Title: The representation theory of the Clifford group, with applications to resource theories

Speakers: David Gross

Collection: Symmetry, Phases of Matter, and Resources in Quantum Computing

Date: November 28, 2019 - 4:45 PM

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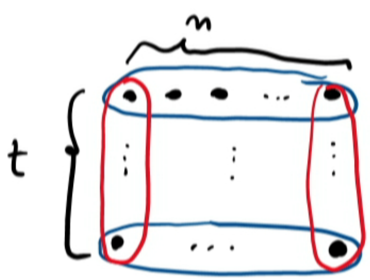
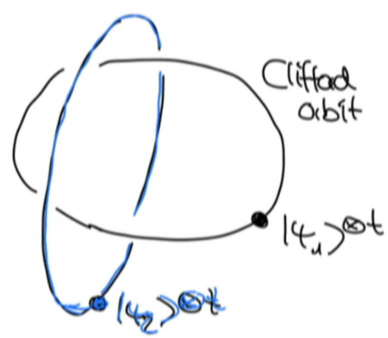
Abstract: I will report on an ongoing project to work out and exploit an analogue of Schur-Weyl duality for the Clifford group. Schur-Weyl establishes a one-one correspondence between irreps of the unitary group and those of the symmetric group. A similar program can be carried out for Cliffords.
The permutations are then replaced by certain discrete orthogonal maps.
As is the case for Schur-Weyl, this duality has many applications for problems in quantum information. It can be used, e.g., to derive quantum property tests for stabilizerness and Cliffordness, a new direct interpretation of the sum-negativity of Wigner functions, bounds on stabilizer rank, the construction of designs using few non-Clifford resources, etc.

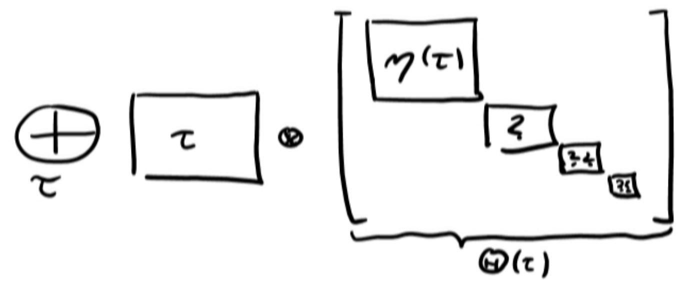
[arXiv:1609.08172, arXiv:1712.08628, arXiv:1906.07230, arXiv:out.soon].

Schur-Weyl duality for the Clifford group

with (many) applications.

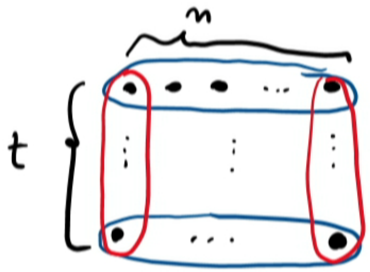


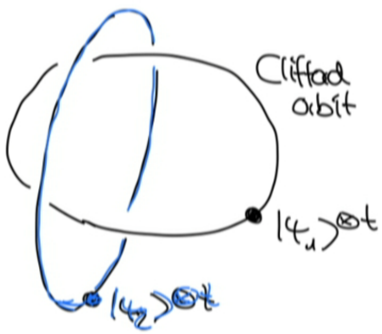


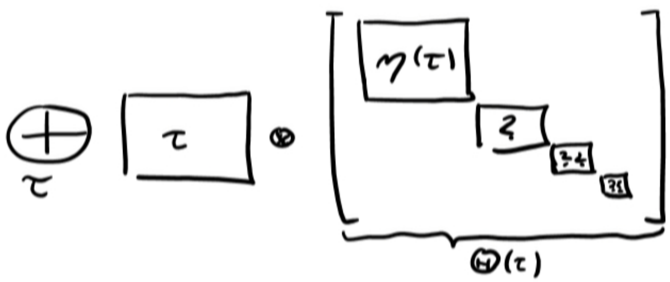


David Gross, University of Cologne

With: Sepehr Nezami, Michael Walter, Felipe Montealegre, Huangjun Zhu







Outline

- Schur-Weyl and the Clifford group
- **Many** applications
- Connections to Howe duality

Look, this is a technical topic.

Comments: 2 figures, 30 pages per figure

Subjects: **Quantum Physics (quant-ph)**; Mathematical Physics

Cit

Iterating Corollary 4 thus gives an isomorphism

$$i_1 : L^2(\mathrm{Hom}(X \to U)) \to \bigotimes_{i=1}^{t} L^2(\mathrm{Hom}(X -$$

defined by

$$\delta_F \mapsto \delta_{f_1 f_1^T F} \otimes \cdots \otimes \delta_{f_{t-1} f_{t-1}^T F} \otimes \delta_{d(U)^{-1}}$$

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Schur-Weyl Duality 1

- On t-th tensor power $\mathcal{H}^{\otimes t}$ of a Hilbert space $\mathcal{H}$, commuting actions:

$$U(\mathcal{H}) \ni U \mapsto U \otimes \cdots \otimes U,$$

$$S_t \ni \pi\colon\ |\psi_1\rangle \otimes \cdots \otimes |\psi_t\rangle \mapsto |\psi_{\pi_1}\rangle \otimes \cdots \otimes |\psi_{\pi_t}\rangle.$$

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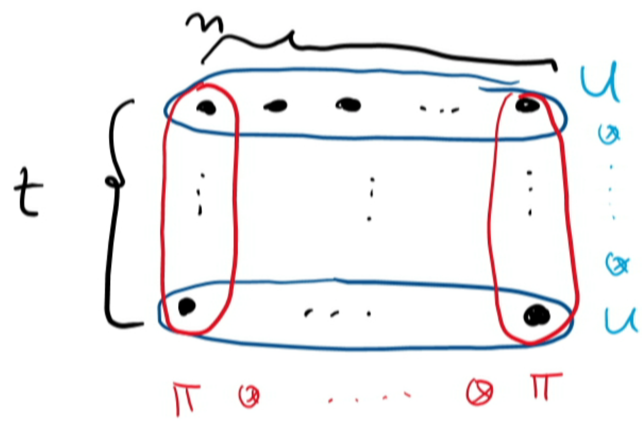
- Operator A commutes with $U^{\otimes t}$ iff

$$A = \sum_{\pi \in S_t} c_\pi\, \pi$$

and vice versa.

