Title: Mapping class group actions on Hopf algebra lattice models

Speakers: Thomas Voss

Series: Quantum Gravity

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Abstract: Hopf algebra lattice models are related to certain topological quantum field theories and give rise to topological invariants of oriented surfaces. Examples are the combinatorial quantisation of Chern-Simons theory and the Kitaev model.

Our main result is the construction of a mapping class group action on these models, formulated under weaker assumptions than the associated topological quantum field theories, namely for pivotal Hopf algebras in symmetric monoidal categories. This description also yields new structures for semisimple finite-dimensional Hopf algebras. The action of the mapping class group is defined in terms of simple graph transformations and allows one to compute quantum Dehn twists explicitly.

Mapping class group actions on Hopf algebra lattice models

Thomas Voß

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November 28, 2019

Kitaev lattice model [Kitaev, Buerschaper, Mombelli,...]:

- Ingredients:
  - semi-simple, finite-dimensional Hopf algebra H over $\mathbb{C}$
  - graph (V, E) embedded on oriented surface with boundary $\Sigma_{g,n}$
- Structures:
  - vector space $H^{\otimes E}$
  - excitations at boundaries $\rightarrow$ $D(H)^{\otimes n}$-module structure on $H^{\otimes E}$
  - topological invariant: protected space $\mathcal{H} \rightarrow D(H)^{\otimes n}$-invariants
  - Projector $H^{\otimes E} \rightarrow \mathcal{H}$
- $2d$-part of extended Turaev-Viro-TQFT [Balsam,Kirillov].

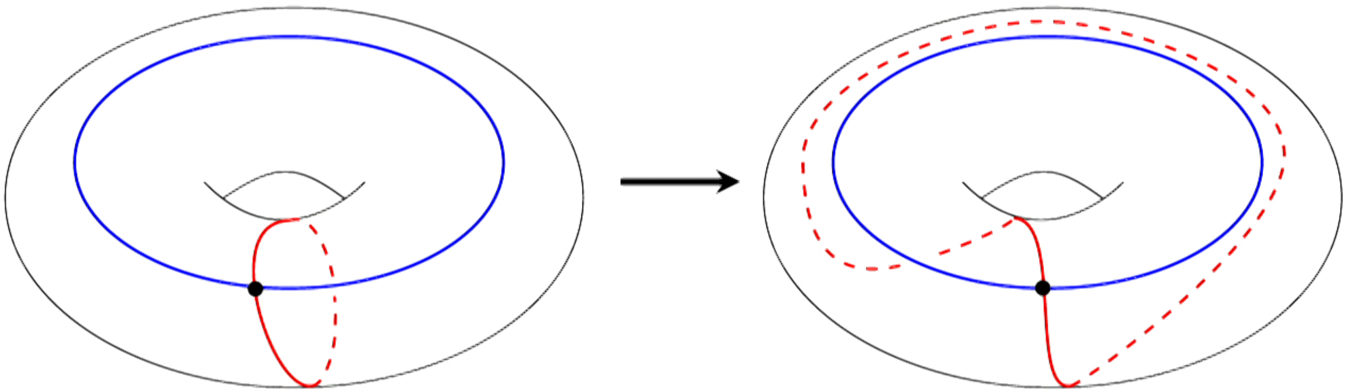
Results:

- Generalization (of aspects of the model) to pivotal Hopf algebras in symmetric monoidal categories
- Toolkit of simple graph transformations
- Explicit expressions for mapping class group actions on this model

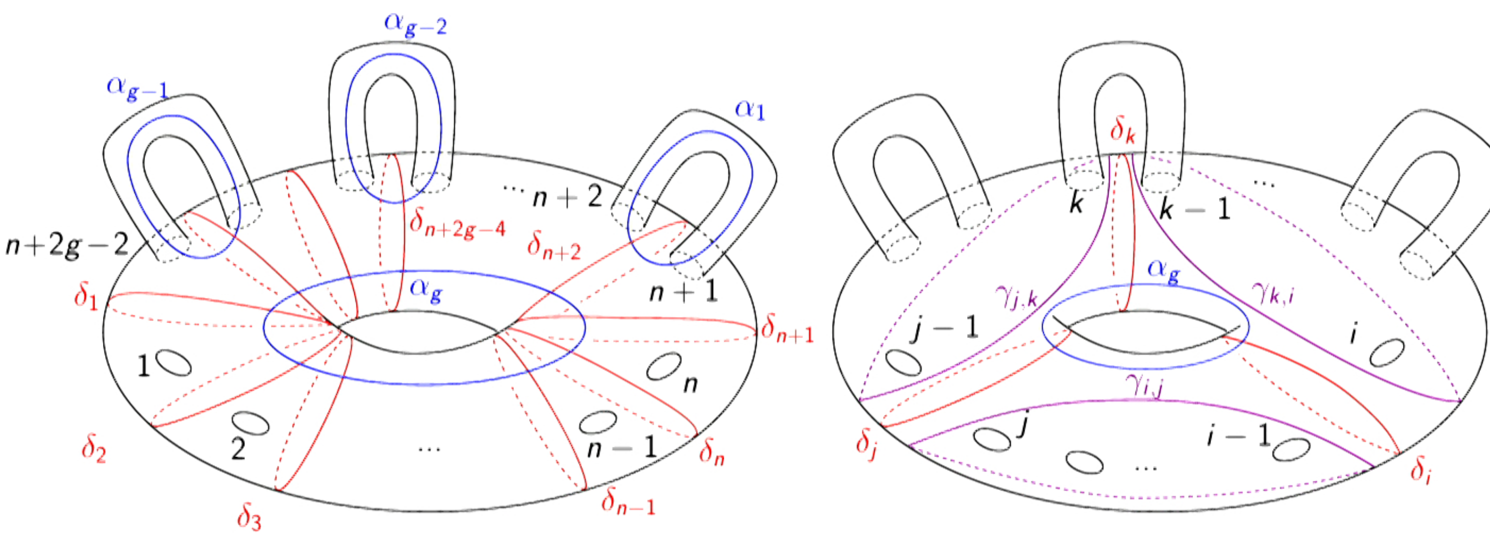
The mapping class group

Surface $\Sigma = \Sigma_{g,n} \rightarrow \mathrm{Map}(\Sigma) := \mathrm{Diff}^{+}(\Sigma, \partial\Sigma)/\mathrm{Diff}_0(\Sigma, \partial\Sigma)$

- symmetries for structures on Σ up to isotopy.
- e.g. on moduli space of flat connections.
- non-isotopic embeddings of $\Gamma \sim \mathrm{Map}(\Sigma)$.
- generated by Dehn twists around simple loops.



Gervais' presentation - generators



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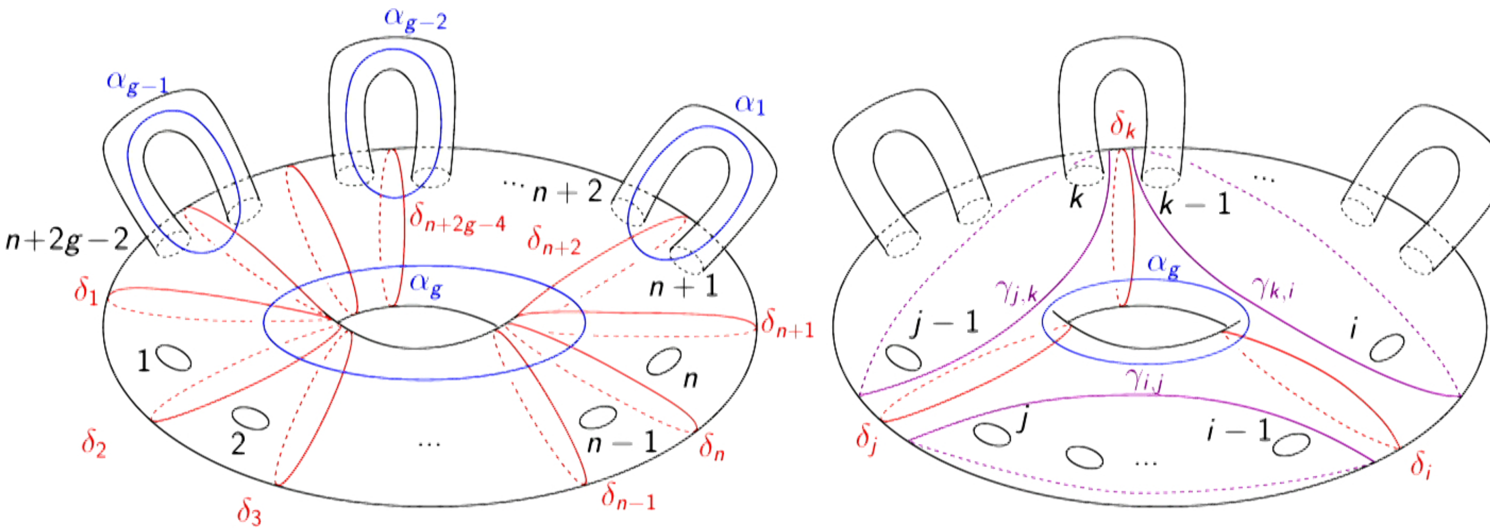


