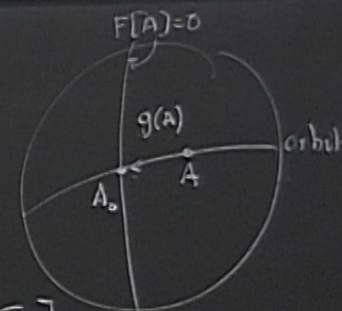
For any $A \in \mathcal{A}$, there is $g(A) \in \mathcal{G}$ such that $A \rightarrow A_0 : \mathcal{F}[A_0] = 0$

$A_0 = A_{g(A)}$ $g(A)$ depends on A (and on the gauge fixing $\mathcal{F}$)
 denote it $g_{\mathcal{F}}[A]$

start from:

$$\int_{\mathcal{A}} \mathcal{D}[A] \times \mathbb{I} = \int_{\mathcal{A}} \mathcal{D}[A] \cdot \int_{\mathcal{G}} \mathcal{D}[g] \delta[g - g(A)]$$

all gauge fields all gauge transformations



Pollev.com/danwohns778

pollev.com/perimeter167

$$\partial^\mu A_\mu^a(x) = 0$$

$$x = (t, \vec{x})$$

$$a = 1, 2, 3$$

Choose a F

$$\int_{\mathcal{A}} D[A] \rightarrow \int_{\mathcal{A}} D[A] \cdot \delta[F[A]] \cdot$$

For any $A \in \mathcal{A}$, there is $g(A) \in G$ such that $A \rightarrow A_0 : F[A_0] = 0$

$A_0 = A_{g(A)}$ $g(A)$ depends on A (and on the gauge fixing F)
 $\hookrightarrow$ denote it $g_F[A]$

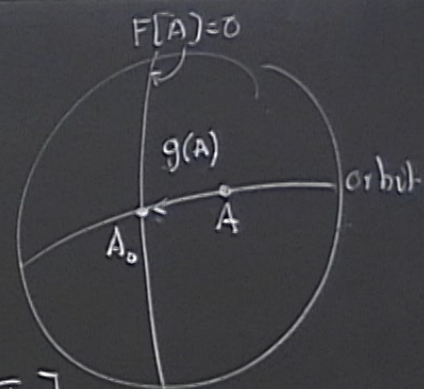
start from.

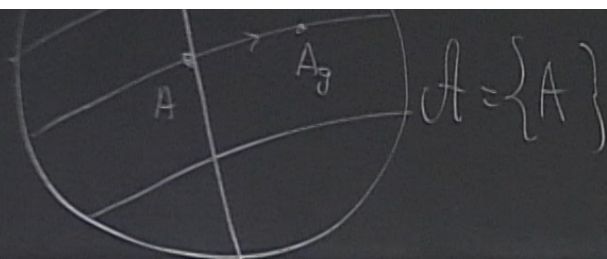
$$\int_{\mathcal{A}} D[A] \cdot \mathbb{1} = \int_{\mathcal{A}} D[A] \cdot \int_G D[g] \delta[g - g(A)] \quad \text{Trick}$$

all gauge fields

all gauge transfs G

$$D[g] = \prod_{x \in M_4} d\mu(g(x)) \quad \uparrow \text{Haar Measure on } G = SU(2)$$