Schur-Weyl is transversal!

$\bullet = \mathbb{C}^d$



Both algebras:

- Generated by *groups*

and

- *by tensor powers!*

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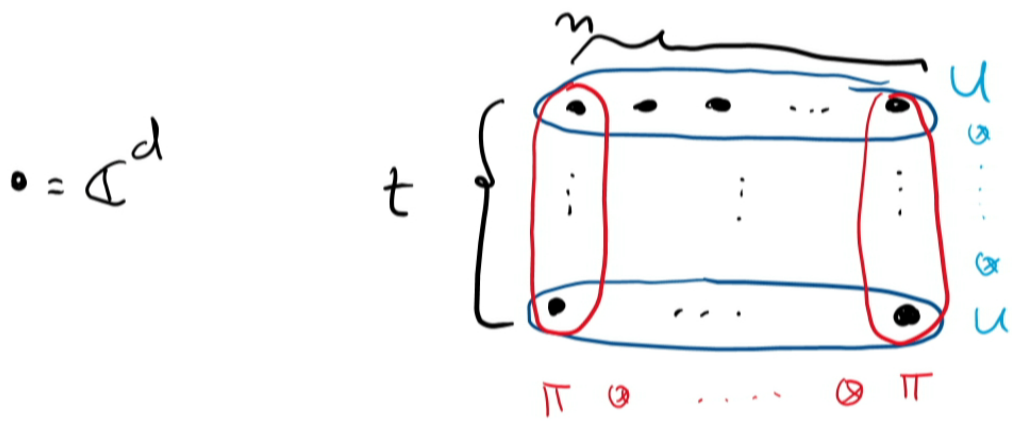

Both algebras:

- Generated by *groups*

and

- *by tensor powers!*

⇒ Often, results are *independent of n!*

Clifford group, prior results

Q: What happens if one restricts to Cliffords $\subset U(\mathcal{H})$?

Clifford group, prior results

as asked in

THE CITY OF Waterloo

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Commutant remains S_t for

- t=2
  [Dankert, Emerson 2005]

Clifford group, prior results

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Commutant remains S_t for

- t=2
  [Dankert, Emerson 2005]

- t=3
  [Zhu; Webb; Gross and Kueng 2015; implicit in Nebe, Rains, Sloane 2006]

Applications of prior results

Representation theory of *t*-th tensor powers used in, e.g.:

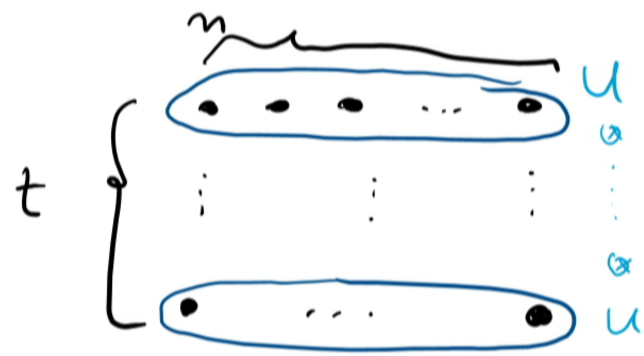
- Randomized benchmarking
- Decoupling technique
- Non-malleable quantum one-time pads

$t = 2$

- Variance bounds for randomized benchmarking
- Stabilizer POVM optimal state-independent measurement for pure states

$t = 4$

Now let's solve the problem for all t.



- Operator A commutes with $U^{\otimes t}$ iff

$$A = \sum_{\pi \in S_t} c_\pi \, \pi$$

and vice versa.

[Nezami, Walter, DG 18]

- Under action of $S_t \times U(\mathcal{H})$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\lambda} S_\lambda \otimes U_\lambda$$

- S_λ irrep of S_t, U_λ irrep of $U(\mathcal{H})$.

[Montealegre-Mora, DG 19]

The commutant

Theorem [Nezami, Walter, DG 18]

The commutant algebra of t-th tensor powers of the Clifford group over d^n is generated by t-th tensor powers of:

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The commutant algebra of t-th tensor powers of the Clifford group over d^n is generated by t-th tensor powers of:

- Stochastic orthogonal transformations
- Self-orthogonal CSS code projections

Stochastic orthogonal transformations

A $t \times t$ matrix O, entries in $\mathbb{Z}_d$ is

- *orthogonal* if

$$O^T O = \mathrm{Id} \quad \mathrm{mod\ d}$$

- *stochastic* if

$$O\underline{1} = \underline{1}, \qquad \underline{1} = (1, \dots .1) \in \mathbb{Z}_d^t.$$

Stochastic orthogonal transformations

Example: **Anti-permutations**

- Binary complement of permutation matrices

$$\overline{\mathbb{1}}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

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$$|x_1 \ldots x_6\rangle \mapsto |\bar{x} - x_1 \ldots \bar{x} - x_6\rangle \hat{=}$$ flip bits if parity is odd

