Title: Hamiltonian tomography for many-body systems

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Abstract: We show that a generic many-body Hamiltonian can be uniquely reconstructed from a single pair of initial-final states under the unitary time evolution. Interesting it is, this method is not practical due to its high complexity. We then propose a practical method for Hamiltonian reconstruction from multiple pairs of initial-final states. The stability of this method is mathematically proved and numerically verified.

This work is joint with Liujun Zou and Timothy Hsieh.

Hamiltonian Tomography for Many-body Systems

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(on going...)

Motivation 1

- Experiments try to realize various Hamiltonians.
- Need to verify the Hamiltonian: Tomography.
- Standard quantum states/process tomography is exponentially expensive.

Motivation 2

- Many-body states know a lot about their mother Hamiltonians.
  - Correlation decay $\sim$ gapped/gapless
  - Eigenstate thermalization hypothesis
  - Topological entanglement entropy knows quantum dimension of excitations
- Qi&Ranard 1712.01850: Generic Hamiltonian (many-body, with local interaction) can be uniquely determined by any of its eigenstate!
- Can the tomography process be simplified for many-body systems?

Problem Setting

- Quantum mechanics time evolution:

$$|\psi(0)\rangle \xrightarrow{U} |\psi(t)\rangle = e^{-\mathbb{i}tH} |\psi(0)\rangle$$

- Question: is it possible to determine H from $|\psi(0)\rangle$ and $|\psi(t)\rangle$?
- No! $\sim D$ equations, D^2 unknowns
- However, for many-body systems, H is often made of local interaction terms.
  - Tranverse Ising model: ZZ, X
  - Heisenberg model: XX, YY, ZZ
  - SYK model: $\psi\psi\psi\psi$
- poly(L) v.s. exp(L)

Problem Setting

- Assuming $H = \sum_{i=1}^{n} \omega_i L_i$, L_i local, $n = \mathrm{poly}(L)$. Try to solve $\{\omega_i\}$ from

$$|\psi(t)\rangle = e^{-\mathbb{i}t \sum \omega_i L_i} |\psi(0)\rangle .$$

- Nonlinear, transcendental.
- Idea: extract polynomial equations of ω.

Solving ω!

- Energy Conservation:

$$\langle\psi(0)|\, H^k\, |\psi(0)\rangle = \langle\psi(t)|\, H^k\, |\psi(t)\rangle$$

- $k = 1$:

$$\sum_{i=1}^{n} \Big[\langle\psi(0)|\, L_i\, |\psi(0)\rangle - \langle\psi(t)|\, L_i\, |\psi(t)\rangle \Big] \omega_i = 0.$$

- $k = 2$:

$$\sum_{i,j=1}^{n} \Big[\langle\psi(0)|\, L_i L_j\, |\psi(0)\rangle - \langle\psi(t)|\, L_i L_j\, |\psi(t)\rangle \Big] \omega_i \omega_j = 0.$$

Naive counting

- Number of unknowns: n (homogeneous), $n-1$ independent unknowns.
- Number of equations: up to $D-1$.
- Let's pick the first n equations. Hopeful they are "generic enough" to determine $\{\omega\}$ uniquely (up to scalar).
- The scalar can also be determined generically (no exact recurrence).

When Fail?

- Will fail if somehow $M \equiv 0$.
  e.g.: if $|\psi(0)\rangle = |\psi(t)\rangle$ (e.g.: eigenstate)
- Will fail if $[H, \mathcal{O}] = 0$, $\mathcal{O}$ = sum of L_i.
  e.g.: classical/fine-tuned MBL cases/fine-tuned integrable cases

The above method is not practical...

- Number of M coefficients $\sim \exp(n)$. Experiments not possible.
  (recall the last equation:
  $\sum_{i_1 i_2 \cdots i_n} M_{i_1 i_2 \cdots i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = 0$)
- NP-complete to solve polynomial equations.
- Even numerical method is expensive.