function $M_4 \rightarrow \text{Lie Algebra}$

$g(A)$ is the solution of $F[A_g] = 0 \Rightarrow g = g(A)$

$$\Rightarrow \int_A D[A] \int_g D[g] \delta[F[A_g]] \cdot |\det[F'[A_g]]|$$

$$\prod_{x \in M_4} \prod_{a=1}^3 \delta[F^a[A_g](x)]$$

$$F'[A_g]_{ab}^a(x,y) = \frac{F^a[A_{g(1+i\alpha)}](x)}{\partial \alpha^b(y)}$$

3×3 matrix + kernel of some linear operator

$\alpha = \{\alpha(x) = \alpha^a(x) t_a\}$
element of the Lie Algebra

function $M_4 \rightarrow \text{Lie Algebra}$

$$\int_A D[A] \times 1 = \int_A D[A] \cdot \int_G D[g] \delta[g - g(A)]$$

all gauge fields all gauge fields

$$D[g] = \prod_{x \in \pi_4} d\mu$$

$$\Rightarrow g = g(A)$$

$$\text{at } [F'[A_g]]$$



$$\alpha = \{\alpha(x) = \alpha^a(x) t_a\}$$

element of the Lie Algebra

$$F'[A_g]_{ab}^a(x, y) = \frac{F^a[A_{g(1+i\alpha)}](x)}{\partial \alpha^b(y)} = J(x, y)_{ab}^a$$

3x3 matrix: Kernel of some linear operator

$$\text{Function } \Psi(y) = \{\Psi^b(y); y \in \pi_4\}$$

Ψ function from $\pi_4 \rightarrow \text{Lie Algebra}$

$F'[A]$ is a linear operator acting on the Ψ

$$F'[A] \cdot \Psi = \Phi \quad \Phi^a(x) \quad x \in \pi_4, a=1,3$$

$$\Phi^a(x) = \int_{\pi_4} d^4y \, J(x, y)_{ab}^a \Psi^b(y)$$

function $M_4 \rightarrow \text{Lie Algebra}$

$$\int_A D[A] \times 1 = \int_A D[A] \cdot \int_A D[g] \delta[g - g(A)]$$

all gauge fields all gauge fields

$$D[g] = \prod_{x \in \pi_4} d\mu$$

$$\Rightarrow g = g(A)$$

$$\text{at } [F'[A_g]]$$



$$\alpha = \{\alpha(x) = \alpha^a(x) t_a\}$$

element of the Lie Algebra

$$F'[A_g]_{ab}^a(x, y) = \frac{F^a[A_{g(1+i\alpha)}](x)}{\partial \alpha^b(y)} = J(x, y)^a_b$$

3x3 matrix: kernel of some linear operator

$$\text{Function } \Psi(y) = \{\Psi^b(y); y \in \pi_4\}$$

Ψ function from $\pi_4 \rightarrow \text{Lie Algebra}$

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$g(A)$ is the solution of $F[A_g] = 0 \Rightarrow g = g(A)$

$$\Rightarrow \int_A D[A] \int_g D[g] \delta[F[A_g]] \cdot |\det[F'[A_g]]|$$

$$\prod_{x \in M_4} \prod_{a=1}^3 \delta[F^a[A_g](x)]$$

$$F'[A_g]_{(x,y)}^a = \frac{F^a[A_{g(1+i\alpha)}](x)}{\partial \alpha^b(y)}$$

3x3 matrix + Kernel of some linear op

$\alpha = \{\alpha(x) = \alpha^a(x)\}$
element of the Lie algebra

$$\prod_{x \in \mathbb{M}_g} \prod_a \delta[F[A_g](x)]$$

$$F'[A_g]_{(x,y)}^a = \frac{\partial [A_{g(1+i\alpha)}](x)}{\partial \alpha^b(y)} = J(x,y)^a_b$$

3x3 matrix + kernel of some linear operator

Last step: use Gauss Invariance: invert the integrations

$$\int_g D[g] \cdot \int_A D[A_g] \cdot \delta(F[A_g]) \cdot |\det[F'[A_g]]| \exp(i S_{YM}[A_g])$$

