Title: Two-dimensional AKLT states as ground states of gapped Hamiltonians and resource for universal quantum computation

Speakers: Tzu-Chieh Wei

Collection: Symmetry, Phases of Matter, and Resources in Quantum Computing

Date: November 28, 2019 - 3:30 PM

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Abstract: Affleck, Kennedy, Lieb, and Tasaki (AKLT) constructed one-dimensional and two-dimensional spin models invariant under spin rotation. These are recognized as paradigmatic examples of symmetry-protected topological phases, including the spin-1 AKLT chain with a provable nonzero spectral gap that strongly supports Haldaneâ€™s conjecture on the spectral gap of integer chains. These states were shown to provide universal resource for quantum computation, in the framework of the measurement-based approach, including the spin-3/2 AKLT state on the honeycomb lattice and the spin-2 one on the square lattice, both of which display exponential decay in the correlation functions. However, the nonzero spectral in these 2D models had not been proved analytically for over 30 years, until very recently. I will review briefly our understanding of the quantum computational universality in the AKLT family. Then I will focus on demonstrating the nonzero spectral gap for several 2D AKLT models, including decorated honeycomb and decorated square lattices, and the undecorated degree-3 Archimedean lattices. In brief, we now have universal resource states that are ground states of provable gapped local Hamiltonians. Such a feature may be useful in creating the resource states by cooling the system and might further help the exploration into the quantum computational phases in generalized AKLT-Haldane phases.

Two-dimensional AKLT states as (1) ground states of gapped Hamiltonians and (2) resource for universal quantum computation

Tzu-Chieh Wei (魏子傑)

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Stony Brook University

2019/11/28 @ Perimeter Institute: Workshop on
"Symmetry, phases of matter and resources in quantum computing"

Acknowledgment

Collaborators: Robert Raussendorf, Ian Affleck, Valentin Murg, Artur Garica-Saez, Ching-Yu Huang, Abhishodh Prakash, **Nikko Pomata**, Hendrik Poulsen Nautrup, David Stephen, Dong-Sheng Wang,…

Nikko Pomata

support:

Outline

I. Introduction

II. AKLT models and states for universal quantum computation (in MBQC framework)

Review: e.g. Adv. Phys. X 3, 1 (2018)

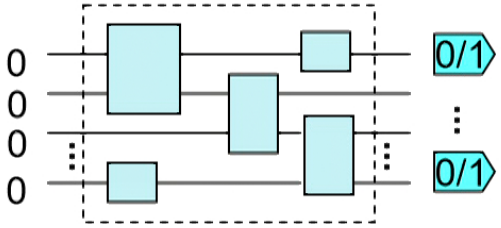
III. Nonzero gap for some 2D AKLT models

Ref: Phys. Rev. B 100, 094429 (2019)
& arXiv:1911.01410

IV. Summary

(Frameworks of) Quantum Computation

I. Circuit:



- ✓ Major scheme by most labs: IBM, Intel, Rigetti, IonQ, Alibaba, Google

(Frameworks of) Quantum Computation

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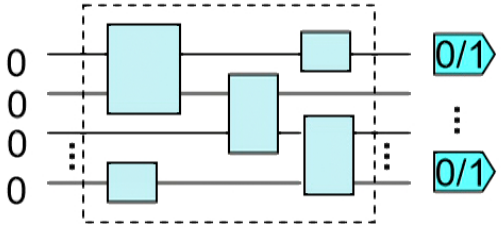

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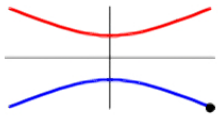
II. Adiabatic:



$$H(t) = \left(1 - \frac{t}{T}\right) H_{\text{initial}} + \frac{t}{T} H_{\text{final}}$$

- ✓ Approach by D-Wave

(Frameworks of) Quantum Computation

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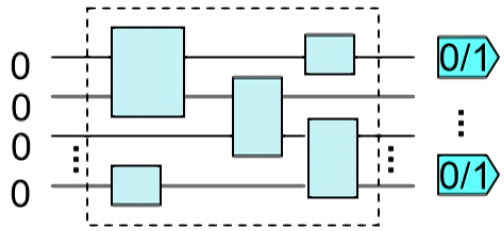

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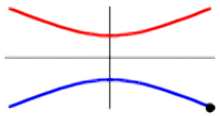

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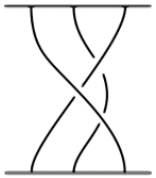
III. Topological:



quantum gates = braiding anyons

- ✓ Approach by Microsoft, Google plans to use a hybrid of III and I

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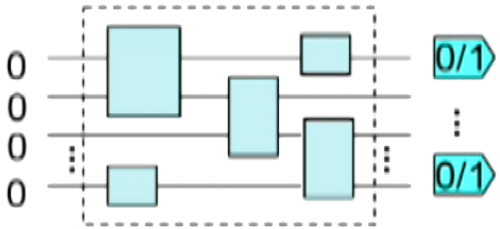

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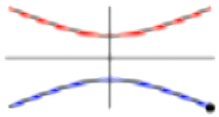

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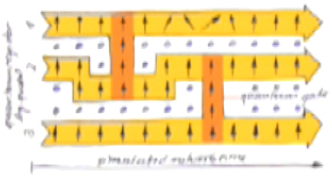
IV. Measurement -based:



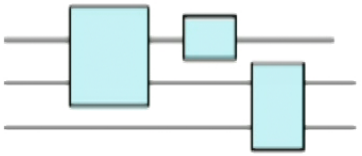
local measurement is the only operation needed

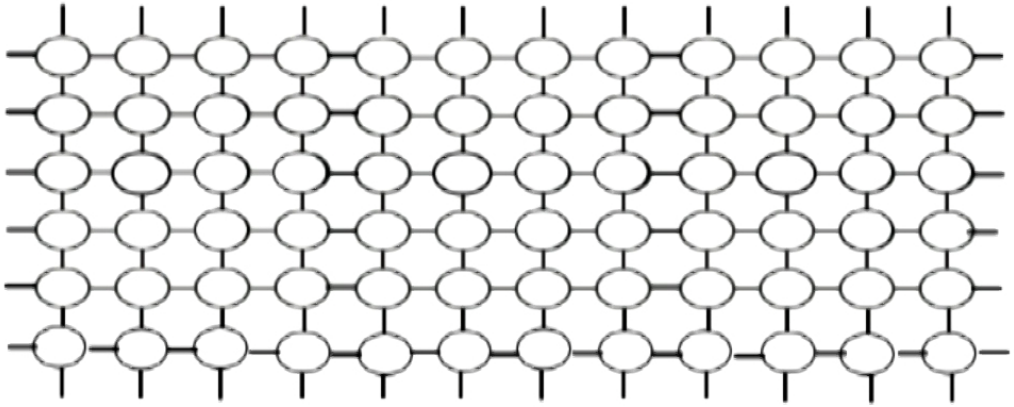
- Used in photonic systems, such as PsiQuantum