Gervais' presentation -relations

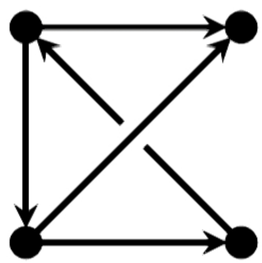
Theorem (Gervais)

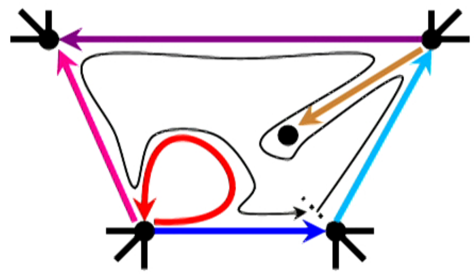
$\mathrm{Map}(\Sigma_{g,n})$ *generated by twists around paths* $\alpha_k, \delta_i, \gamma_{i,j}$ *with* $k = 1, \ldots, g$, $i, j = 1, \ldots, n + 2g - 2$ *with relations*

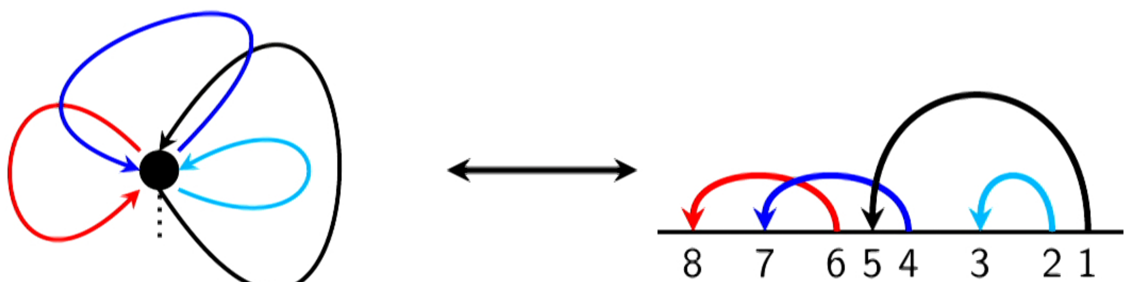
$$\begin{aligned}
D_\rho D_\pi &= D_\pi D_\rho && \text{if } \pi \cap \rho = \emptyset \\
D_\rho D_\pi D_\rho &= D_\pi D_\rho D_\pi && \text{if } |\pi \cap \rho| = 1 \\
D_{\gamma_{i,j}} D_{\gamma_{j,k}} D_{\gamma_{k,i}} &= \left(D_{\delta_i} D_{\delta_j} D_{\delta_k} D_{\alpha_g}\right)^3 && \text{for } i \leq j \leq k \text{ up to cyclic permutation} \\
D_{\gamma_{n+2i,n+2i-1}} &= D_{\gamma_{n+2i-1,n+2i-2}} && \text{for } i = 1, \ldots, g \\
D_{\gamma_{i,i}} &= id && \text{for } i = 1, \ldots, n + 2g - 2
\end{aligned}$$

Ribbon graphs

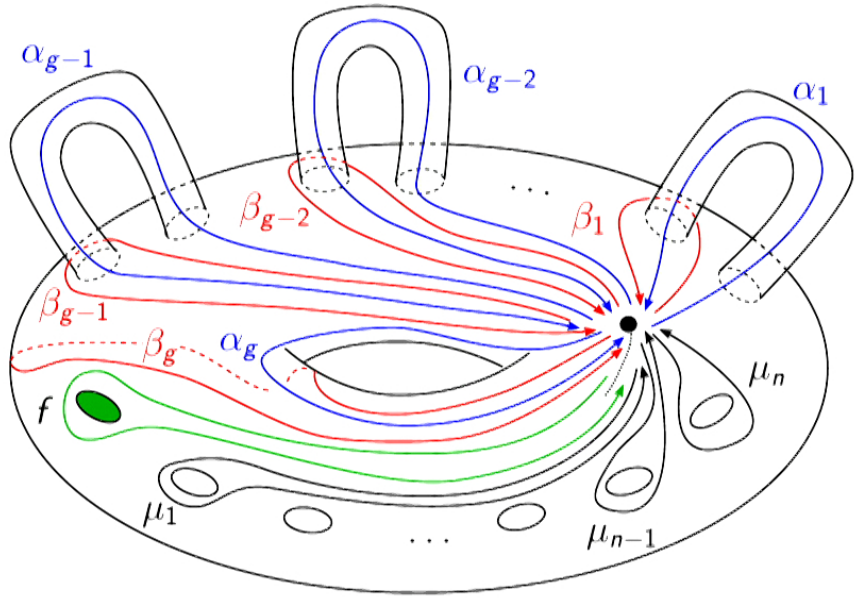
- oriented graph Γ embedded in oriented surface Σ $\sim$ ribbon graph
- edge ends at a vertex are cylcically ordered
- components of $\Sigma \setminus \Gamma$ $\sim$ faces of Γ
- e.g. triangulation of Σ
- single vertex $\rightarrow$ chord diagram

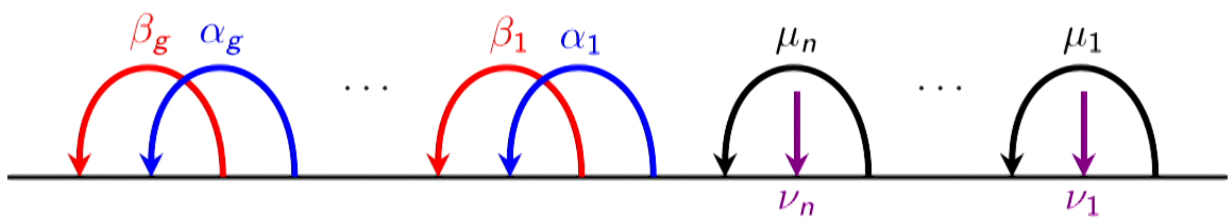






Ribbon graph for $\Sigma_{g,n+1}$

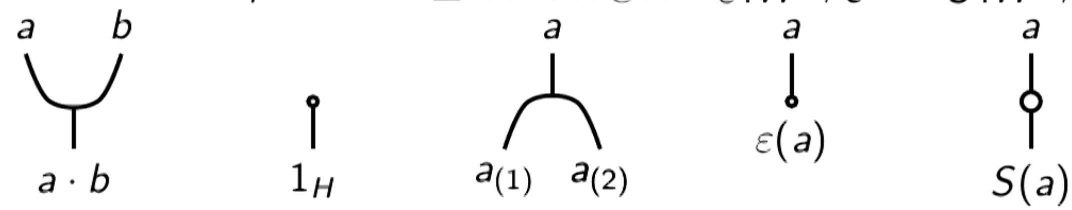




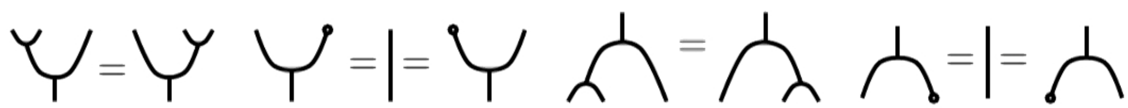
Hopf algebras in symmetric monoidal categories

- object H
- morphisms:

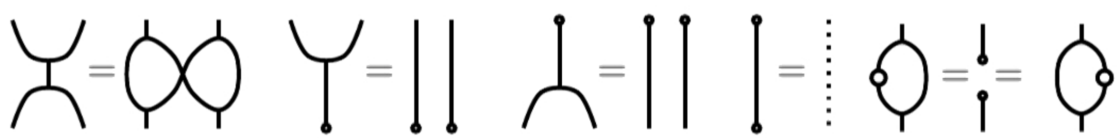
$\mu : H \otimes H \to H$ $\quad$ $\eta : e \to H$ $\quad$ $\Delta : H \to H \otimes H$ $\quad$ $\varepsilon : H \to e$ $\quad$ $S : H \to H$



- conditions:
  - (co-)algebra:



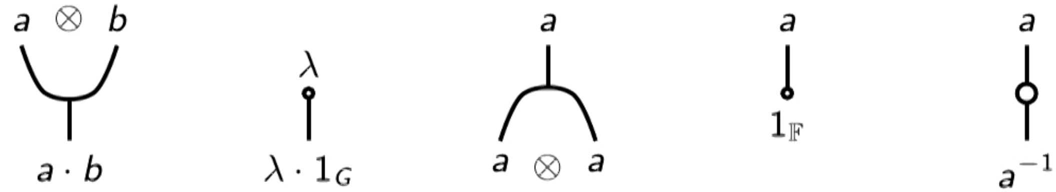
  - Hopf algebra:

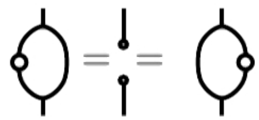


Examples of Hopf algebras

- group $G \to$ group algebra $H = \mathbb{F}[G]$
  - morphisms:

$\mu : H \otimes H \to H$ $\quad \eta : \mathbb{F} \to H$ $\quad \Delta : H \to H \otimes H$ $\quad \varepsilon : H \to \mathbb{F}$ $\quad S : H \to H$



  - antipode condition  translates to $a^{-1}a = 1_G = aa^{-1}$.
- group G in Set, Top, ...
- Lie algebra $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$.
- semisimple, finite-dimensional Lie algebra $\mathfrak{g} \to \mathcal{U}_q(\mathfrak{g}), \mathcal{U}_q^{\mathrm{res}}(\mathfrak{g})$.