• replace $A \rightarrow A_g$ in the Action
and in the measure

$$S_{YM}[A] = S_{YM}[A_g]$$

$$D[A] = D[A_g]$$

↓ replace

$$\exp(i S_{YM}[A_g])$$

$$\Rightarrow \int_A D[A] \int_g D[g] \delta[F[A_g]] \cdot |\det[F'[A_g]]|$$

$$\alpha = \left\{ \alpha^a(x) = \mathbb{R}^3(x) t_a \right\}$$

element of the Lie algebra

$F[A]$ is a linear operator acting on the Ψ

$$F[A] \cdot \Psi = \Phi \quad \Phi^a(x) \quad x \in \mathbb{R}^4, \quad a=1,3$$

$$\Phi^a(x) = \int_{\mathbb{R}^4} d^4y \, J(x,y)^a_b \Psi^b(y)$$

$$\prod_{x \in \mathbb{R}^4} \prod_{a=1}^3 \delta[F^a[A_g](x)]$$

$$F'[A_g]_{ab}^{xy} = \frac{F^a[A_{g(1+y)}](x)}{\partial \alpha^b(y)} = J(x,y)^a_b$$

3x3 matrix + kernel of the linear operator

Last step: use Gauss' Theorem: invert the integrations

$$Z = \int_g D[g] \cdot \int_A D[A] \cdot \delta(F[A_g]) \cdot |\det[F'[A_g]]| \exp(i S_{YM}[A_g])$$

• replace $A \rightarrow A_g$ in the Action and in the measure

$$S[A] = S[A_g]$$

$$D[A] = D[A_g]$$

• change of variable

$$Z = \int_g D[g] \cdot \int_A D[A] \delta[F[A]] |\det[F'[A]]| \exp(i S_{YM}[A])$$

↓ replace

$$\exp(i S_{YM}[A_g])$$

factors out

Gauss fixed functional integral

F.P. determinant ensure gauge invariance

$$= \text{Vol}(g) \times \int D[A] \delta[F[A]] |\det[F'[A]]| \exp(i S_{YM}[A])$$

$$x \in \mathbb{M}_4, a =$$

$$F[A_g]_b(x,y) = \frac{\partial \alpha^b(y)}{\partial x^b(y)}$$

3x3 matrix: Kernel of some linear operator

Last step: use Gauge Invariance: invert the integrations

$$Z = \int_g D[g] \cdot \int_A D[A_g] \cdot \delta(F[A_g]) \cdot |\det[F'[A_g]]| \exp(i S_{YM})$$

↓ replace

• replace $A \rightarrow A_g$ in the Action
and in the measure

$$S[A] = S[A_g]$$

$$D[A] = D[A_g]$$

• change of variable

$$Z = \int_g D[g] \cdot \int_A D[A] \delta[F[A]] |\det[F'[A]]| \exp(i S_{YM}[A]) = \left[\int \right]$$

$-j(x,y)$
 b
 pseudo

$S_{YM}[A]$
 $\frac{1}{g}$

replace

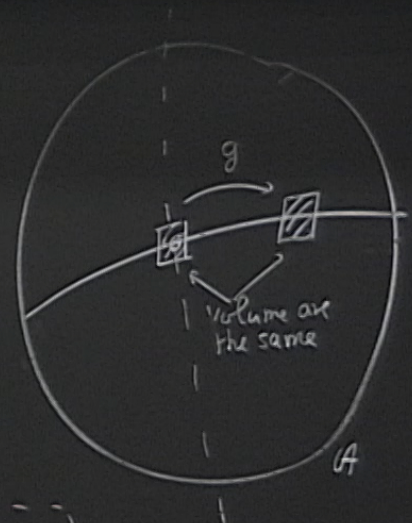
$[A_g]$

factors out

$$= \boxed{Vol(g)}$$

$$\times \int D[A] \delta[F[A]] |det[F'[A]]| \exp(i S_{YM}[A]) W[A]$$

gauge fixed functional integral
 $\Downarrow$
 F.P determinant ensure gauge invariance



$$x \in \mathbb{M}_4, a =$$

$$[A_g]_b^{(xy)} = \partial \alpha^b(x)$$

3x3 matrix + Kernel of some linear operator

Last step: use Gauge Invariance: invert the integrations

$$Z = \int_{\mathcal{G}} D[g] \cdot \int_{\mathcal{A}} D[A_g] \cdot \delta(F[A_g]) \cdot |\det[F'[A_g]]| \exp(i S_{YM})$$