Calderbank-Shor-Steane codes

Let $N \subset \mathbb{Z}_d^t$ be *self-orthogonal*:

$$\sum_i u_i v_i = 0 \mod d, \qquad u, v \in N.$$

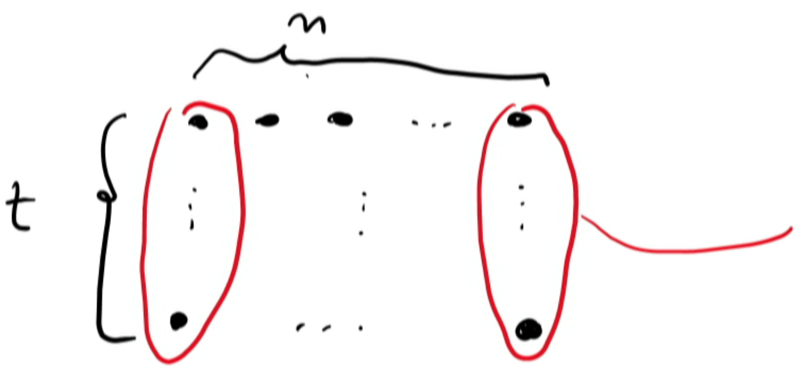
The commutant

Theorem [Nezami, Walter, DG 18]

Commutant generated by tensor powers of:

- Stochastic orthogonal transformations
- Self-orthogonal CSS code projections

- Transversal! ☺
- Not *quite* a group.
  ⇒ (mild) failure of finite *Howe duality*

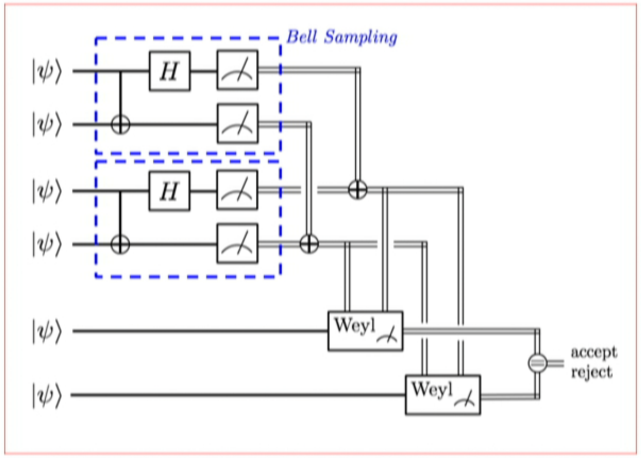


Stocastic orthogonal $O^{\otimes n}$

or

CSS code projector $P^{\otimes n}$.

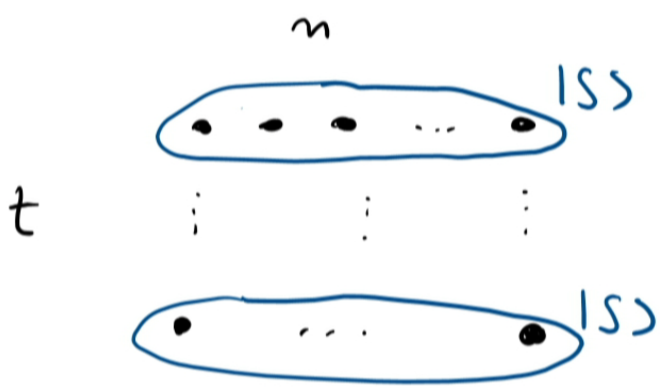
Applications



Stabilizer states have additional symmetry

Consider stabilizer state $|s\rangle$ on n qudits...

...and its t-th tensor power.



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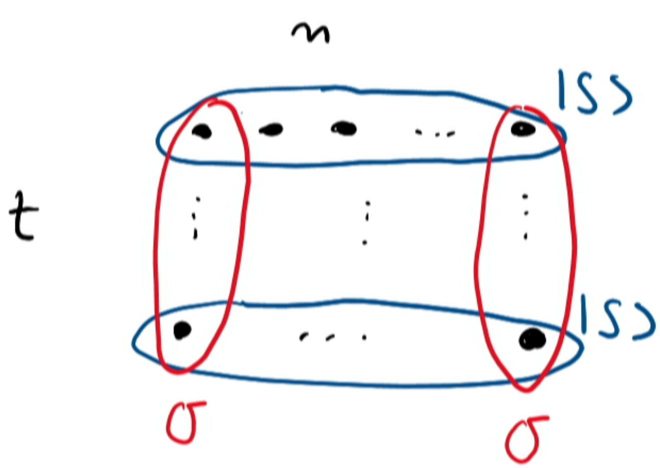


Tensor powers of stabilizer states are invariant under the stochastic orthogonal group.

Proof: True for $|S\rangle = |0, \ldots, 0\rangle$:

$$\left| \sigma \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \right\rangle = \left| \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \right\rangle$$

$$\Rightarrow \sigma^{\otimes n} |S\rangle^{\otimes t} = \sigma^{\otimes n} U^{\otimes t} |\underline{0}\rangle^{\otimes t}$$

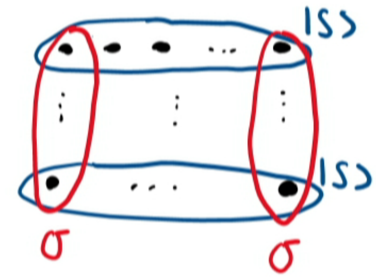
$$= U^{\otimes t} \sigma^{\otimes n} |\underline{0}\rangle^{\otimes t} = |S\rangle^{\otimes t}$$