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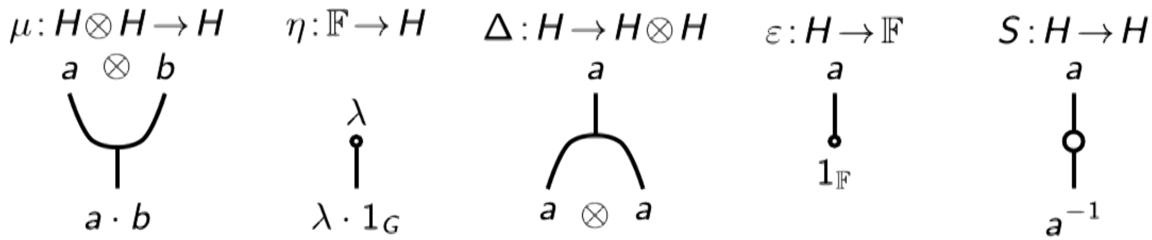
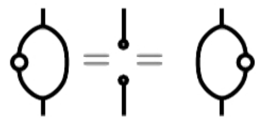


  - antipode condition  translates to $a^{-1}a = 1_G = aa^{-1}$.
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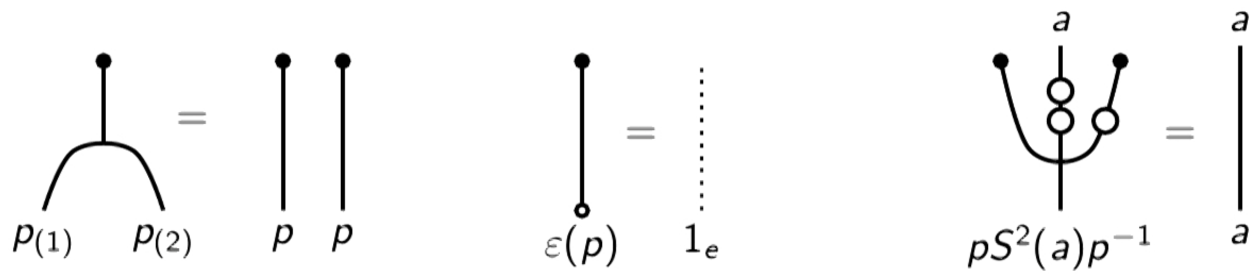
Pivotal Hopf algebras

- choice of $p : e \to H =$  with



- defines involution $T : H \to H :=$  $:=$ 
- **examples:**
  - group G or group algebra $\mathbb{F}[G]$ with $p \in Z(G)$.
  - $\mathcal{U}_q(\mathfrak{g})$ or $\mathcal{U}_q^{res}(\mathfrak{g})$ with p associated to the sum of positive roots.
  - involutive H with pivot 1_H, e.g. $\mathcal{U}(\mathfrak{g})$, even for non-semisimple $\mathfrak{g}$.
  - ribbon Hopf algebras.

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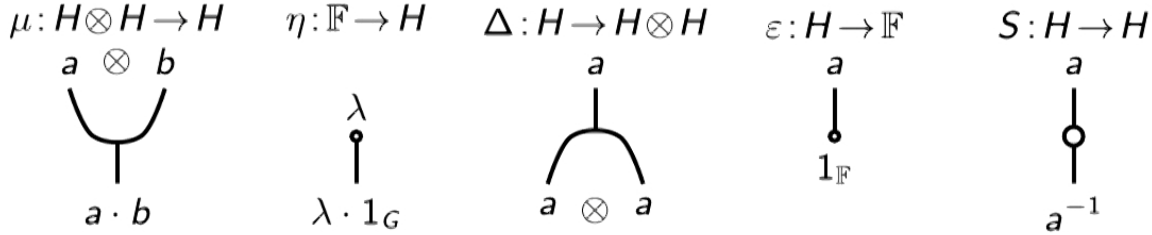
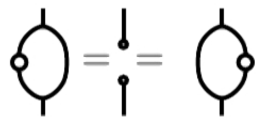


  - antipode condition  translates to $a^{-1}a = 1_G = aa^{-1}$.
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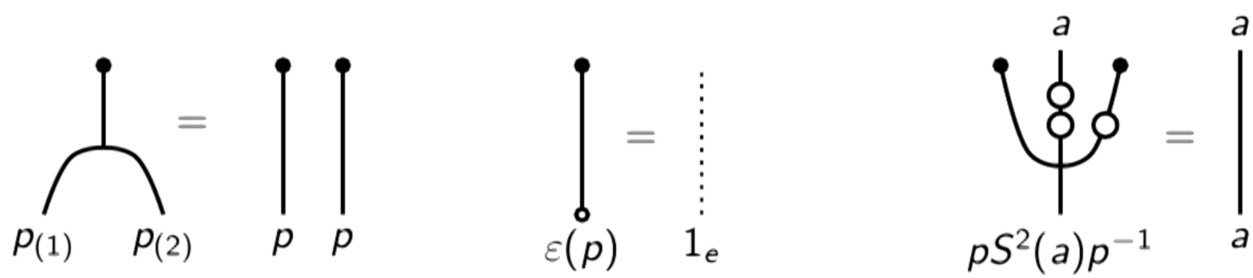

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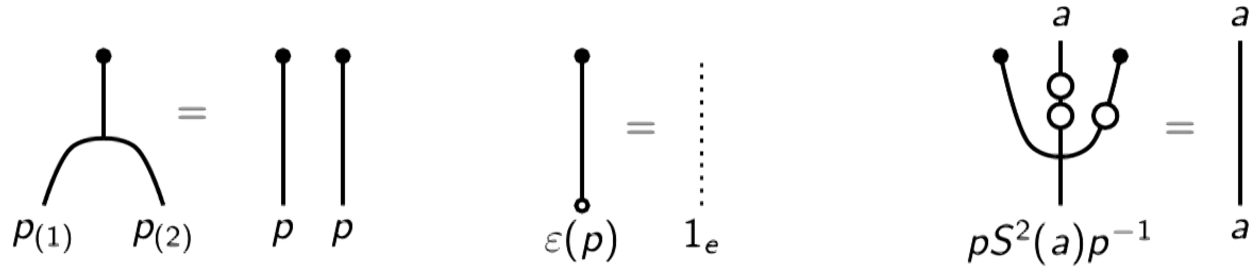
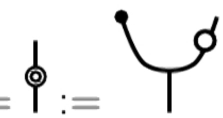


- defines involution $T : H \to H :=$ 
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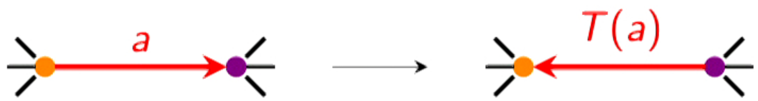
Hopf algebra lattice model

- **ingredients:**
  - graph $\Gamma = (V, E)$ embedded in a surface.
  - pivotal Hopf algebra (H, p) in a symmetric monoidal category $\mathcal{C}$.
- **construct:**
  - object $H^{\otimes E} \sim$ label edges with 'elements' of H
  - triangle operators.
  - face and vertex operators.
  - independence of edge orientation.