↓ repl

• replace $A \rightarrow A_g$ in the Action
and in the measure

$$S[A] = S[A_g]$$

$$D[A] = D[A_g]$$

• change of variable

$$Z = \int_{\mathcal{G}} D[g] \cdot \int_{\mathcal{A}} D[\tilde{A}] \delta[F[\tilde{A}]] |\det[F'[\tilde{A}]]| \exp(i S_{YM}[\tilde{A}]) = \int_{\mathcal{A}} W[A]$$

gauge invariant observable

Explicit calculation

when $\epsilon = 0$

↓

$SU(2)$: Lorentz-Landau - Feynman

$$\partial^\mu A_\mu^a(x) - \epsilon^a(x) = 0 \Rightarrow F[A]^a(x)$$

↑ some function

Gauge transformation generated by $\alpha^a(x)$

structure constant
of $SU(2)$
↓

$$A_\mu^a(x) \rightarrow A_\mu^a(x) + D_\mu \alpha^a(x) = A_\mu^a(x) + \partial_\mu \alpha^a(x) + \epsilon_{bc}^a A_\mu^b(x) \alpha^c(x) = \tilde{A}_\mu^a(x)$$

$$F[\tilde{A}]^a(x) = F[A]^a(x) + \left[\partial^\mu D_\mu \alpha \right]^a(x) = \partial^\mu \partial_\mu \alpha^a(x) + \epsilon_{bc}^a \partial^\mu [A_\mu^b(x) \alpha^c(x)]$$

Explicit calculation

when $\epsilon = 0$

$SU(2)$: Lorentz-Landau-Feynman

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$$\left[\delta^a_c \partial^\mu \partial_\mu + \epsilon^a_{bc} A^b_\mu(x) \partial^\mu + \epsilon^a_{bc} (\partial^\mu A^b_\mu(x)) \right] \alpha^c(x)$$

F.P operator $F[A]$ is a differential operator

$$\left[\delta^a_c \Delta + \epsilon^a_{bc} A^b \cdot \partial + \epsilon^a_{bc} \partial \cdot A^b \right]$$

$$A^a_\mu(x)$$

$$\partial^\mu [A^b_\mu(x) \alpha^c(x)]$$

Explicit calculation when $\epsilon=0$

$SU(2)$: Lorentz-Landau-Feynman

$$\partial^\mu A_\mu^a(x) - \epsilon^a(x) = 0 \Rightarrow F[A]^a(x)$$

↑ some function

structure constant
of $SU(2)$

Gauge transformation generated by

$\alpha^a(x)$

$$A_\mu^a(x) \rightarrow A_\mu^a(x) + D_\mu \alpha^a(x) = A_\mu^a(x) + \partial_\mu \alpha^a(x) + \epsilon_{bc}^a A_\mu^b(x) \alpha^c(x) = \tilde{A}_\mu^a(x)$$

$$F[\tilde{A}]^a(x) = F[A]^a(x) + \underbrace{\left[\partial^\mu D_\mu \alpha \right]}_{\text{divergence}}^a(x) = \partial^\mu \partial_\mu \alpha^a(x) + \epsilon_{bc}^a \partial^\mu [A_\mu^b(x) \alpha^c(x)]$$

$$\left[\delta^a_c \partial^\mu \partial_\mu + \epsilon^a_{bc} A^b_\mu(x) \partial^\mu_x + \epsilon^a_{bc} (\partial^\mu_x A^b_\mu(x)) \right] \alpha^c(x) = (F[A] \cdot \alpha)^a(x)$$