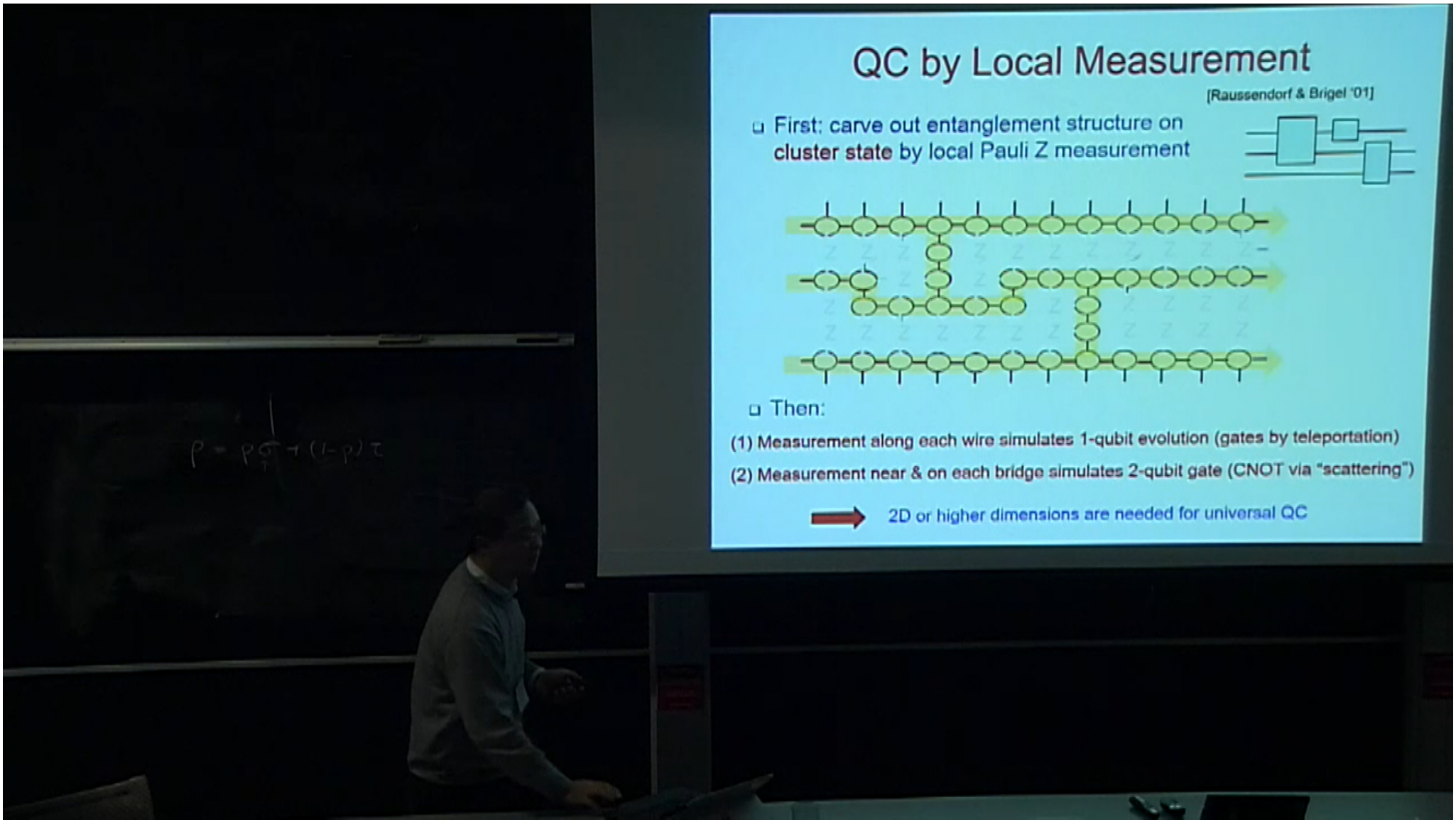
QC by Local Measurement

[Raussendorf & Brigel '01]

- First: carve out entanglement structure on cluster state by local Pauli Z measurement






QC by Local Measurement
[Raussendorf & Brigel '01]
First: carve out entanglement structure on cluster state by local Pauli Z measurement
Then:
(1) Measurement along each wire simulates 1-qubit evolution (gates by teleportation)
(2) Measurement near & on each bridge simulates 2-qubit gate (CNOT via "scattering")
2D or higher dimensions are needed for universal QC

How much entanglement is needed?

[Gross, Flammia & Eisert '09; Bremner, Mora & Winter '09]

- States (n-qubit) with too much entanglement $E_g > n - \delta$ are not universal for QC

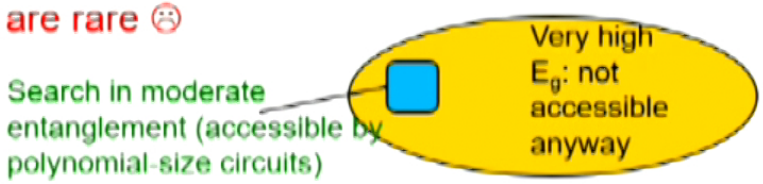
$$E_g(|\Psi\rangle) \equiv -\log_2 \max_{\phi \in \mathcal{P}} |\langle \phi | \Psi \rangle|^2 \qquad \mathcal{P} = \text{set of product states } \{\phi = \phi_1 \otimes \phi_2 \otimes \cdots \phi_n\}$$

- Intuition: for state with high entanglement, every local measurement outcome has low probability

  ➔ whatever local measurement strategy, the distribution of outcomes is so random that one can simulate it with a random coin (thus not more powerful than classical random guessing)

- Moreover, states with high entanglement are typical: those with $E_g < n - 2\log_2(n) - 3$ is rare, i.e. with fraction $< e^{-n^2}$

  ➔ Universal resource states are rare ☹



Some questions for MBQC

- Characterizing all resource states? Still open

- Can they be unique ground state with 2-body Hamiltonians with a finite gap? ➔ If so, create resources by cooling!

  - Affleck-Kennedy-Lieb-Tasaki (AKLT) family of states [*AKLT* '87, '88]