Linear Algebra Warm-up

- n linear equations for n homogeneous unknowns:

$$Ax = 0.$$

- Generic A admits no solution.
- At least one solution $\Leftrightarrow \det A = 0$.
- At least two solutions
  $\Leftrightarrow \text{rank}(A) \leq n-2 \Leftrightarrow$ all $(n-1)$-minors have $\det = 0$.

Polynomial Case

- n homogeneous equations for n homogeneous unknowns:

$$\sum_{i_1 i_2 \cdots i_k} M_{i_1 i_2 \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k} = 0,$$

- Generic M admits no solution.
- At least one solution on $\mathbb{C} \Leftrightarrow \mathrm{poly}_1(M) = 0$ (Macaulay resultant).
- At least two solutions on $\mathbb{C} \Leftrightarrow \mathrm{poly}_2(M) = 0$ for several polys.

Polynomial Case

- In our case, one solution exists automatically ($x = \omega$), $\Rightarrow \text{poly}_1(M) = 0$.
- We want $\text{poly}_2(M) \neq 0$ for at least one poly_2.
- Key observation: $\text{poly}_2(M)$ is real analytic in ω. If a real analytic function is nonzero at some point, it's nonzero almost everywhere.
- One only needs to check one point.

Polynomial Case

Short summary:

Theorem

Fix the system and fix a basis $\{L_i\}$ for local operators. If $H_0 = \sum_i \omega_i^0 L_i$ can be uniquely determined from the above procedure, then almost all $H = \sum_i \omega_i L_i$ can be uniquely determined.

Checks

- (analytical) translational-invariant transverse Ising with next-to-nearest-neighbor interaction ($L = \infty$)

$$H = \lambda_1 \sum ZZ + \lambda_2 \sum X + \lambda_3 \sum YIY$$

- (numerical) translational-invariant Ising model with magnetic field(s)/Heisenberg model ($L = 10$)

$$H = \lambda_1 \sum ZZ + \lambda_2 \sum X + \lambda_3 \sum Z$$
$$H = \lambda_1 \sum XX + \lambda_2 \sum YY + \lambda_3 \sum ZZ$$

- (numerical) tranverse Ising with random couplings ($L = 4$, totally 7 local operators)

$$H = \sum \lambda_i Z_i Z_{i+1} + \sum h_i X_i$$

A practical protocol

- Only use 1st (linear) equation:

$$\langle\psi(0)|\, H\, |\psi(0)\rangle = \langle\psi(t)|\, H\, |\psi(t)\rangle\,.$$

- Use $n-1$ pairs of initial-final instead of one pair.
- Find kernel of a linear system:

$$Mx = 0,$$

where

$$M_{ij} = \langle\psi_i(0)|\, L_j\, |\psi_i(0)\rangle - \langle\psi_i(t)|\, L_j\, |\psi_i(t)\rangle\,.$$

- Same analytic function argument applies.
- Check (numerical): spin chain with random 1,2-body operators ($L = 8$, $8 \times 3 + (8-1) \times 9 = 87$ local operators).

Tomography with errors

- Measurements always come with errors
- Solution might be very unstable
- Formally: given $\sum_j M_{ij}\omega_j = 0$ for each $i = 1, 2, \cdots, k$. Measure M_{ij} with error, try to "guess" $\{\omega\}$.
- Linear regression! (homogeneous version)

$$y = \sum k_i x_i + b$$

$k_i, b \leftrightarrow \omega, \quad x_i, y \leftrightarrow M_{ij}$

- Least square method: minimize $\sum_i (\sum_j M_{ij}\omega_j)^2$ subject to $\sum_j \omega_j^2 = 1$.
- Solution: the eigenvector of $M^T M$ with minimal eigenvalue. Equivalently, singularvector of M with minimal singularvalue.

Tomography with errors

Short summary:

1. For a initial state $|\psi_i(0)\rangle$, measure M_j $(j = 1, 2, \cdots, n)$.
2. Repeat for k different initials $k \geq n - 1$.
3. Write down the $k \times n$ matrix M, find its singularvector with minimal singularvalue.