F.P operator $F[A]$ is a differential operator

$$\left[\delta^a_c \Delta + \epsilon^a_{bc} A^b \cdot \partial + \epsilon^a_{bc} \partial \cdot A^b \right] = \partial \cdot D$$

depends on A

$$p[A^b_\mu(x) \alpha^c(x)]$$

$$\left[\delta^a_c \partial_x^\mu \partial_\mu + \epsilon^a_{bc} A^b_\mu(x) \partial_x^\mu + \epsilon^a_{bc} (\partial_x^\mu A^b_\mu(x)) \right] \alpha^c(x) = (F[A] \cdot \alpha)^a(x)$$

F-P operator $F[A]$ is a differential operator

$$\left[\delta^a_c \Delta + \epsilon^a_{bc} A^b \cdot \partial + \epsilon^a_{bc} \partial \cdot A^b \right] = \partial \cdot D$$

depends on $A \leftarrow SU(2)$ not commutative

det: is trivial $\leftarrow U(1)$
indep on A

$$[A^b_\mu(x) \alpha^c(x)]$$

$$\det [\square] = 0$$

$$\square \psi = 0$$

$\hookrightarrow$ zero mode

The Ghost and Antighost Field

$$\left| \det [F'[A]] \right| = \det [F'[A]] = \int D[\bar{c}, c] \exp(\bar{c} \cdot F'[A] \cdot c)$$

anticommuting fields (Grassman)
↓

Assume ☺
↑
no sign changes
no zero modes

Explicit calculation when $\epsilon=0$

SU(2): Lorentz-Landau - Feynman

$$\partial^\mu A_\mu^a(x) - \epsilon^a(x) = 0 \Rightarrow F[A]^a(x)$$

↑ same function

Gauge transformation generated by $\chi^a(x)$

structure constant of SU(2)

$$A_\mu^a(x) \rightarrow A_\mu^a(x) + D_\mu^a \chi^a(x) = A_\mu^a(x) + \partial_\mu \chi^a(x) + \epsilon_{bc}^a A_\mu^b(x) \chi^c(x) = \tilde{A}_\mu^a(x)$$

$$F[\tilde{A}]^a(x) = F[A]^a(x) + \left[\partial^\mu D_\mu \chi^a(x) \right] = \partial^\mu \partial_\mu \chi^a(x) + \epsilon_{bc}^a \partial^\mu [A_\mu^b(x) \chi^c(x)]$$

ordering

normal

$$\left[\delta_c^a \partial_\mu \partial^\mu + \epsilon_{bc}^a A_\mu^b(x) \partial^\mu + \epsilon_{bc}^a (\partial_\mu A_\mu^b(x)) \right] \chi^c(x) = (F[A] \cdot \chi)^a$$

F.P operator $F[A]$ is a differential operator

$$\left[\delta_c^a \Delta + \epsilon_{bc}^a A^b \cdot \partial + \epsilon_{bc}^a \partial \cdot A^b \right] = \partial \cdot D$$

depends on $A \leftarrow$ SU(2) not commutative

det: is trivial $\leftarrow$ U(1)

$$\det[F[A]] = \det[F[A]] = \int D[\bar{C}, C] \exp(-\bar{C} F[A] C)$$

Assume no sign changes
no zero modes

$$C = \{C^a(x)\}$$

ghost

$$\bar{C} = \{\bar{C}_a(x)\}$$

Antighost

spin changes $\rightarrow u=1, 2, 3$
no-spin, spin=0
 $\in$ Lie Algebra

$$S_{\text{ghost}}(\bar{C}, C, A) = \int_{M_4} d^4x \left[\bar{C}_a(x) \delta_b^a (-\Delta) C^b(x) + \epsilon_{bc}^a \partial_\mu \bar{C}_a(x) \cdot A_\mu^b(x) C^c(x) \right]$$