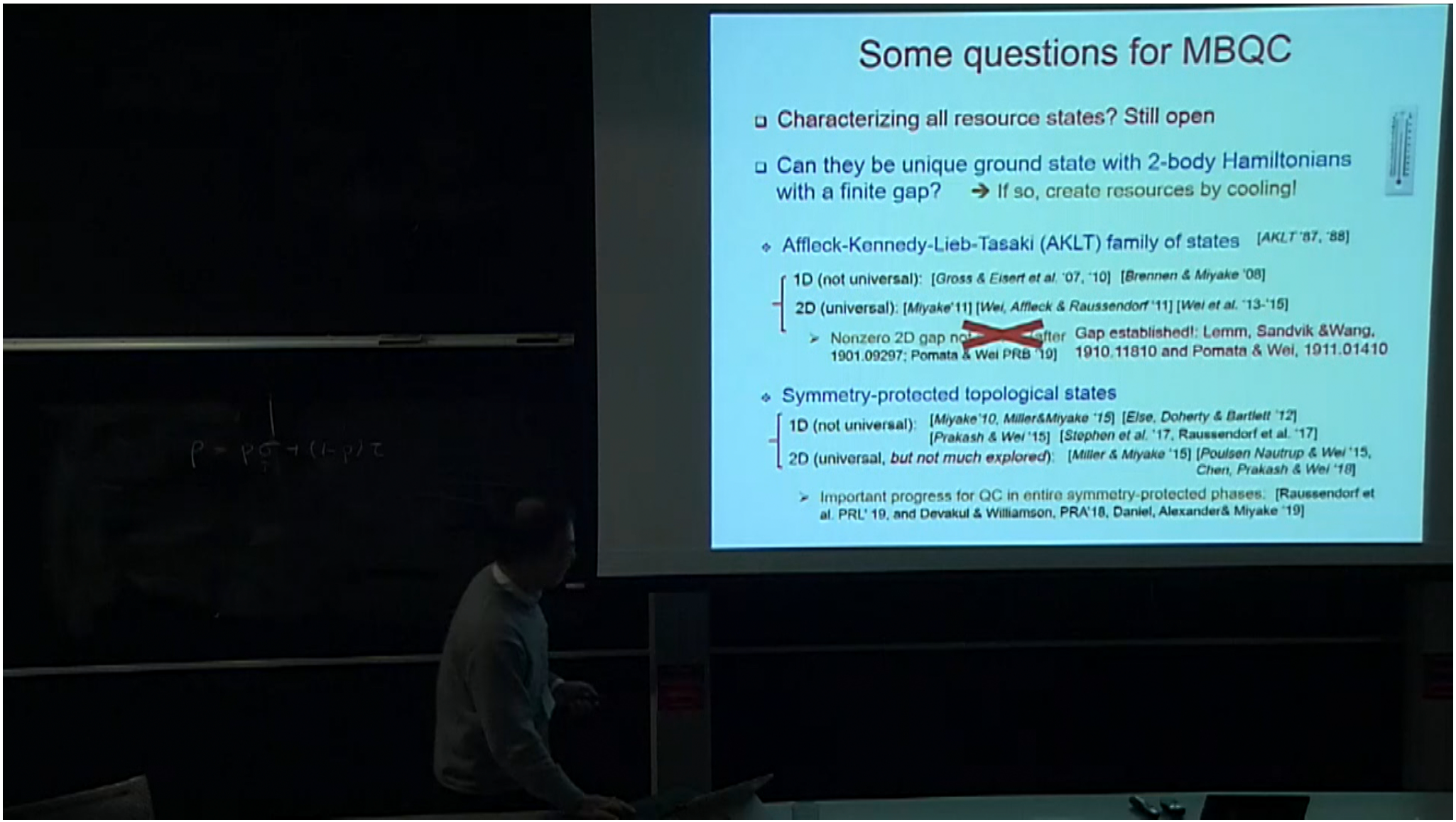
    1D (not universal): [*Gross & Eisert et al.* '07, '10] [*Brennen & Miyake* '08]

    2D (universal): [*Miyake*'11] [*Wei, Affleck & Raussendorf* '11] [*Wei et al.* '13-'15]

    - Nonzero 2D gap no[illegible] after 1901.09297; Pomata & Wei PRB '19]

      Gap established!: Lemm, Sandvik &Wang, 1910.11810 and Pomata & Wei, 1911.01410

  - Symmetry-protected topological states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[Prakash & Wei '15] [Stephen et al. '17, Raussendorf et al. '17]
2D (universal, but not much explored): [Miller & Miyake '15] [Poulsen Nautrup & Wei '15, Chen, Prakash & Wei '18]
Important progress for QC in entire symmetry-protected phases: [Raussendorf et al. PRL' 19, and Devakul & Williamson, PRA'18, Daniel, Alexander& Miyake '19]

Affleck-Kennedy-Lieb-Tasaki states/models

- Valence-bond ground states of isotropic antiferromagnet
  - Importance: provide strong support for Haldane's conjecture on spectral properties of spin chains [*AKLT* '87,88]
  - Provide concrete examples for symmetry-protected topological order [*Gu & Wen* '09, '11, Pollmann et al. '12 …]

- States of local spin S=1,3/2, 2,.. (defined on any lattice/graph)

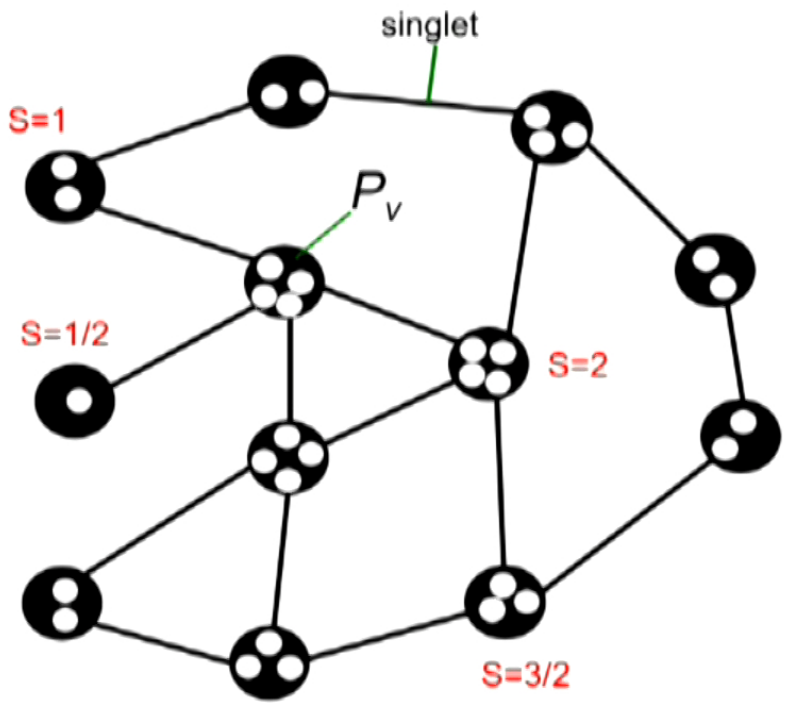
  ➔ Unique* ground states of gapped# two-body isotropic Hamiltonians

$$H = \sum_{\langle i,j \rangle} f(\vec{S}_i \cdot \vec{S}_j)$$ *f(x)* is a polynomial

e.g. 1D: S=1 $$H_{1D} = \sum_i \hat{P}^{(S=2)}_{i,i+1} = \frac{1}{2} \sum_{\text{edge } \langle i,j \rangle} \left[\vec{S}_i \cdot \vec{S}_j + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_j)^2 + \frac{2}{3} \right]$$

*w/ appropriate boundary conditions [Kennedy, Lieb & Tasaki '88]