Error estimation

Theorem

Denote n = number of unknowns, k = number of initial states, λ = the gap of the smallest singularvalue (0) and the 2nd smallest singularvalue of $\frac{1}{\sqrt{k}}M$, ϵ = typical error on each M_{ij}, then the averaged error is controlled by:

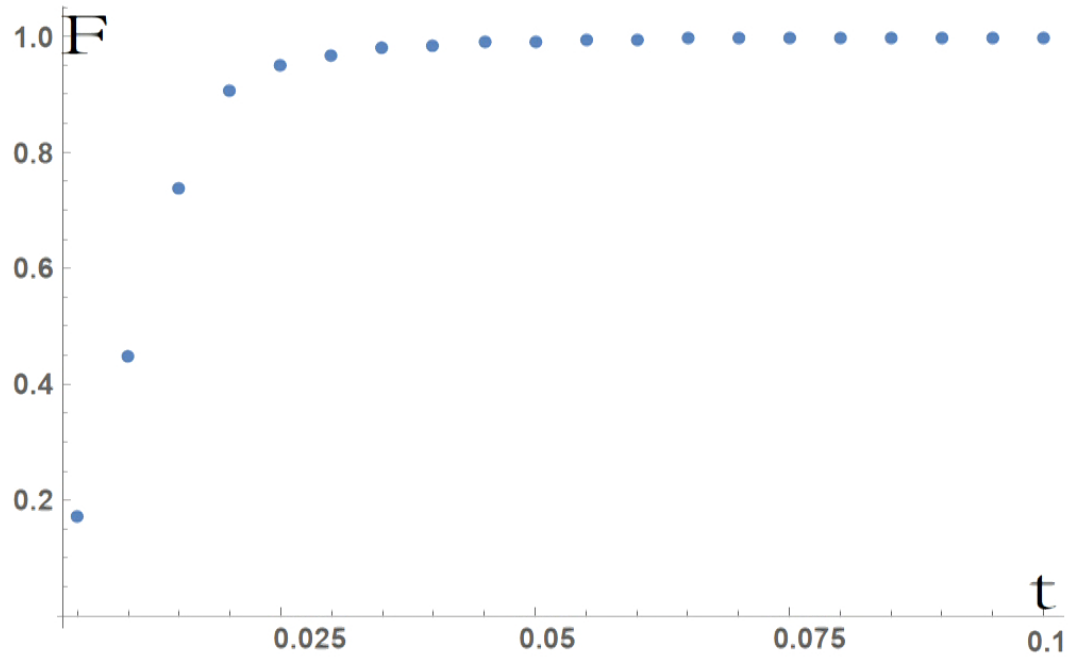
$$\theta \leq C\sqrt{\frac{n}{k}}\frac{\epsilon}{\lambda},$$

where θ is the angle between the reconstructed Hamiltonian and the true Hamiltonian.

Try to understand the gap λ and make it big, by tuning

- time between final and initial
- the initial states ensemble

Understanding the Gap–time



($L = 8$, $k = 256$, error and Hamiltonian averaged)

Reason: recall $M_{ij} = \langle \psi_i(0) | L_j - L_j(t) | \psi_i(0) \rangle$.

Understanding the Gap—initial states ensemble

Choice 1: Haar random states

$$\lambda^2 \leq \frac{1}{n-1}\left(\frac{1}{k}M^T M\right) = \frac{1}{n-1}\sum_j \overline{\langle\psi| L_j - L_j(t) |\psi\rangle^2}$$

$$= \frac{1}{n-1}\sum_j \int d\psi \, \langle\psi| L_j - L_j(t) |\psi\rangle^2$$

$$\sim \frac{1}{D}.$$

Very bad.