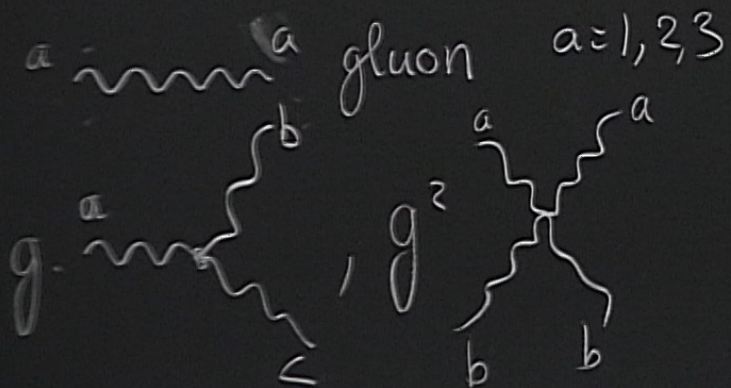
no-spin, spin=0

$$\int D[A] \int D[\bar{c}, c] \delta[F[A]] \exp(i \cdot (S_{YM}[A] + S_{Ghosts}[\bar{c}, c, A]))$$

no-spin, spin=0

$$\int D[A] \int D[\bar{c}, c] \delta[F[A]] \exp(i (S_{YM}[A] + S_{ghost}[\bar{c}, c, A]))$$