(hybrid) AKLT state defined on any graph



P_v = projection to symmetric subspace of n qubit ≡ spin n/2

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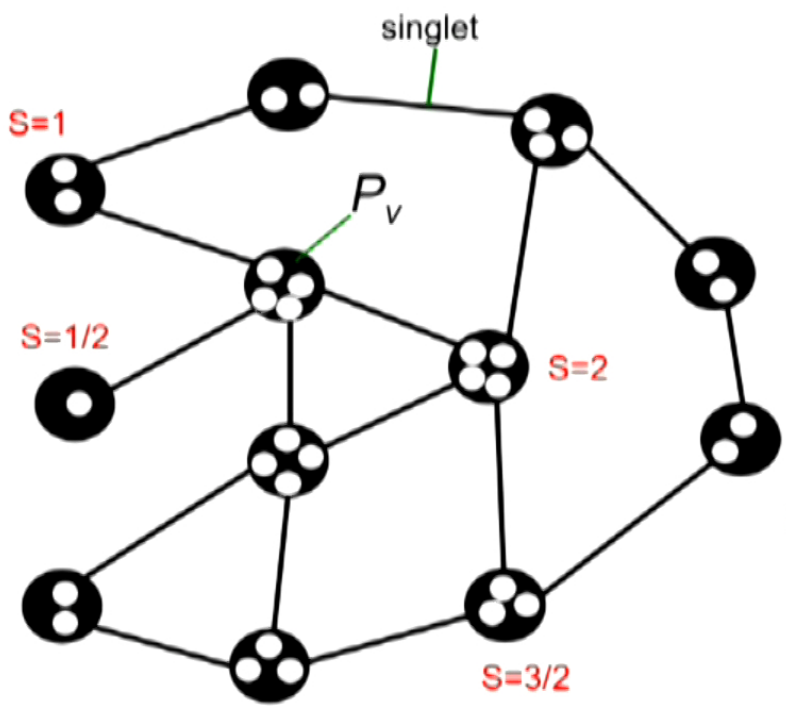
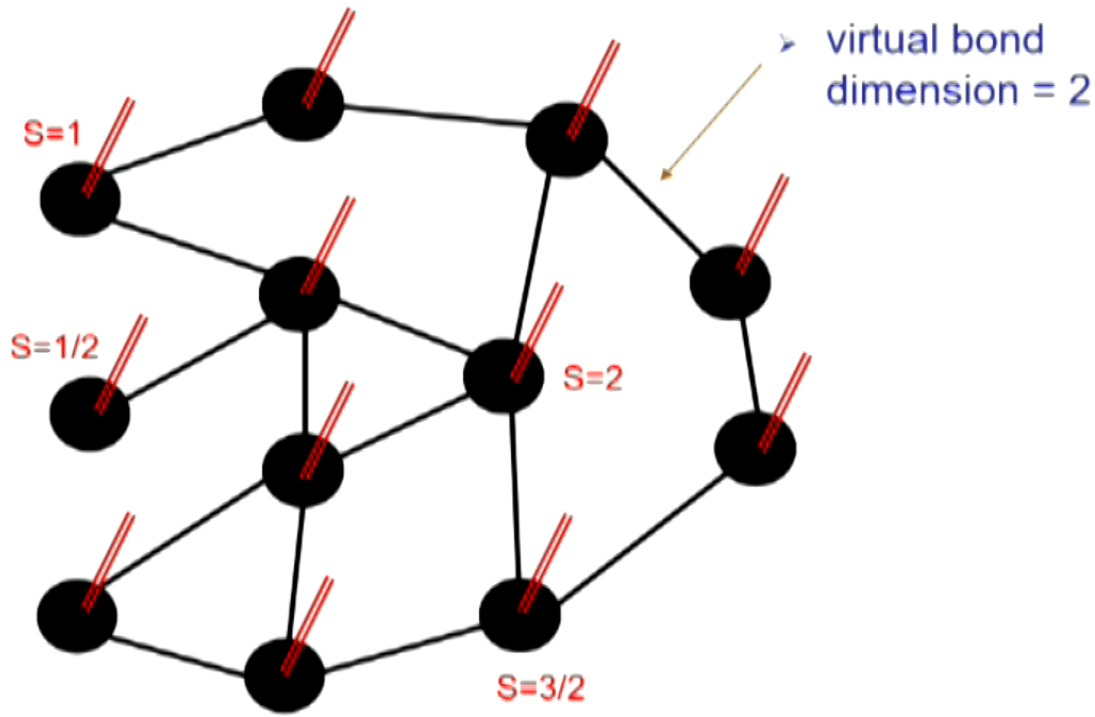


- # virtual qubits /site = # neighbors
- Physical spin Hilbert space = symmetric subspace of qubits
- Local S = # neighbors/2
- Hamiltonian = sum of projectors

$$H = \sum_{\langle v,w \rangle} P_{vw}^{(S=S_v+S_w)}$$

P_v = projection to symmetric subspace of n qubit ≡ spin n/2

AKLT has tensor-network representation



- virtual bond dimension = 2

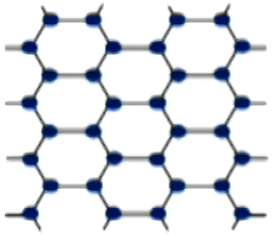
2D AKLT states for quantum computation?

- On various lattices

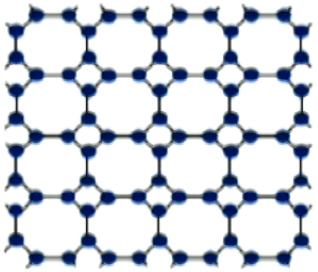
Miyake '11; ***Wei***, *Affleck & Raussendorf*, PRL '11
Wei, PRA '13, ***Wei***, *Haghnegahdar& Raussendorf, PRA '14*
Wei *& Raussendorf, PRA '15*

spin-3/2:

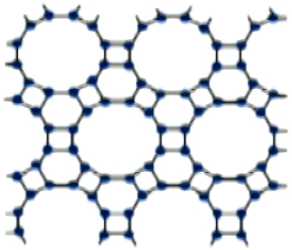
honeycomb



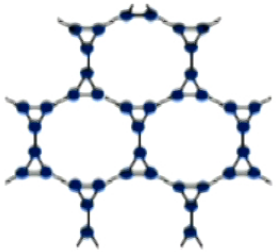
square-octagon



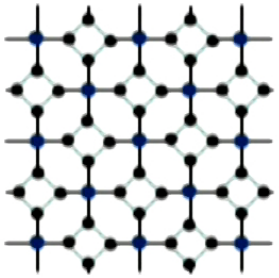
'cross'



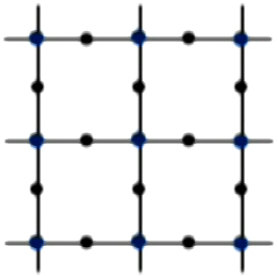
star



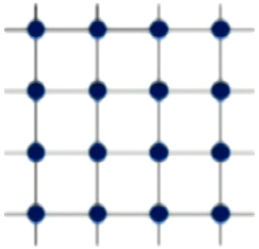
square-hexagon
(spin-2 spin-3/2 mixture)



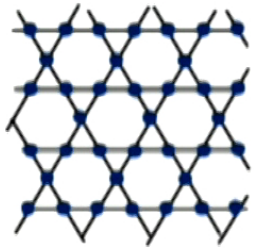
decorated-square
(spin-2 spin-1 mixture)



square
(spin-2)