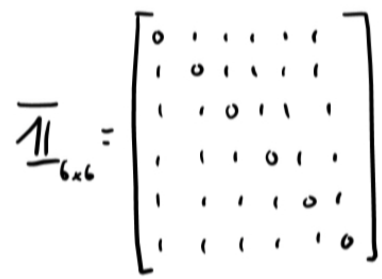
$\square$

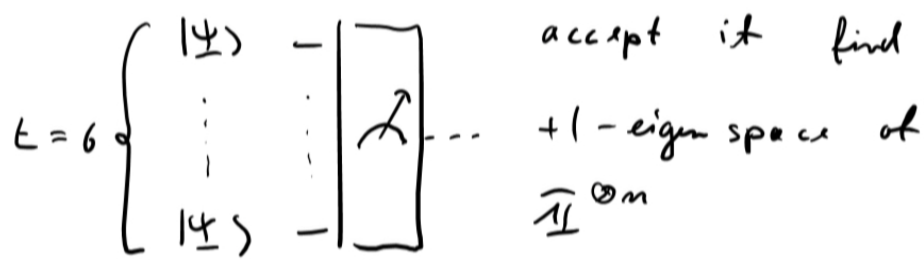
Application 1: Stabilizer testing

Thm. [Nezami, Walter, DG 18]

Let ψ be state on n qubits.







(Solves open pro. in q. property testing).

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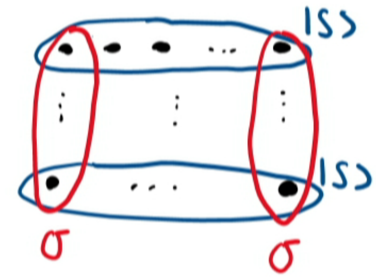
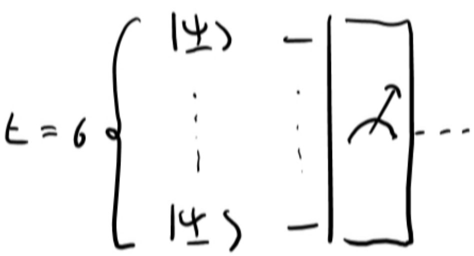


$$\bar{\mathbb{1}}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



accept if find $+1$-eigenspace of $\bar{\mathbb{1}}^{\otimes n}$

Application 2: Robust Hudson

Thm. [Nezami, Walter, DG 18]
Pure ψ on n qudits, d odd.

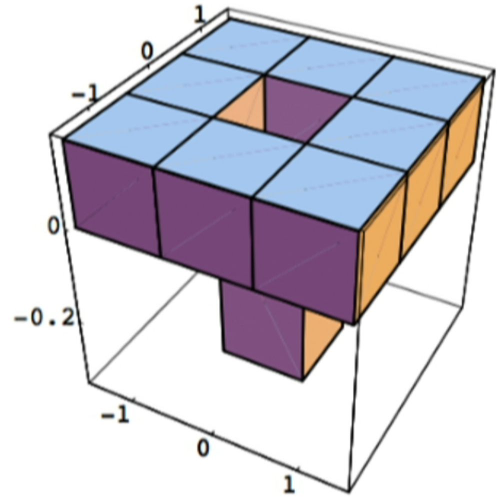
Wigner sum negativity for pure state:

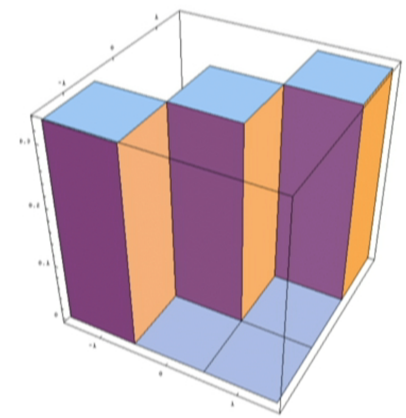
$$\mathrm{sn}(\psi) = \sum_{v, W_\psi(v) \leq 0} |W_\psi(v)|.$$

Then

$$\max_S |\langle \psi | S \rangle|^2 \leq 1 - d^2 \, \mathrm{sn}(\psi),$$

independent of n.





Application 3: exponential de Finetti

Thm.

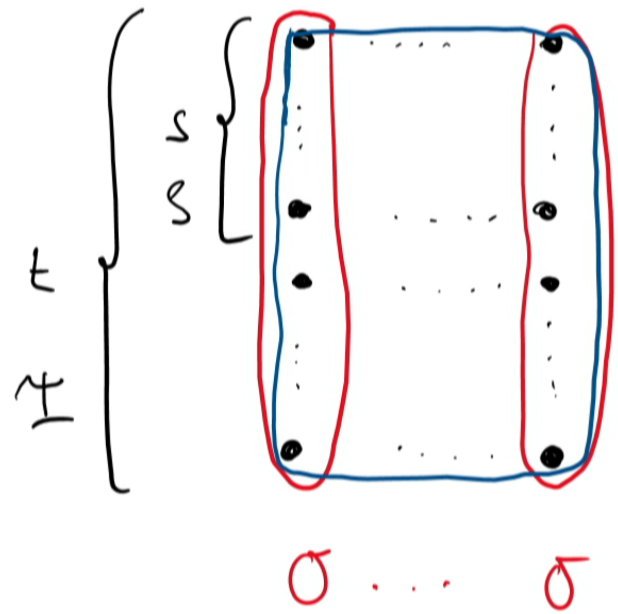
Let $\psi \in \left(\mathbb{C}^{2^n}\right)^{\otimes t}$ be invariant under stochastic orthogonal group.

Let ρ be the reduction to the first s copies.