Quantize: Perturbation Theory



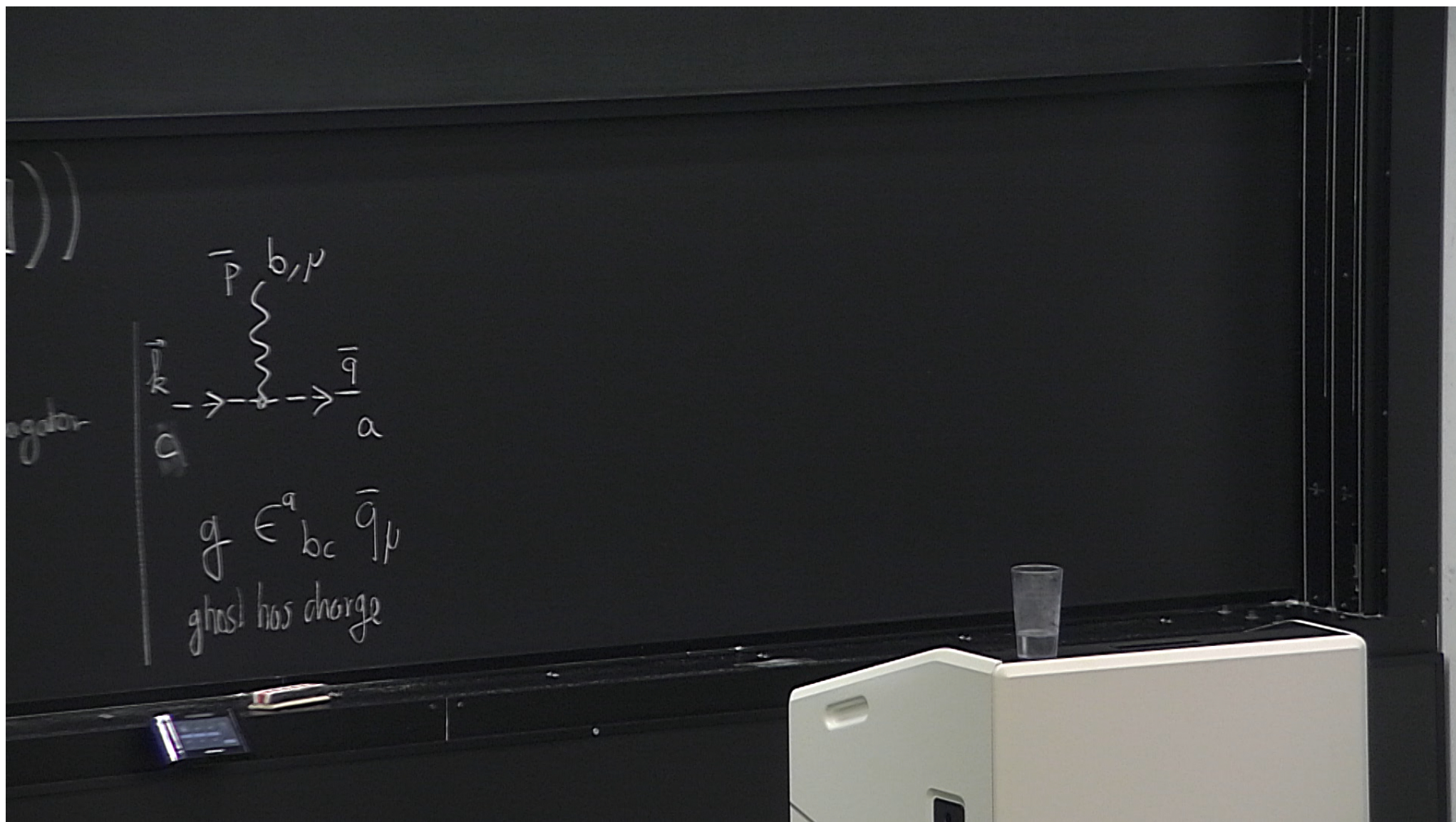
$A \rightarrow gA$

ghost

$$c^a \dashrightarrow \bar{c}^b$$

$a=1,2,3$

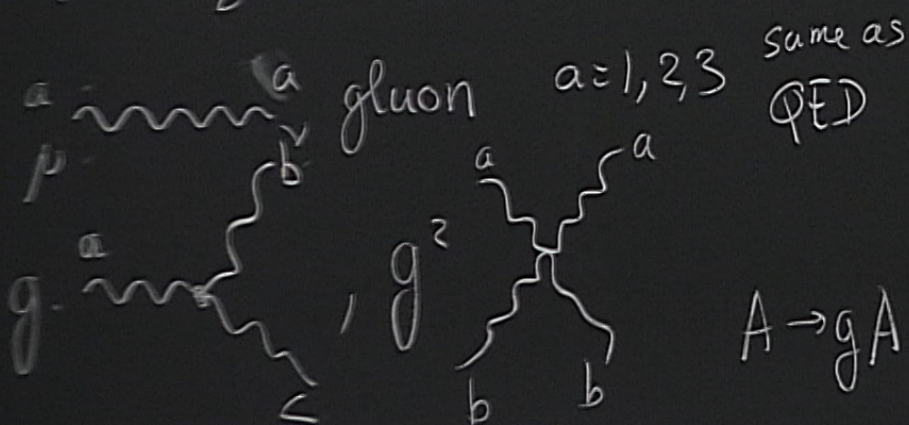
$\frac{\delta^{ab}}{p^2}$ propagator



no-spin, spin=0

$$\int D[A] \int D[\bar{c}, c] \delta[F[A]] \exp(i \cdot (S_{YM}[A] + S_{ghost}[\bar{c}, c, A]))$$