Kagome
(spin-2)



AKLT states on trivalent lattices

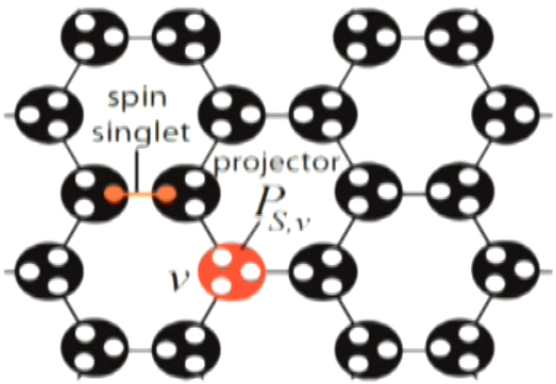
- Each site: three virtual qubits  ≡ spin 3/2 (in general: S= #nbr /2)

  ➔ physical spin = symmetric subspace of qubits

- Two virtual qubits on an edge form a singlet 

$|01\rangle - |10\rangle$

$$P = |3/2\rangle\langle 000| + |-3/2\rangle\langle 111| + |1/2\rangle\langle W| + |-1/2\rangle\langle \overline{W}|$$



$$|000\rangle \leftrightarrow \left|S = \frac{3}{2}, S_z = \frac{3}{2}\right\rangle$$

$$|111\rangle \leftrightarrow \left|\frac{3}{2}, -\frac{3}{2}\right\rangle$$

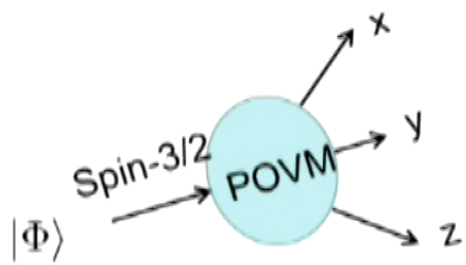
$$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left|\frac{3}{2}, \frac{1}{2}\right\rangle$$

$$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left|\frac{3}{2}, -\frac{1}{2}\right\rangle$$

Reduction to 2D graph states

[*Wei, Affleck & Raussendorf* '11]

- A specific generalized measurement (POVM) on all sites converts AKLT to a graph state
  (graph depends on random x, y and z outcomes)



$$|\Phi\rangle \longrightarrow F_{\alpha=x,y,\text{or } z}|\Phi\rangle$$

$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z = I \quad \text{(Completeness)}$$

$$F_z = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_z + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_z\right) \sim \left|000\right\rangle\left\langle 000\right| + \left|111\right\rangle\left\langle 111\right|$$

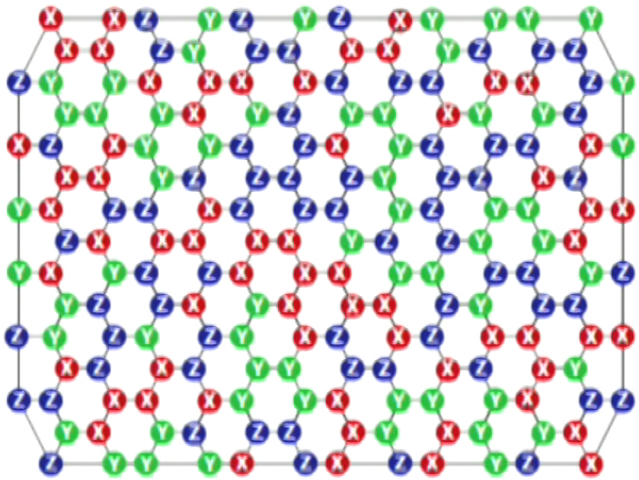
(effective 2 levels)

$$F_x = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_x + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_x\right)$$

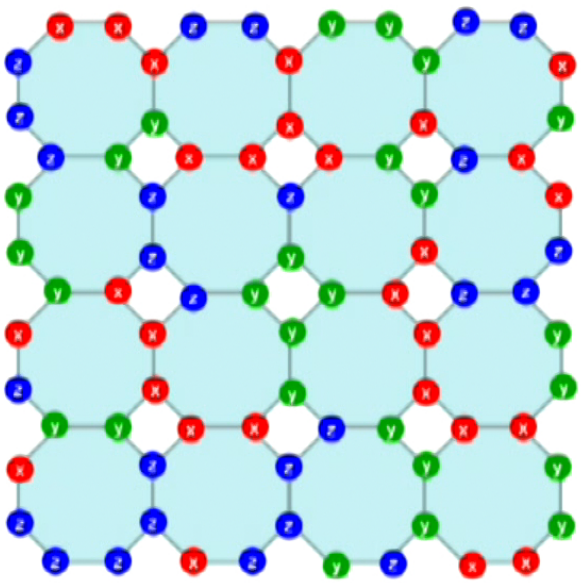
$$F_y = \sqrt{\frac{2}{3}}\left(\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|_y + \left|-\frac{3}{2}\right\rangle\left\langle-\frac{3}{2}\right|_y\right)$$

Recipe: construct graph for 'the graph state'

- Examples: random POVM outcomes x, y, z



honeycomb



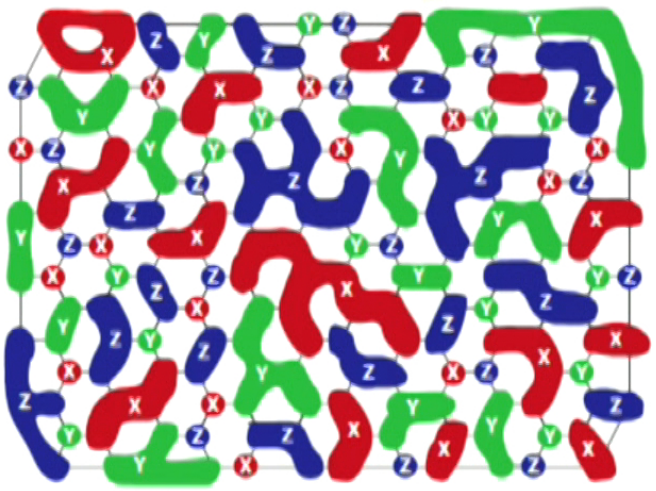
square octagon

$$P(\{\alpha(v)\}) \sim 2^{|V|-|\mathcal{E}|}$$

V: domains, $\mathcal{E}$: inter-domain edges

Step 1: Merge sites to "domains" ➔ vertices

- 1 domain = 1 logical qubit



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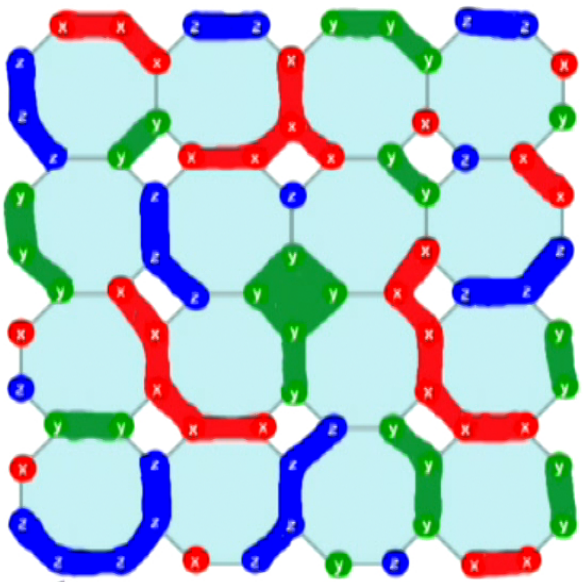