There is a distribution over stabs s.t.:

$$\left\| \rho - \sum_S |S\rangle\langle S|^{\otimes s} p(S) \right\|_{tr}$$

$$\leq \exp\left(n^2 - (t-s) \right)$$



Finite analogue of [Leverrier 2017]

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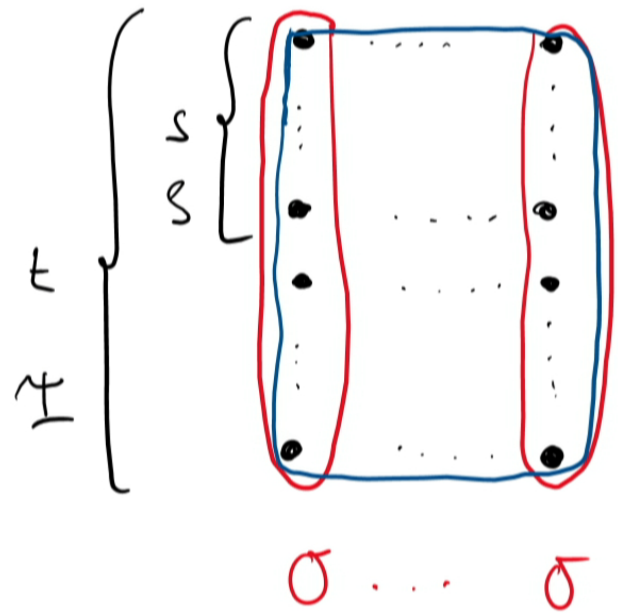

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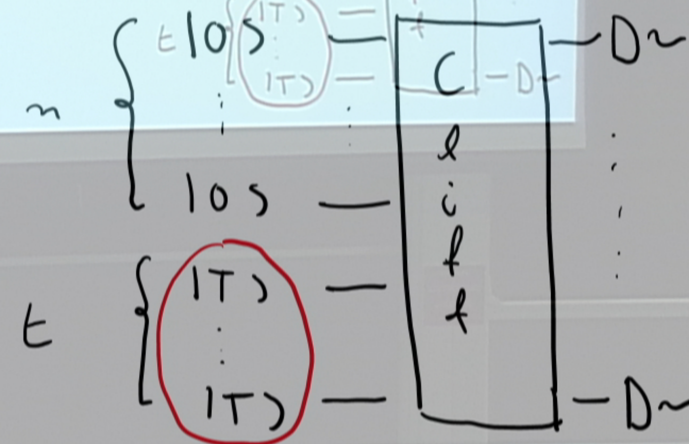
Application 4: Stabilizer rank

Theorem [Nezami, Walter, DG 18; Zhu, Grassl, Kueng, DG 16]

- For $t \leq 5$, the powers $|S\rangle^{\otimes t}$ of stabilizer states span symmetric space $\mathrm{Sym}^t(\mathbb{C}^{2^n})$.
- This fails for $t \geq 6$.



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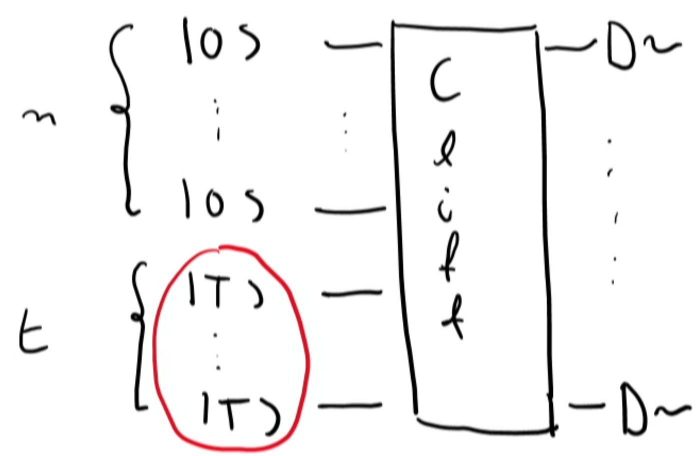
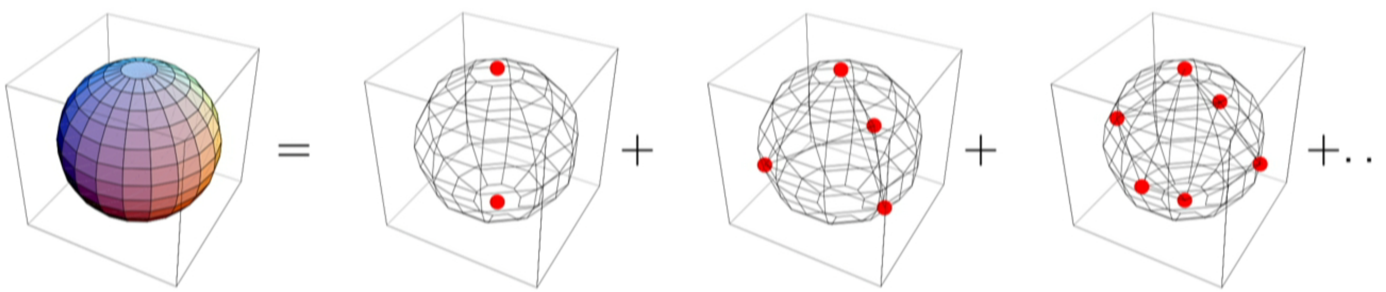
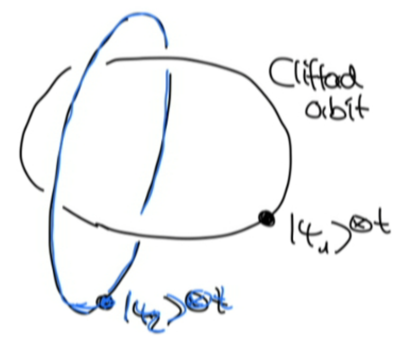


Application 5: Designs

Def.: *t*-designs

finite set of points on sphere / unitaries that reproduce *t*-th moments.