Quantize: Perturbation Theory

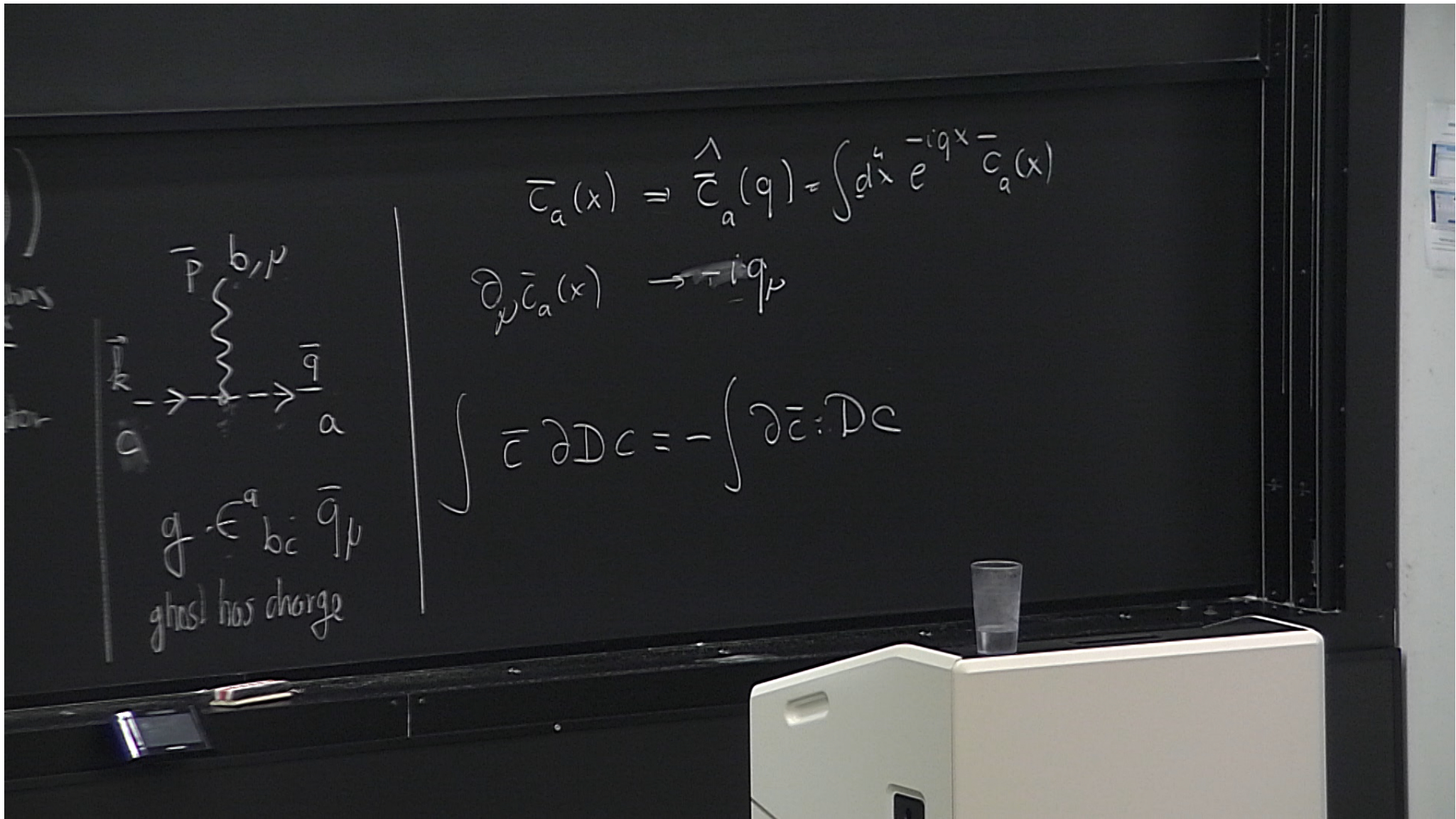


ghost: Fermions $\Rightarrow$ Anticommuting relations

$$c^a \dashrightarrow \bar{c}^b$$

$a=1,2,3$

$$\frac{\delta^{ab}}{p^2} \text{ propagator}$$



change of variable

$$Z = \int_g D[g] \cdot \int_{\mathcal{H}} D[\tilde{A}] \delta[F[\tilde{A}]] \left| \det[F'[\tilde{A}]] \right| \exp(i S_{\text{YM}}[\tilde{A}]) = \boxed{Vol(\mathcal{G})} \times \underbrace{\int D[A] \delta[F[A]] \left| \det[F'[A]] \right|}_{\text{the measure (S) field}}$$

$D[A] = D[A_g]$

$W[A]$

$$\text{tadpole} = \text{line} + g^2 \left(\underbrace{\text{one-loop} + \text{two-loop}}_{\substack{\text{troubles!} \\ \text{cancellations}}} - \underbrace{\text{bubble}}_{\substack{\text{troubles} \\ \text{unitarity, renormalizability}}} \right)$$