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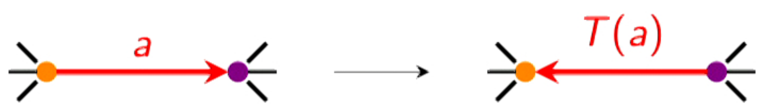
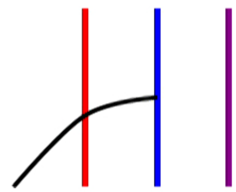


Triangle operators

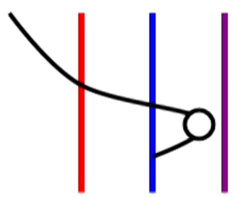
edge $\alpha \rightarrow 2$ H-left module & H-left comodule structures



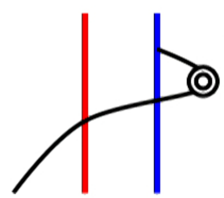
$$\delta_{\alpha+}(a) = a_{(1)} \otimes a_{(2)}$$

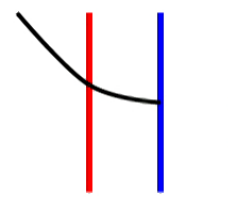
$$h \triangleright_{\alpha-} a = a \cdot S(h)$$

$$h \triangleright_{\alpha+} a = h \cdot a$$



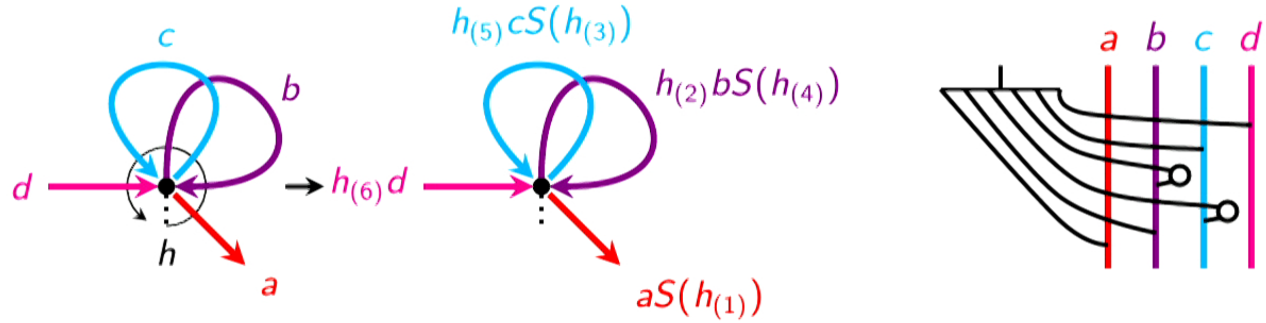
$$\delta_{\alpha-}(a) = T(a_{(2)}) \otimes a_{(1)}$$



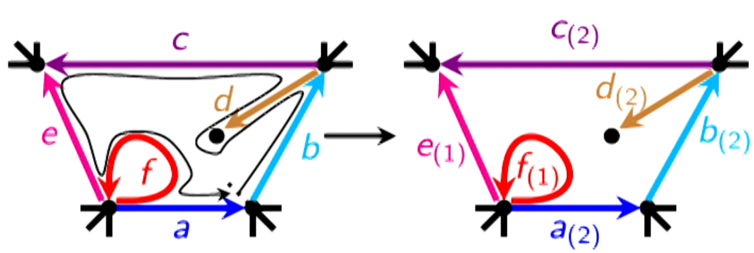


Face and vertex operators

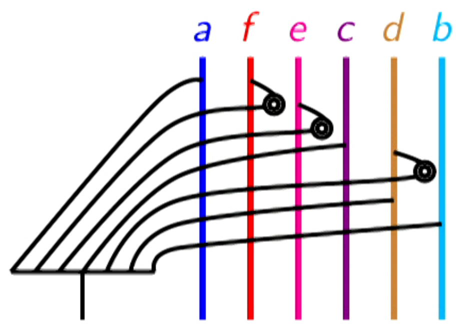
- vertex operators:



- face operators:



$$a_{(1)}T(f_{(2)})T(e_{(2)})c_{(1)}T(d_{(3)})d_{(1)}b_{(1)}$$



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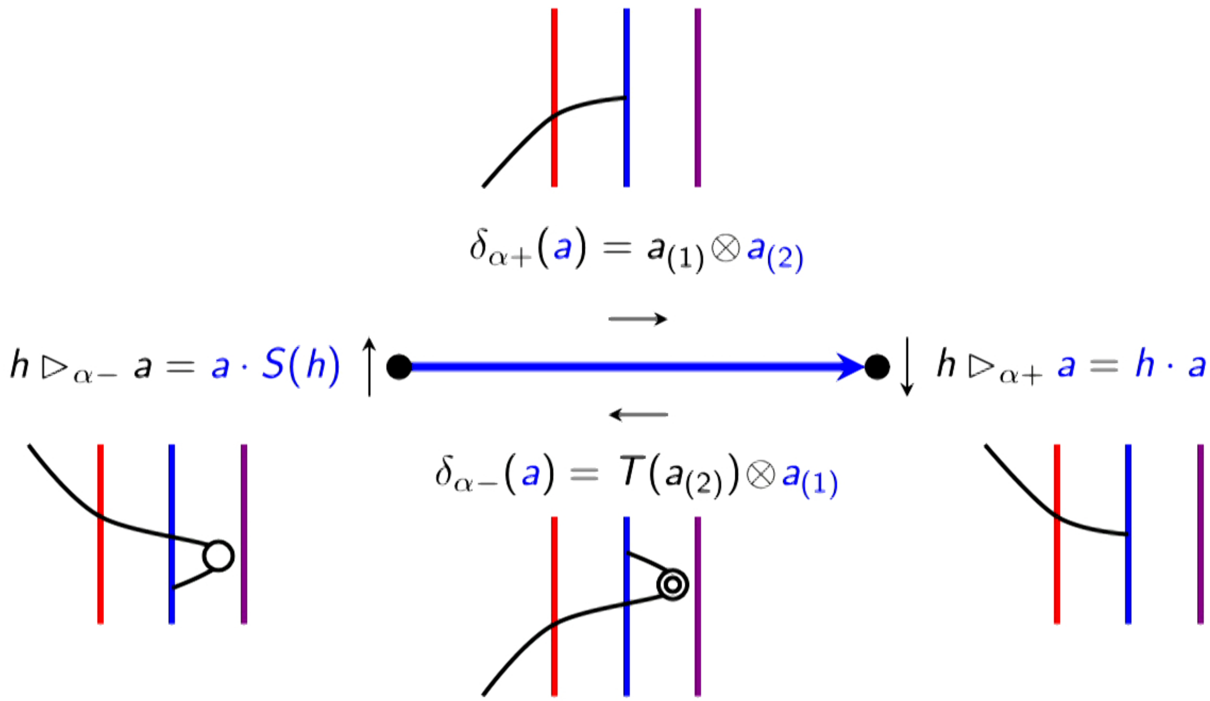
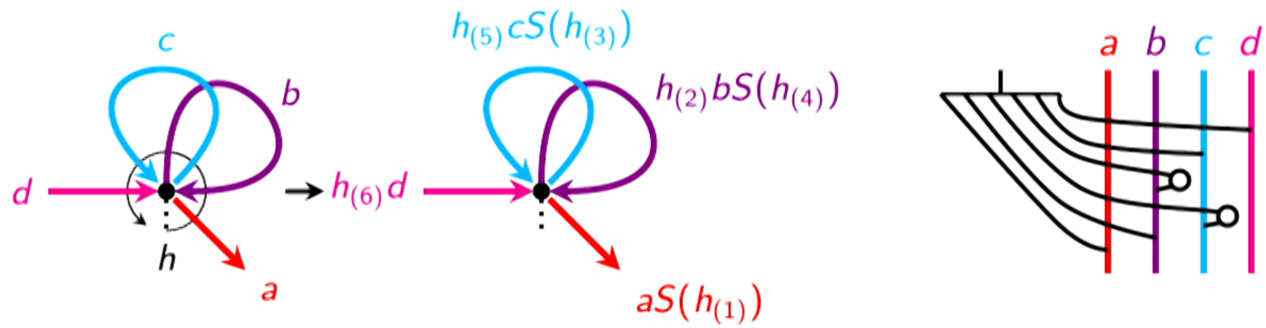
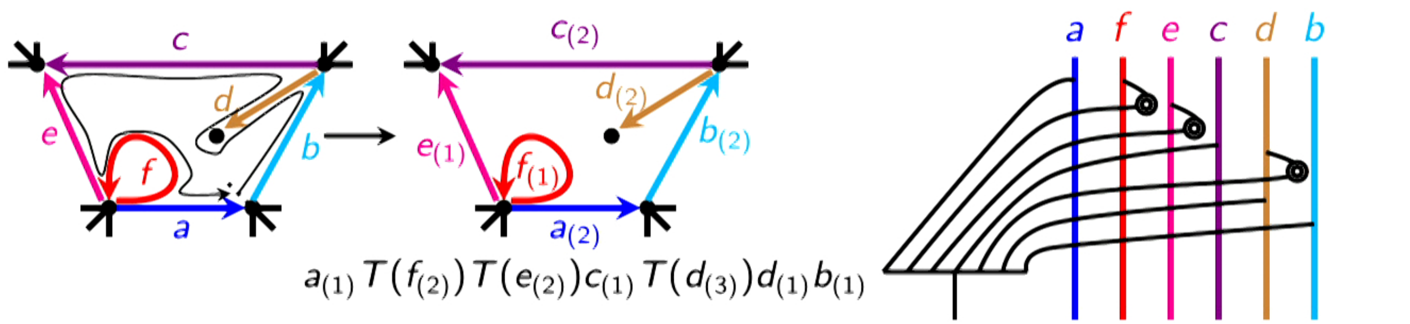