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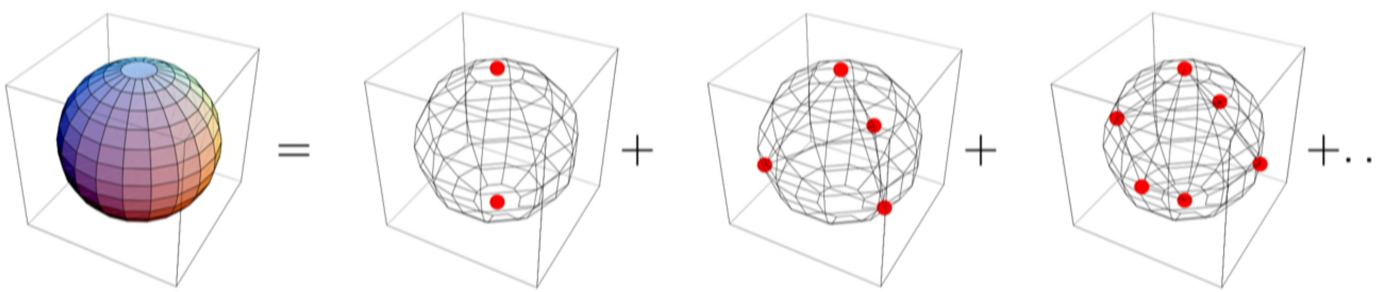
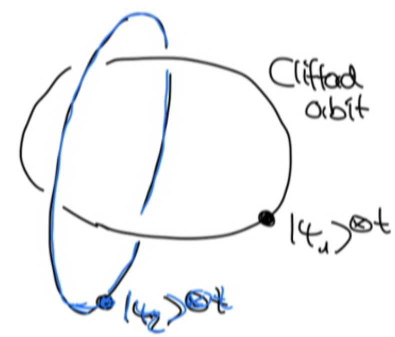
Theorem [Nezami, Walter, DG 18]

- Can construct exact *t*-designs from ***n*-independent** number of Clifford orbits





Summary of Quantum Information results

We have found the commutant of tensor powers of the Clifford group.

- Generated by stochastic orthogonal transformations and CSS code projections
- Many applications: Stabilizer rank, stabilizer testing, exponential de Finetti, …

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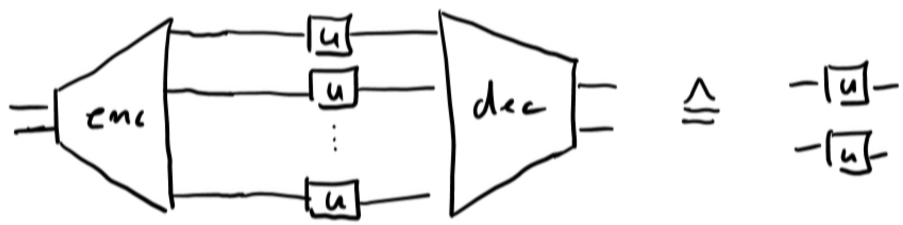
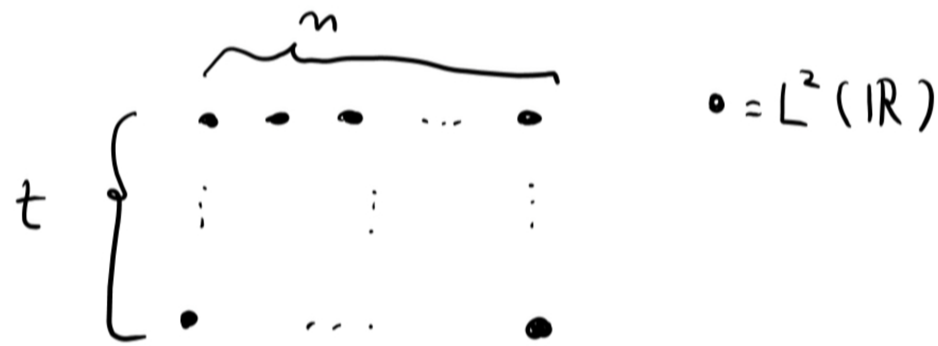
Howe duality



Howe Duality – Continuous Variables

- Consider *metaplectic* representation

$$\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n), \quad \mu: \mathrm{Sp}(\mathbb{R}^{2n}) \to U(\mathcal{H})$$



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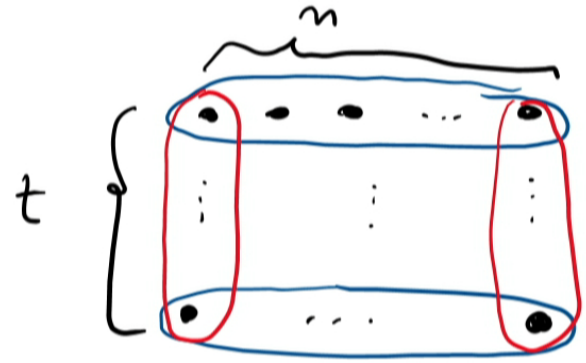
- ...commutes with O(t) $\supset S_t$.