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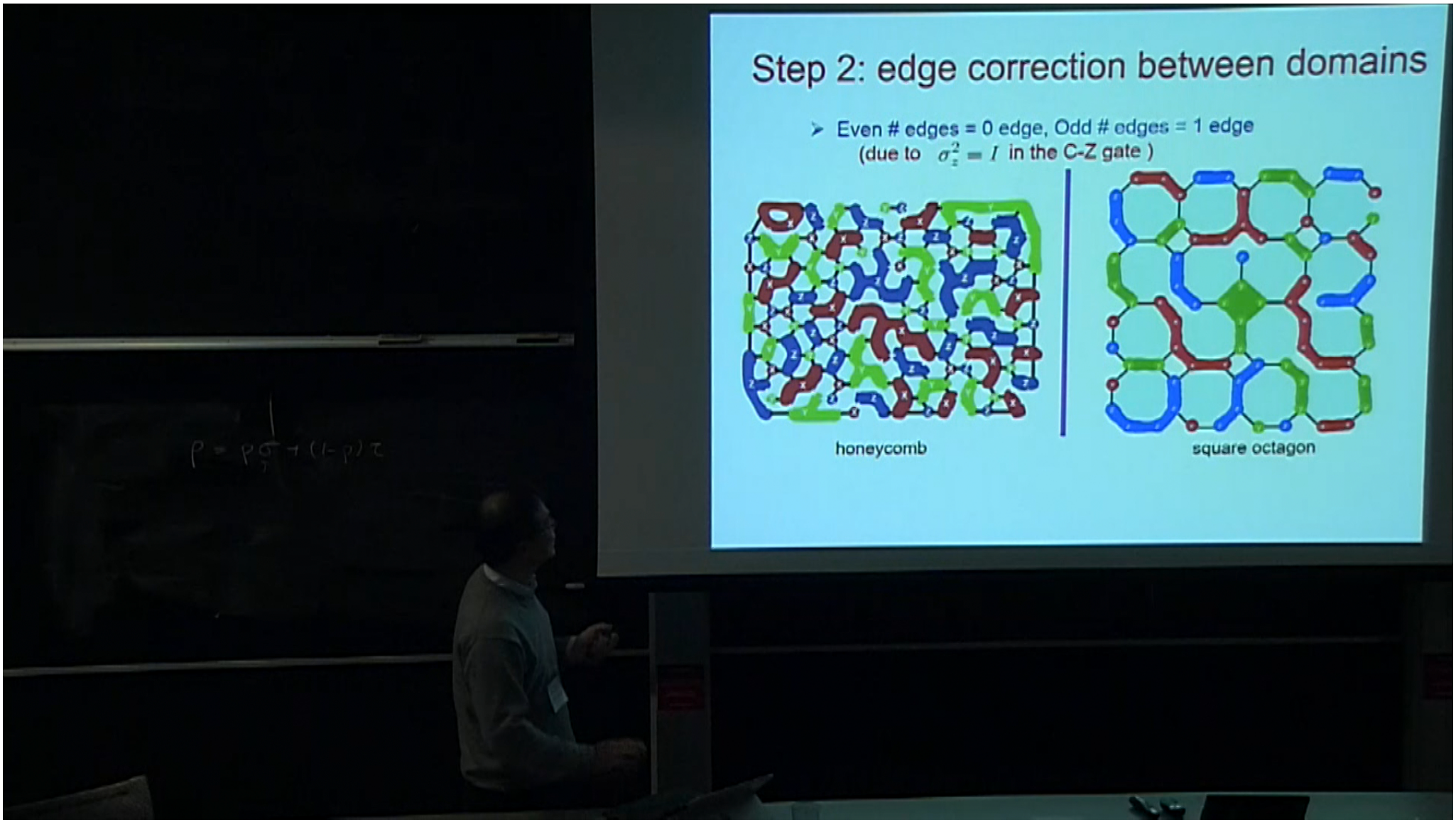
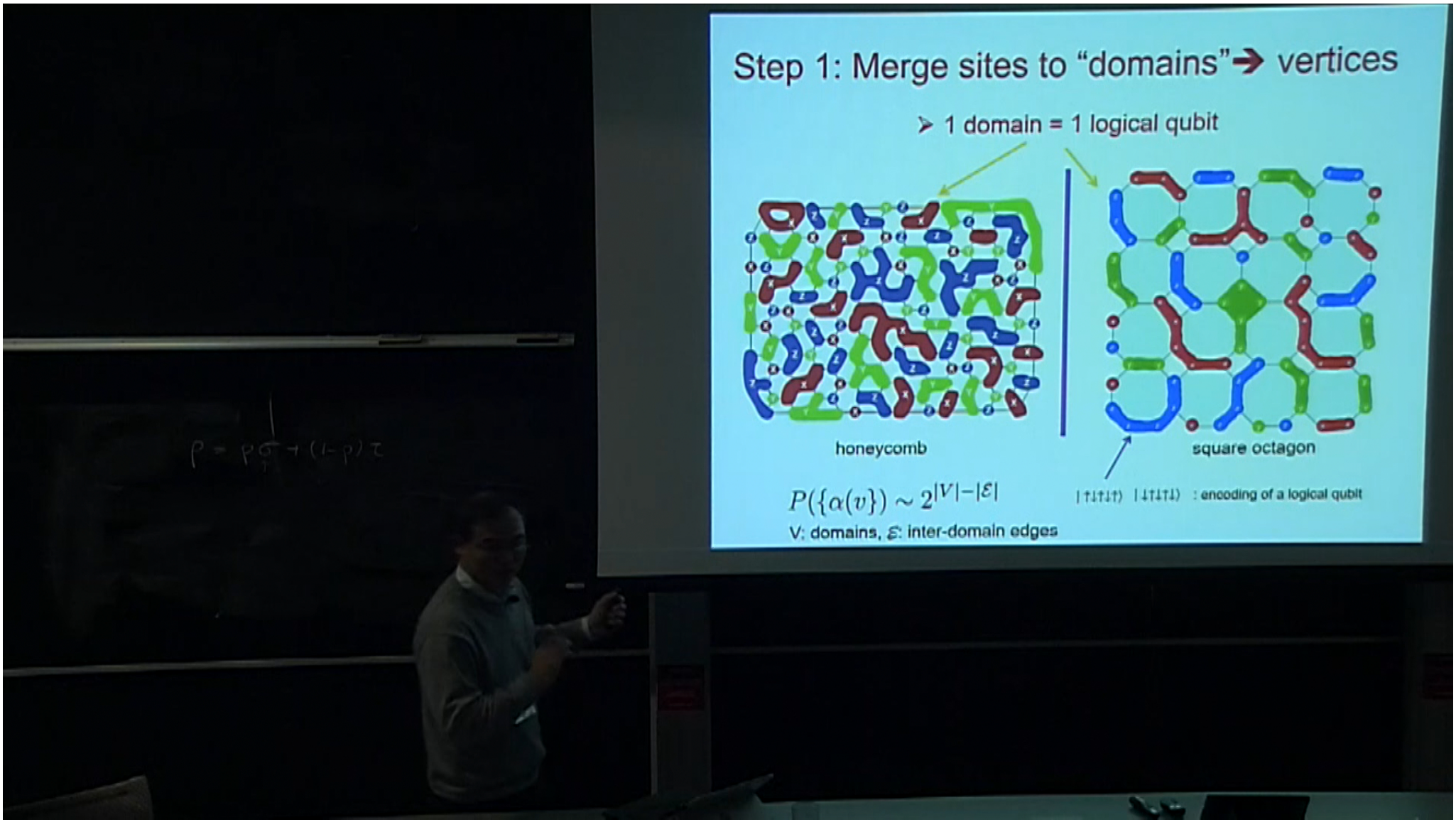
$|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$ $|\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$: encoding of a logical qubit