Face and vertex operators

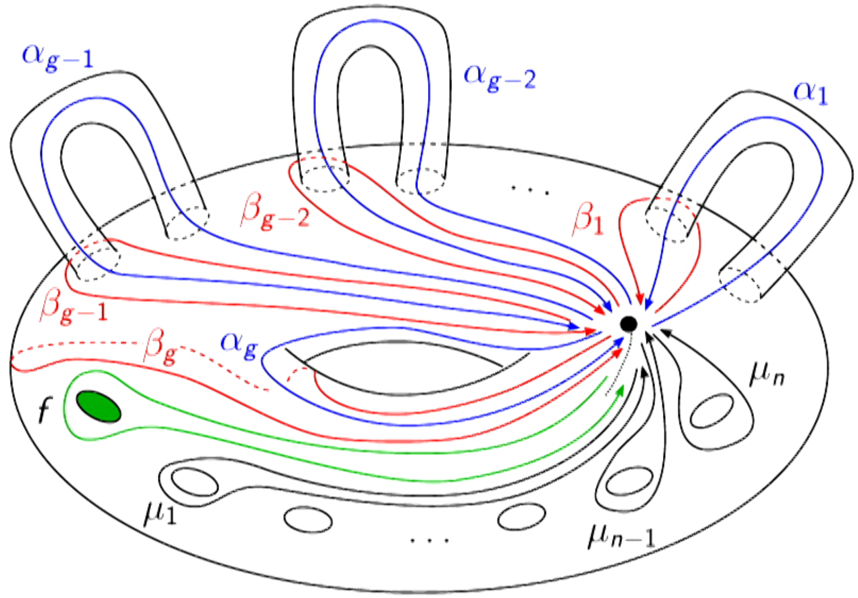
- vertex operators:

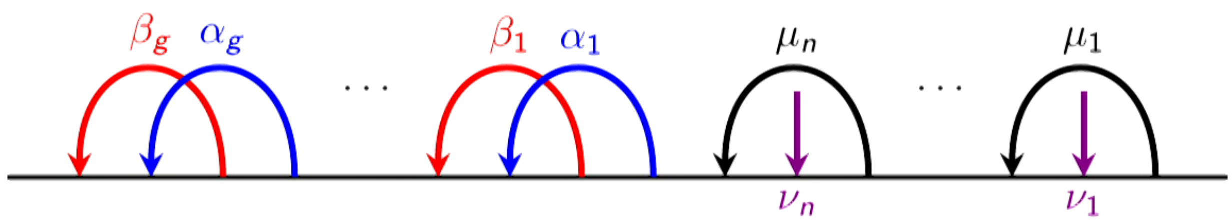


- face operators:



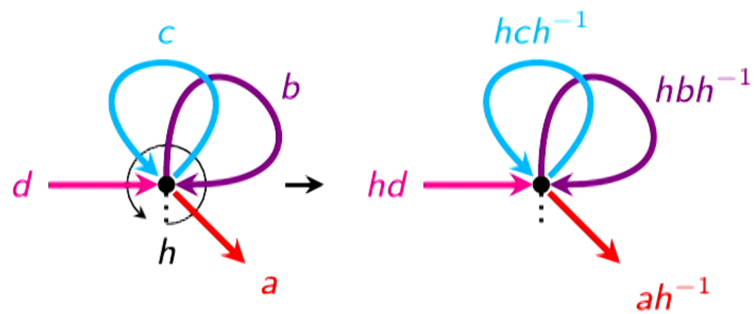
Reminder: Ribbon graph for $\Sigma_{g,n+1}$

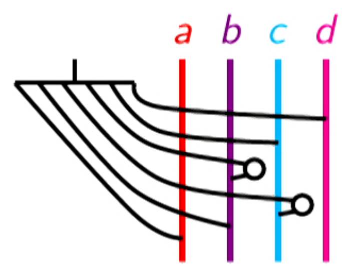




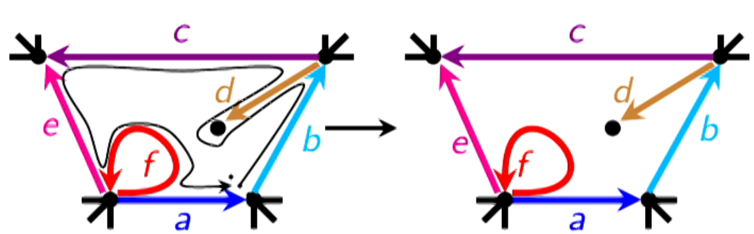
Face and vertex operators (group case)

- vertex operators:

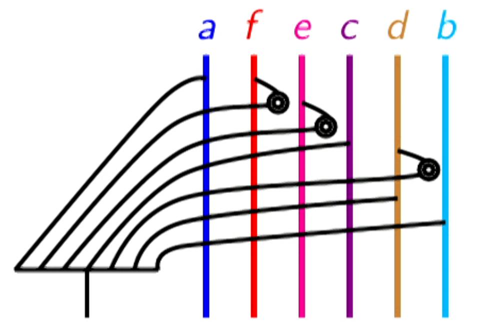




- face operators:

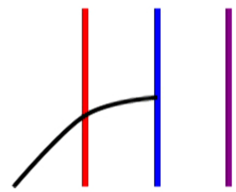


$p^3 a f^{-1} e^{-1} c d^{-1} d b$



Triangle operators

edge $\alpha \to 2$ H-left module & H-left comodule structures



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$\longrightarrow$

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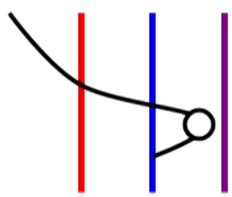
$\longleftarrow$



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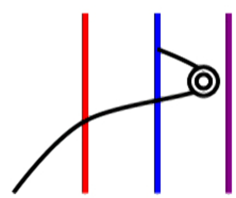

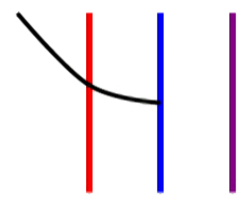

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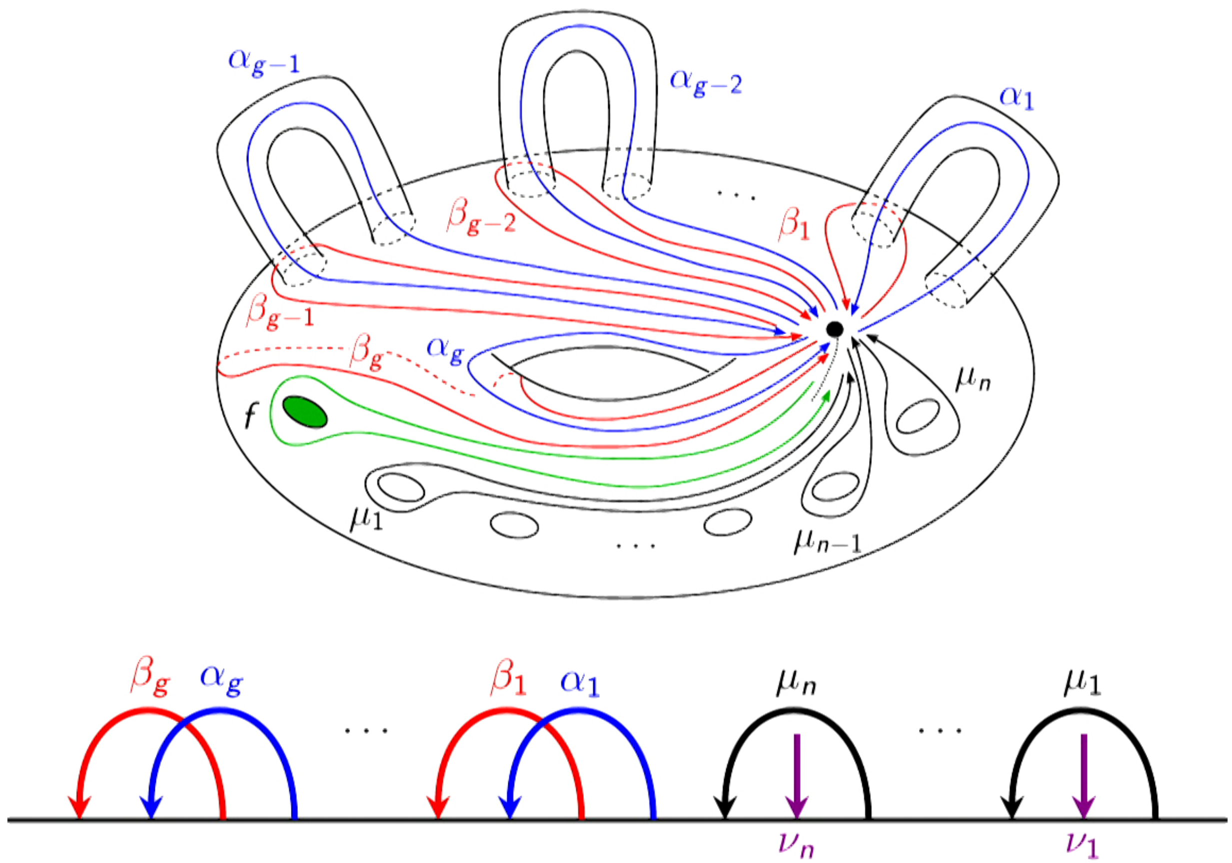

Main result

Theorem (C.M, T.V)

For (H, p) a pivotal Hopf algebra in a symmetric monoidal category, the Dehn twist along the paths $\alpha_k, \delta_i, \gamma_{i,j}$ for $k = 1, \ldots, g$, $i, j = 1, \ldots, n + 2g - 2$ on the graph for $\Sigma_{g,n+1}$

- *define $Map(\Sigma) \circlearrowleft H^{\otimes E}$.*
- *commute with face and vertex operators.*

- can be extended to any graph with pairing $\{\text{vertices}\} \to \{\text{faces}\}$.
- explains how Kitaev models with different embeddings of the same graph are related.
- greater pool of examples.
- $H = G$, $p = 1 \to$ action on moduli space of flat connections (up to vertex operators).
- explicit description in terms of simple graph transformations.