$\bullet = L^2(\mathbb{R})$

Ψ

$|F\rangle \quad F \in \mathbb{R}^{t\times n}$

$O: |F\rangle \mapsto |OF\rangle$

Howe Duality – Continuous Variables

- Consider *metaplectic* representation

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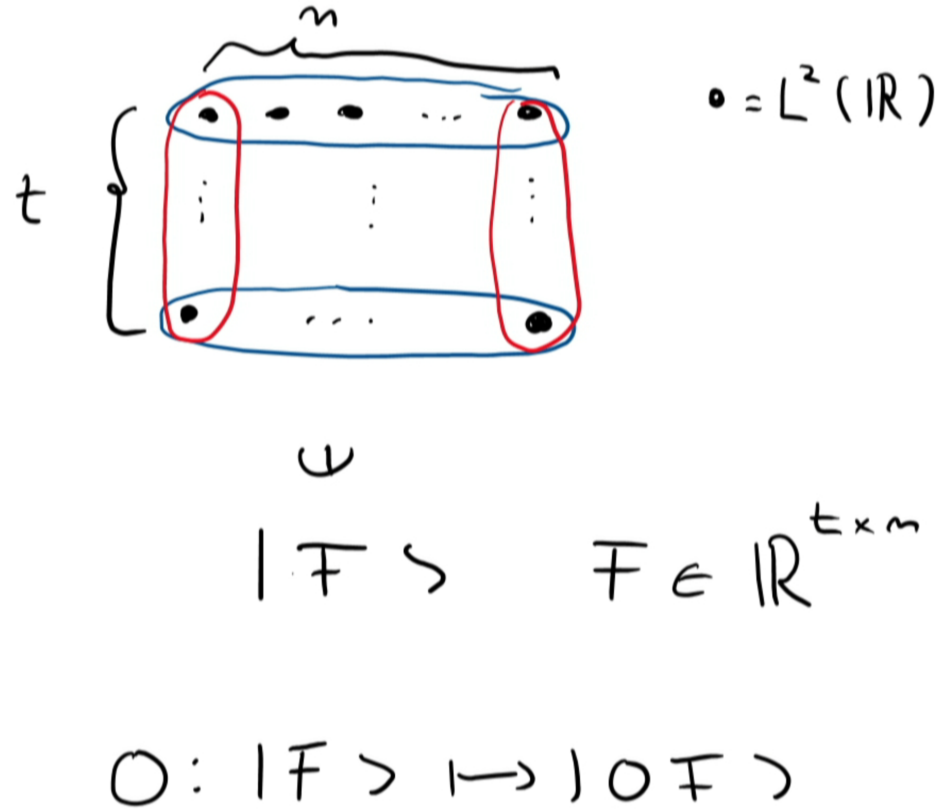
- Tensor power $\mu^{\otimes t}$…
- …commutes with O(t) $\supset S_t$.

- Under $O(t) \times \mathrm{Sp}(\mathbb{R}^{2n})$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ irrep of $\mathrm{Sp}(\mathbb{R}^{2n})$.



Howe Duality – finite (and odd) dimensions

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ reducible.
- Failure of Howe duality over finite fields known since 70s…
- …building on Nezami-Walter-DG, Gurevich-Howe 2016…

Rank of Sp(V)-representations

- Sp(V) contains a large Abelian subgroup

$$\begin{bmatrix} \mathbb{1} & A_1 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & A_2 \\ 0 & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & A_1 + A_2 \\ 0 & \mathbb{1} \end{bmatrix}$$

[Gurevich-Howe 2017]

The rank of Sp(V)-representations

Def.: $\operatorname{rank}\pi = \max_B \operatorname{rank} B$

Fact.: The rank of $\mu^{\otimes t}$ is t.

$\underline{t=1:}$

$$\mu\begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix} |\underset{\in \mathbb{Z}_d^n}{\underline{x}}\rangle = \omega^{(\underline{x}, A\underline{x})} |\underline{x}\rangle = \omega^{\operatorname{tr} A \underbrace{\underline{x}\,\underline{x}^T}_{B}} |\underline{x}\rangle.$$

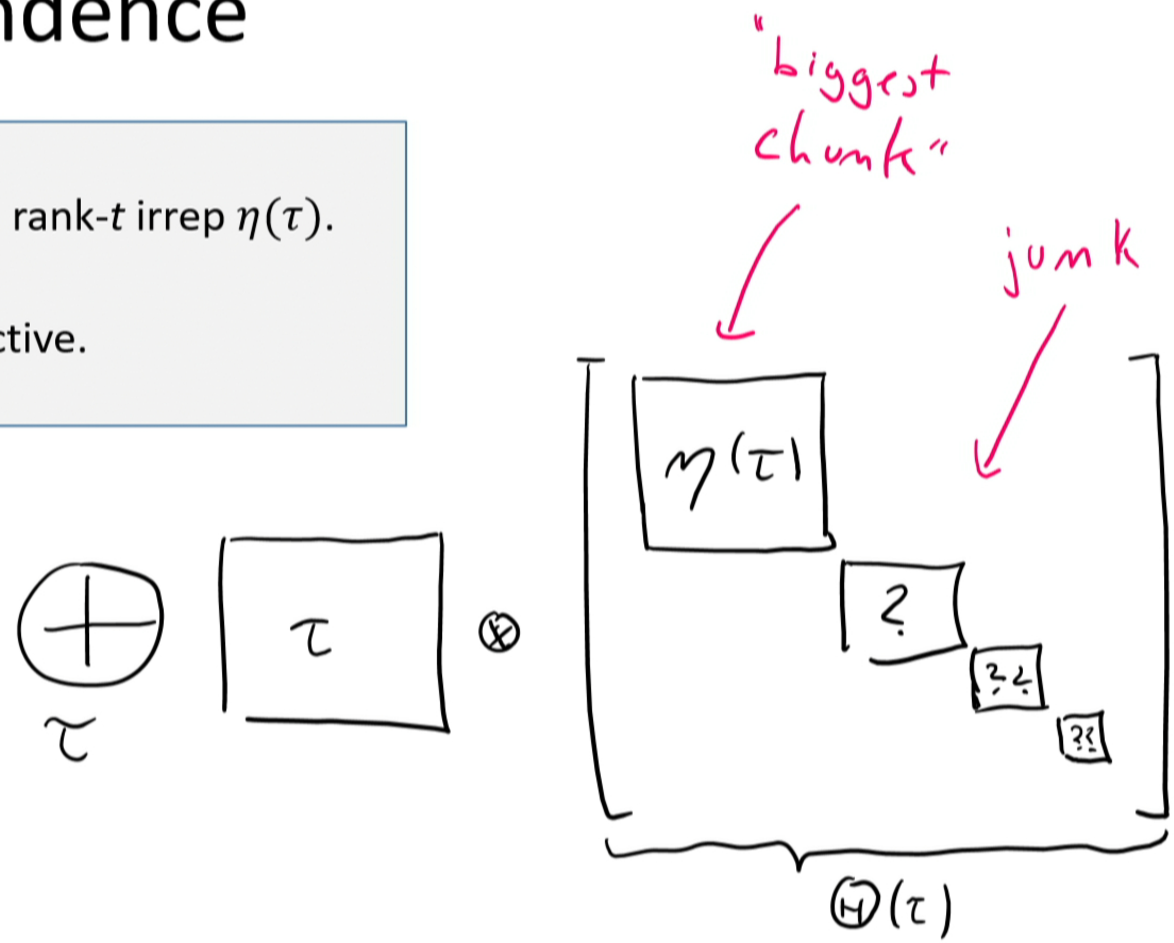
[Gurevich-Howe 2017]