Understanding the Gap—initial states ensemble

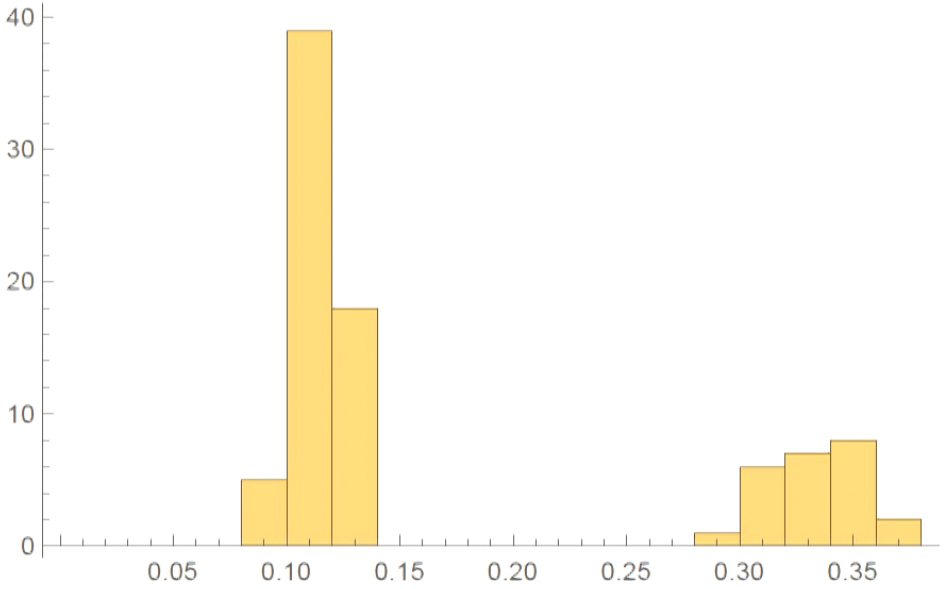
Choice 2: random product states

Claim

Assuming the eigenstate thermalization hypothesis (ETH) as well as some numerically verified approximation, singlarvalues of $\frac{1}{\sqrt{k}}M$ *are distributed near* $(\frac{1}{\sqrt{3}})^l$, *gap*$=(\frac{1}{\sqrt{3}})^{l_{max}}$. *Here* l *are possible lengths of the local operators,* l_{max} *is the max of them.*

Understanding the Gap—initial states ensemble

A sample distribution of eigenvalues $(\frac{1}{k}M^T M)$:



(operator basis $\{L_i\}$: 1-body and 2-body Pauli)

Another type of stability

- Interactions in real world are not strictly local:

$$H = \sum_i \omega_i L_i + \sum_{i'} \nu_{i'} L_{i'}.$$

- If reconstruct as

$$H = \sum_i \bar{\omega}_i L_i,$$

how far will $\bar{\omega}$ be different from ω?

Another type of stability

Recall

$$M_{ij} = \langle \psi_i(0) | L_j | \psi_i(0) \rangle - \langle \psi_i(t) | L_j | \psi_i(t) \rangle ,$$

block matrix $M = (M_1; M_2)$.

Theorem

Denote the true Hamiltonian is $\begin{pmatrix} \omega \\ \nu \end{pmatrix}$, *the reconstructed Hamiltonian with our method is* $\begin{pmatrix} \bar{\omega} \\ 0 \end{pmatrix}$. *Then the angle between* ω *and* $\bar{\omega}$ *is controlled by:*

$$\sin\theta \leq \frac{\left\| \sqrt{\frac{1}{k}} M_2 \nu \right\|}{\lambda \|\omega\|} \leq \frac{\left\| \sqrt{\frac{1}{k}} M_2 \right\|}{\lambda} \frac{\|\nu\|}{\|\omega\|}.$$

Another type of stability

- Interactions in real world are not strictly local:

$$H = \sum_i \omega_i L_i + \sum_{i'} \nu_{i'} L_{i'}.$$

Summary

1. Generic many-body Hamiltonian can be uniquely determined from a single pair of initial-final states.
2. By using multiple initial-final pairs, one gets practical protocol Hamiltonian tomography.
3. This protocol has complexity $O(n^2)$, experimental feasible.
4. This protocol is stable
   (1) against errors
   (2) against ignorance of longer-range/many-body interactions.