Step 2: edge correction between domains
Even # edges = 0 edge, Odd # edges = 1 edge
(due to $\sigma_z^2 = I$ in the C-Z gate)
honeycomb
square octagon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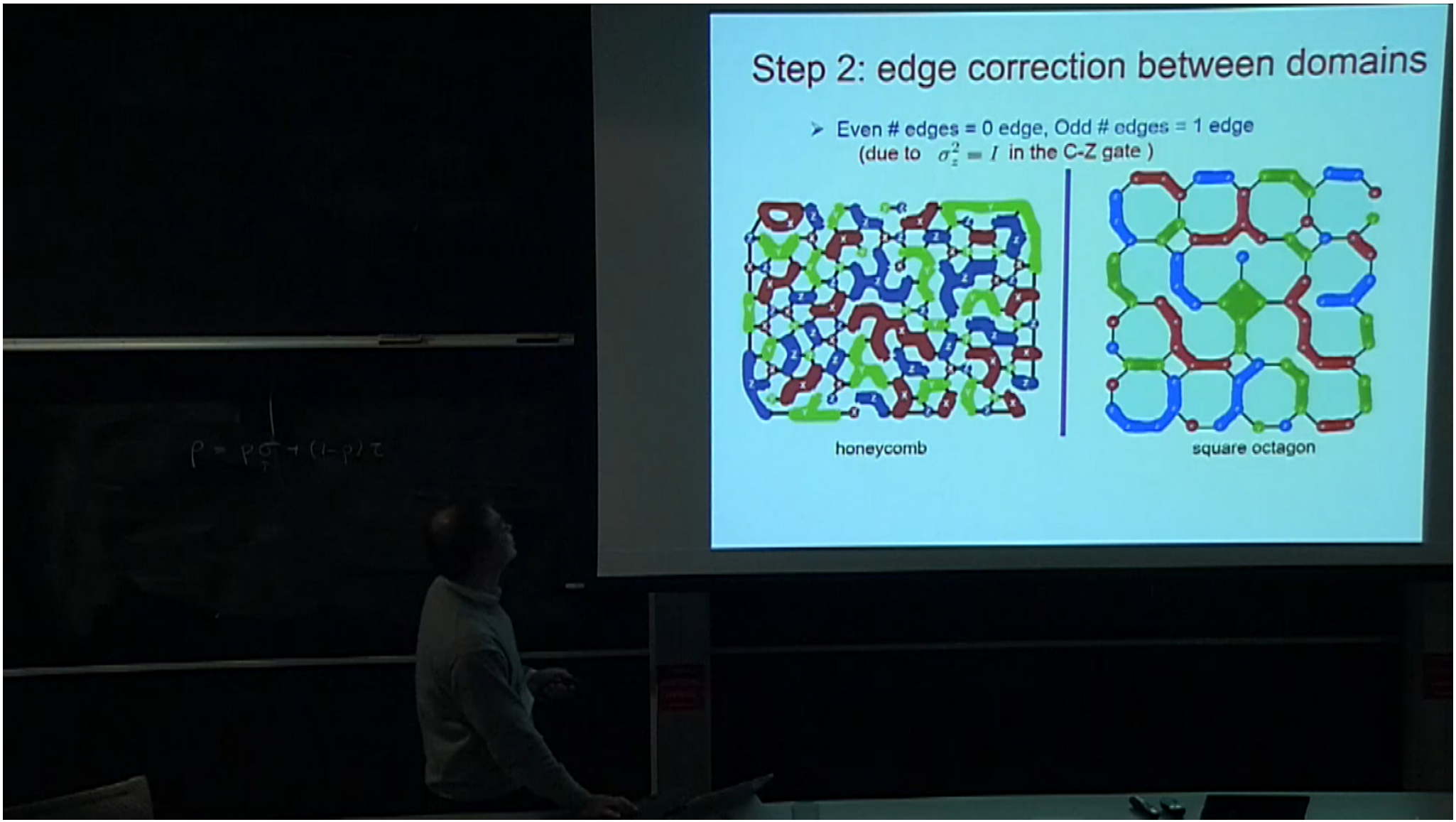
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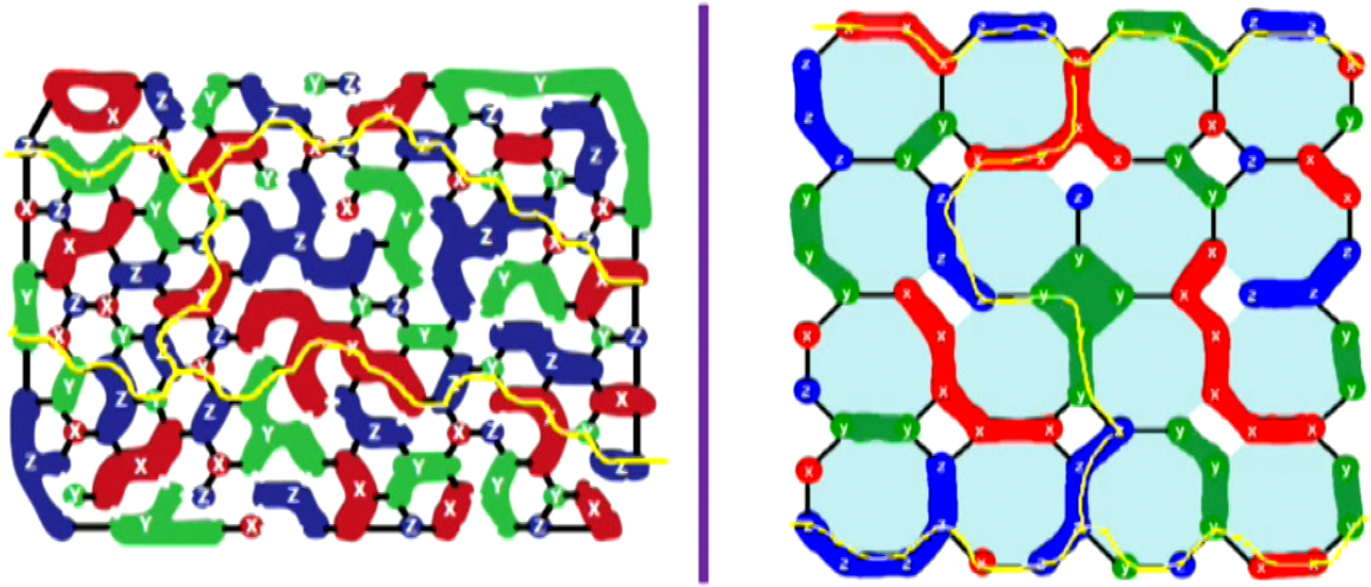
honeycomb
square octagon