The η-correspondence

Thm [Gurevich-Howe `17]

- $\Theta(\tau)$ contains exactly one rank-t irrep $\eta(\tau)$.
- The map $\tau \mapsto \eta(\tau)$ is injective.

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$



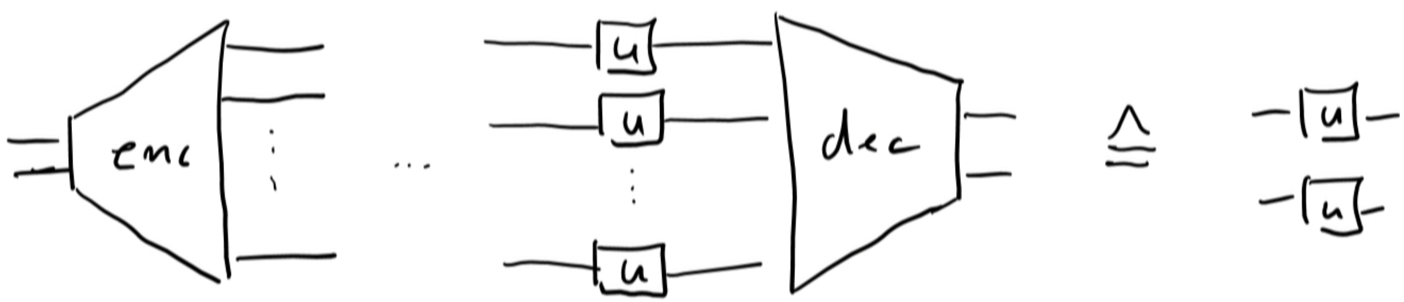
Where do the rank-deficient reps come from?

Idea: Can one "imbed lower tensor powers into t-th tensor power"?

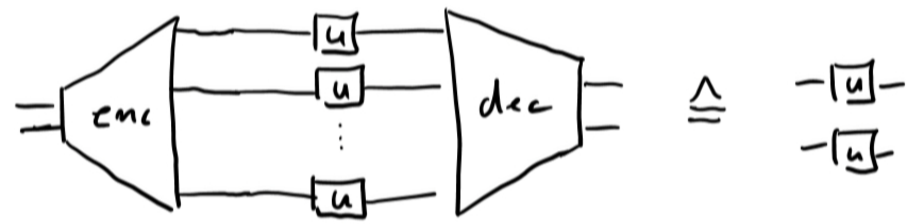
Where do the rank-deficient reps come from?

Idea: Can one “imbed lower tensor powers into *t*-th tensor power”?

…that’s what transversal gates on quantum codes do!



...from CSS codes!

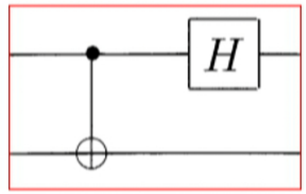


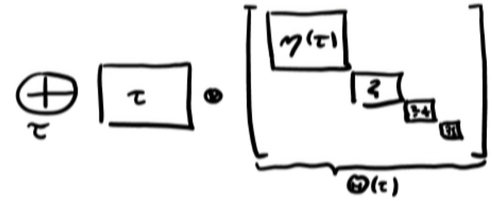
Thm [Montealegre-Mora, DG]

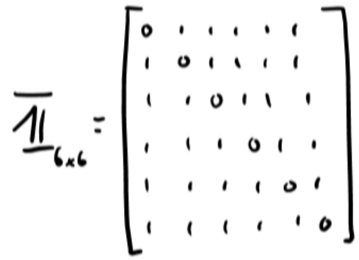
Let $N \subset \mathbb{Z}_d^t$ be isotropic, let C_N be the associated CSS code.

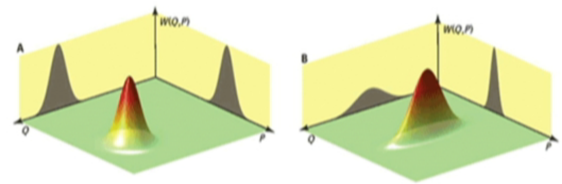
- Then $C_N^{\otimes t}$ is isomorphic to $\mu^{\otimes s}$, $s = t - 2\dim N$.

Thank you!









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