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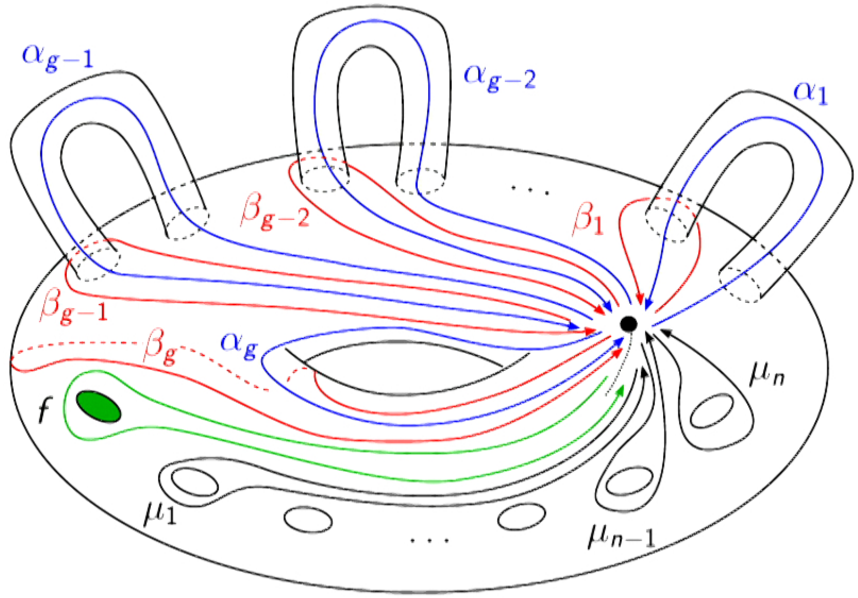

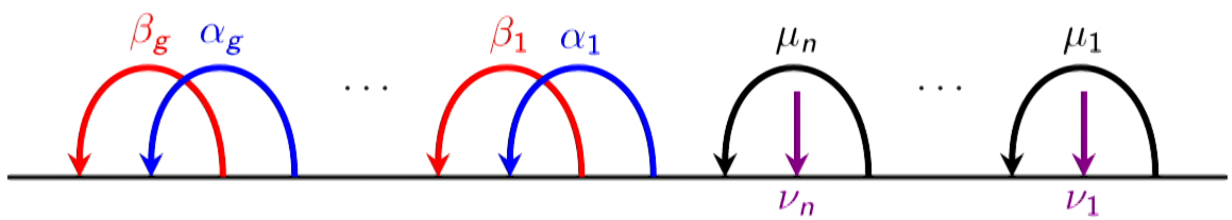

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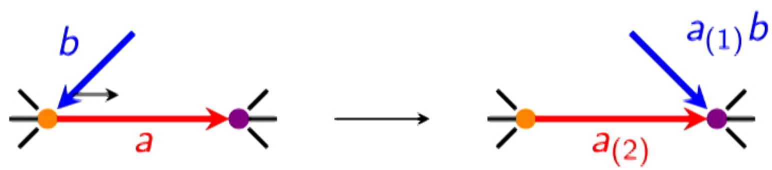
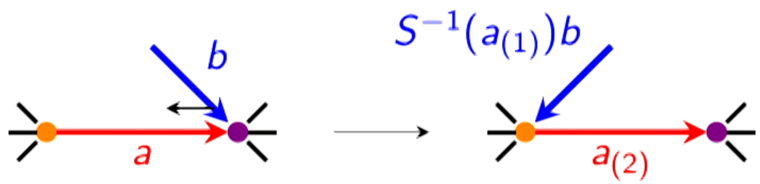
Sliding move

- slide:



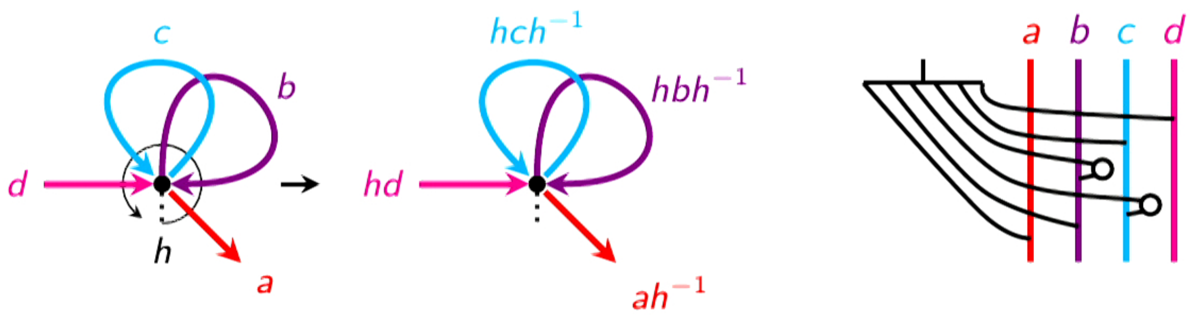
- inverse slide:



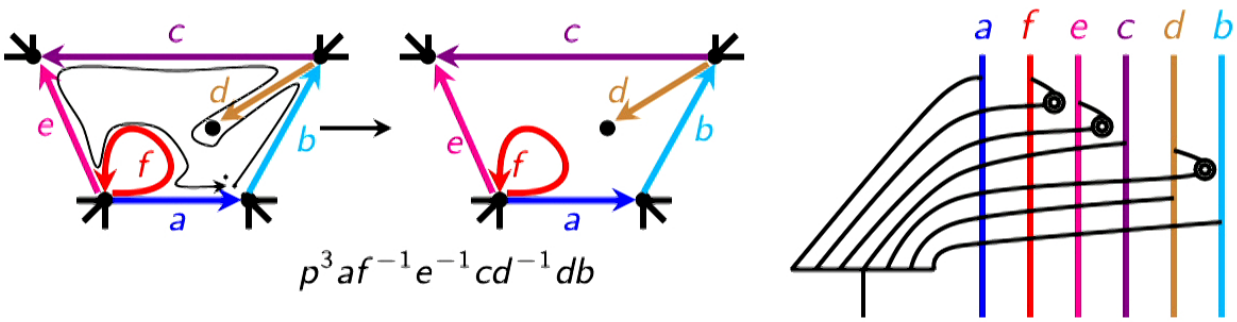
- different orientations via T.
- commutes with face, vertex operators (condition: not sliding over cilium)

Face and vertex operators (group case)

- vertex operators:

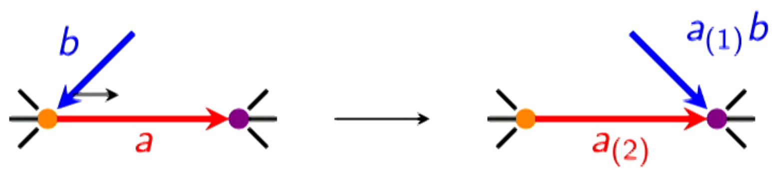


- face operators:

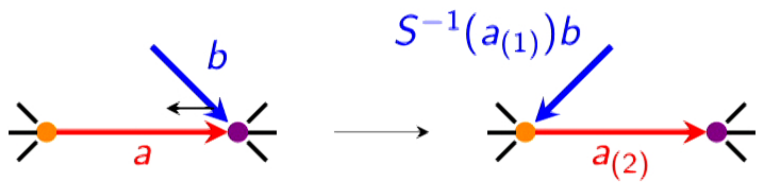


Sliding move

- slide:



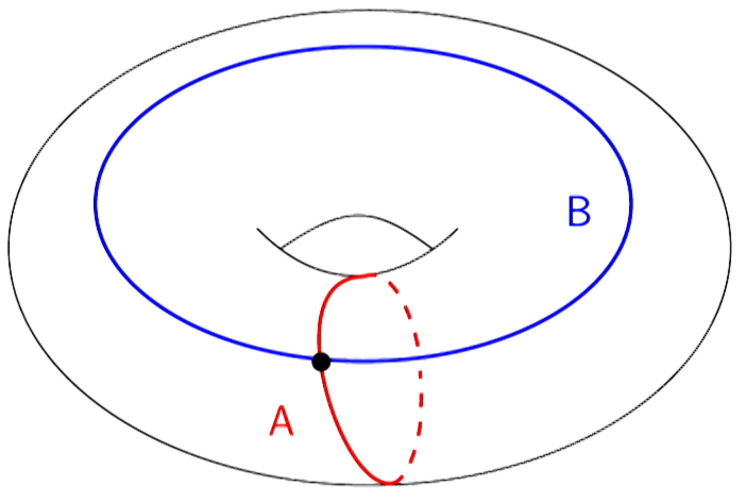
- inverse slide:



- different orientations via T.
- commutes with face, vertex operators (condition: not sliding over cilium)

Mapping class group of the torus

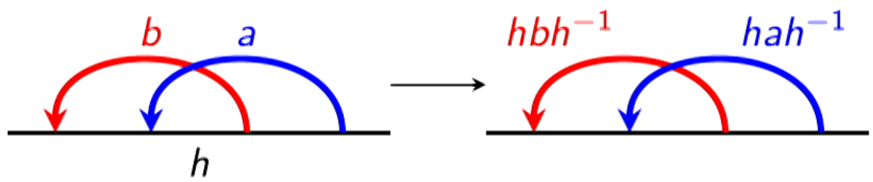
- $\mathrm{Map}(\Sigma_{1,1}) \cong B_3 = \langle A, B \mid ABA = BAB \rangle$
- $\mathrm{Map}(\Sigma_{1,0}) \cong \mathrm{SL}_2(\mathbb{Z}) \cong B_3/Z(B_3) = \langle A, B \mid ABA = BAB\,,\, (ABA)^4 = \mathrm{id} \rangle$



Vertex and face operators(group case)

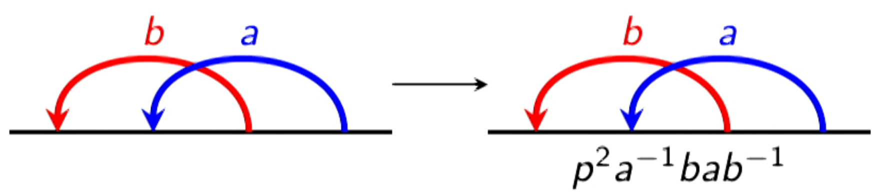
- vertex operator:

$\triangleright$:



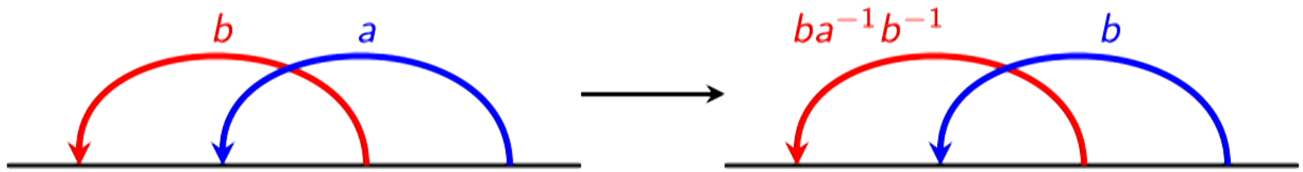
- face operator:

δ :

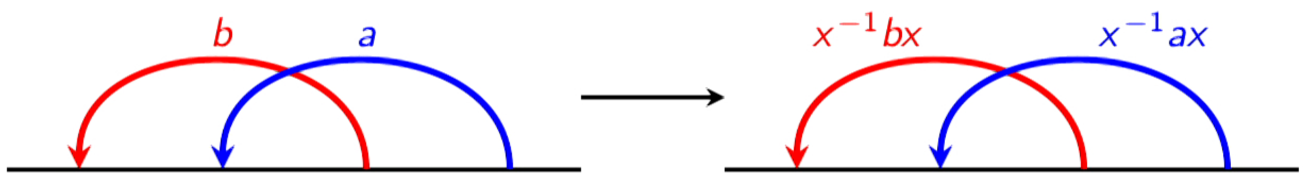


Relations of the Dehn twists (group case)

$B \circ A \circ B = A \circ B \circ A$:



$(B \circ A \circ B)^4 = \triangleright_v \circ (T \otimes 1^{\otimes 2}_{\mathbb{F}[G]}) \circ \delta_f$:

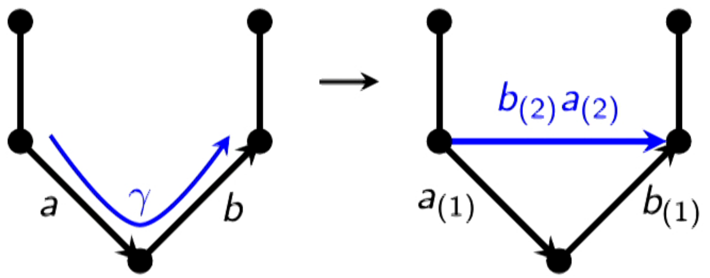


$x = a^{-1}bab^{-1}p^{-1}$

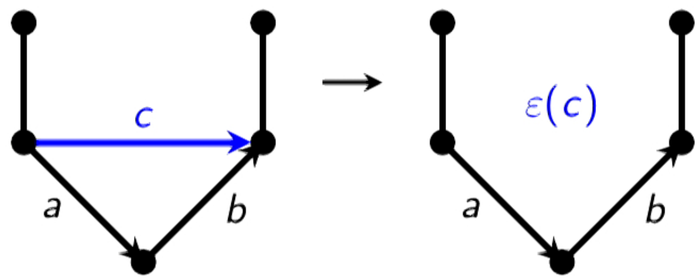
- $\mathrm{Map}(\Sigma_{1,1})$-action always.
- $\mathrm{Map}(\Sigma_{1,0})$-action e.g. on coinvariants.

Adding edges and removing edges

- adding edges

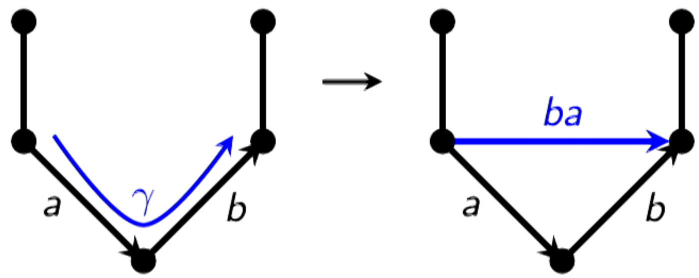


- commutes with face, vertex operators (condition: γ does not traverse cilium)
- left inverse (removing edges):

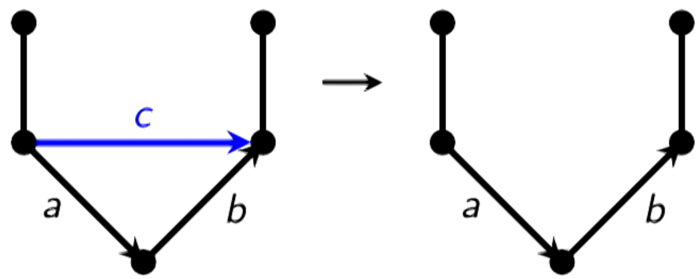


Adding edges and removing edges (group case)

- adding edges

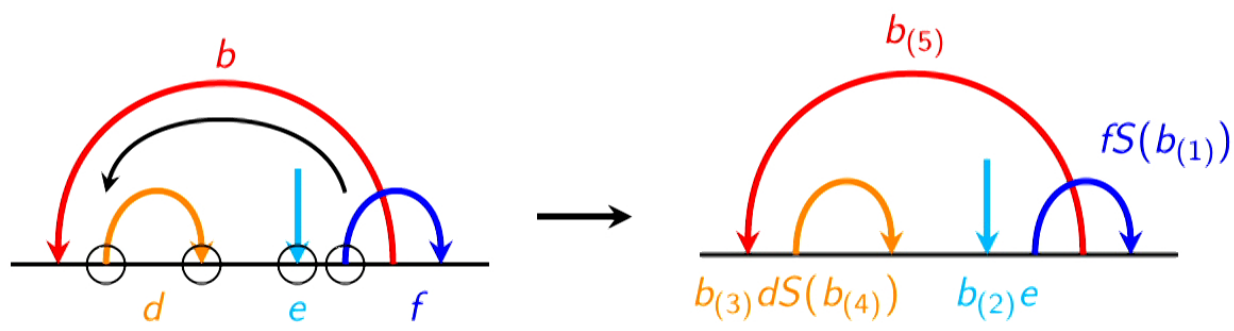


- commutes with face, vertex operators (condition: γ does not traverse cilium)
- left inverse (removing edges):

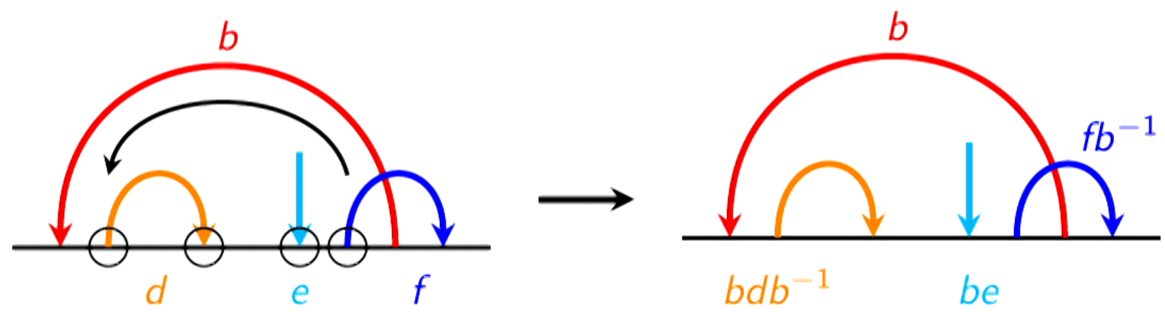


Action of Dehn twists

- Dehn twist along edge:

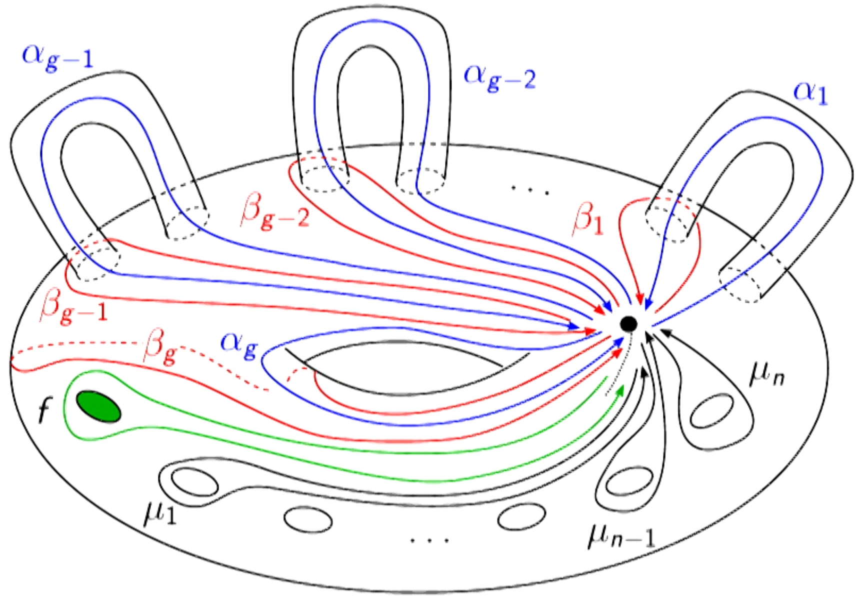


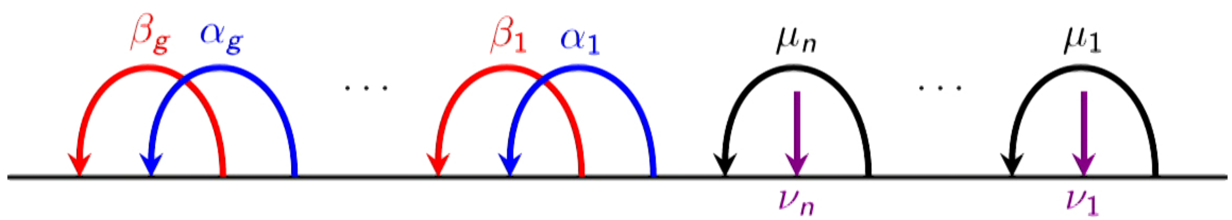
- group case



- general paths $\rightarrow$ replace with loop

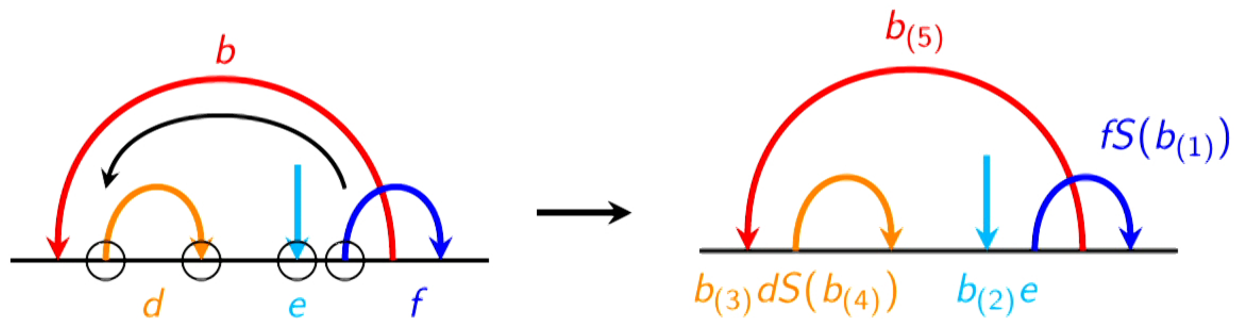
Reminder: Ribbon graph for $\Sigma_{g,n+1}$



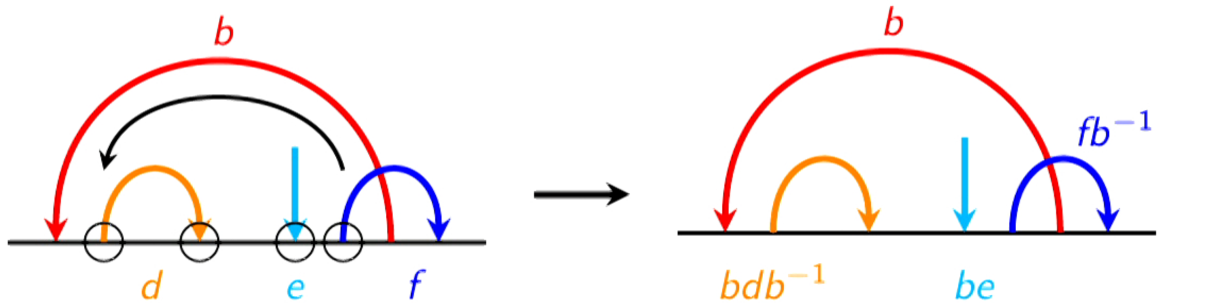


Action of Dehn twists

- Dehn twist along edge:



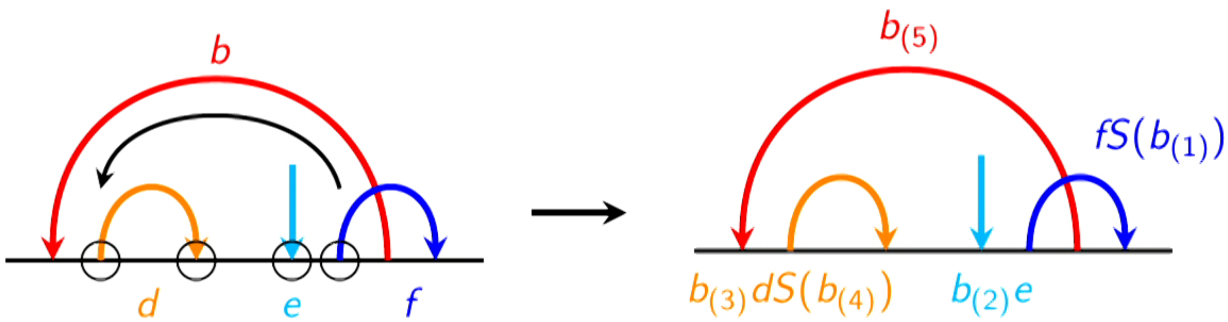
- group case



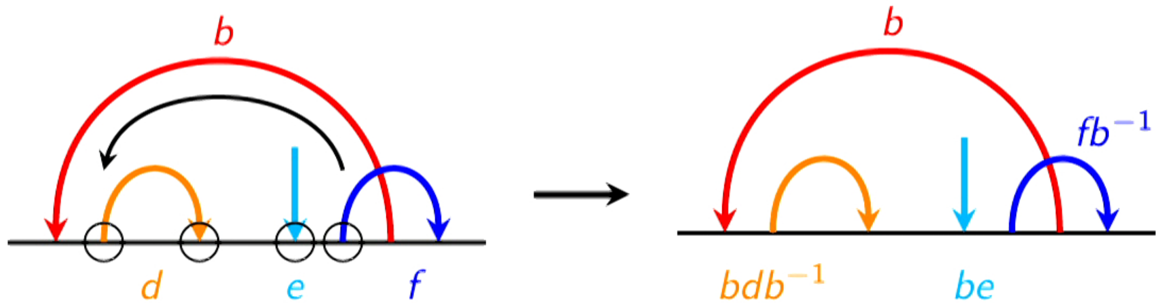
- general paths $\rightarrow$ replace with loop

Action of Dehn twists

- Dehn twist along edge:



- group case



- general paths $\rightarrow$ replace with loop