Step 3: Check connections (percolation)

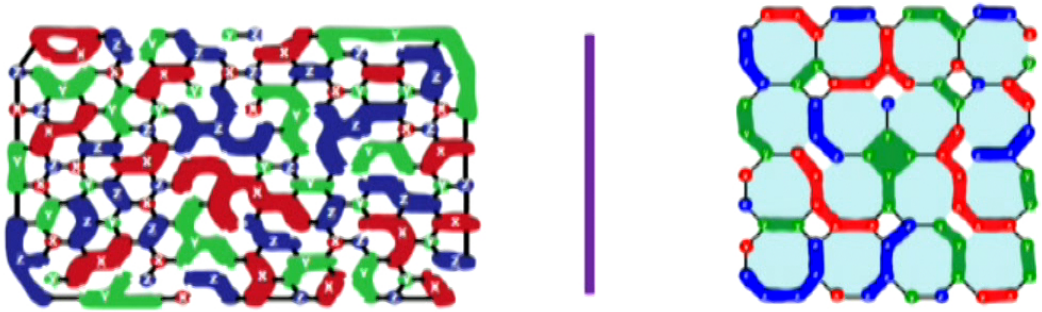
- Sufficient number of wires if graph is in supercritical phase (percolation)



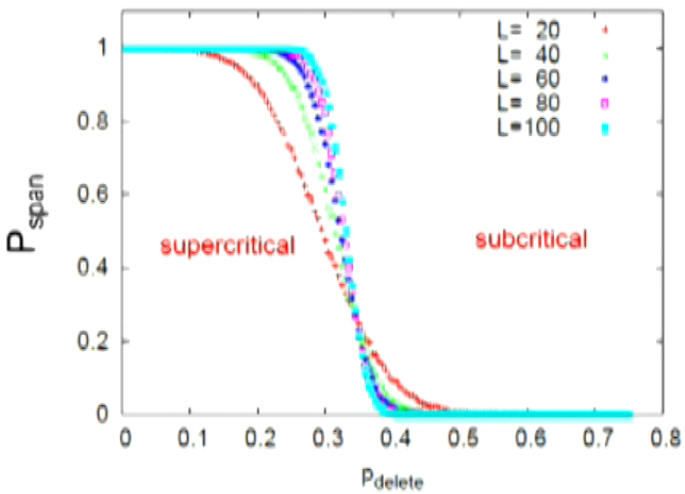
✓ Verified this for honeycomb, square octagon and cross lattices
➔ AKLT states on these are universal resources

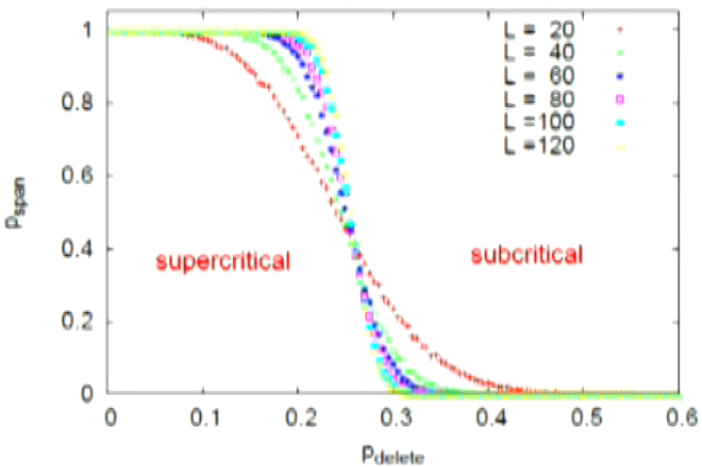
Robust connectivity?

- Characterized by artificially removing domains to see when connectivity collapses (phase transition)



[*Wei* '13]





Difficulty for spin-2

- Technical problem: trivial extension of POVM does NOT work!

$$
\begin{aligned}
F_z &= |2\rangle\langle 2|_z + |-2\rangle\langle -2|_z \\
F_x &= |2\rangle\langle 2|_x + |-2\rangle\langle -2|_x \\
F_y &= |2\rangle\langle 2|_y + |-2\rangle\langle -2|_y
\end{aligned}
$$

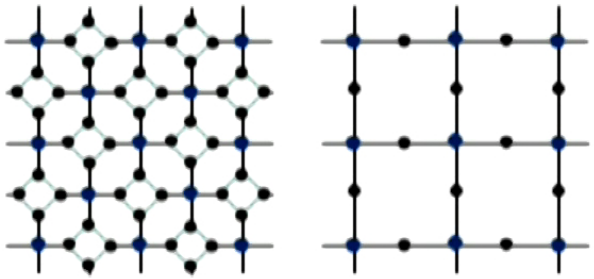
$$F_x^\dagger F_x + F_y^\dagger F_y + F_z^\dagger F_z \neq c \cdot I$$

➔ Possible leakage out of logical subspace!

- How to calculate probability distribution?

$$p(\{F,K\}) = \langle \mathrm{AKLT}| \bigotimes_u F^\dagger_{\alpha(u)} F_{\alpha(u)} \bigotimes_v K^\dagger_{\beta(v)} K_{\beta(v)} |\mathrm{AKLT}\rangle = ?$$

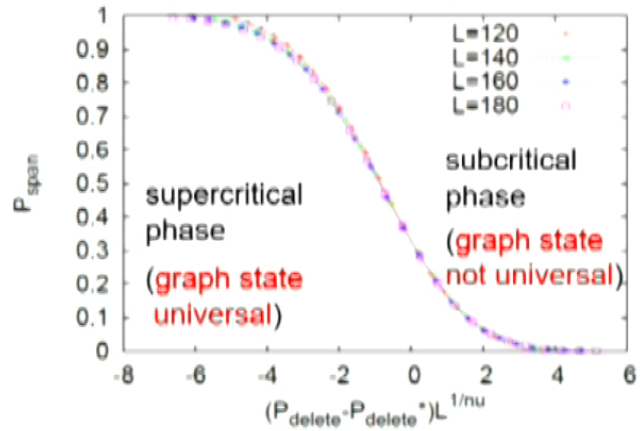
- We first solved the hybrid AKLT states (on mixed & decorated lattices) then the square lattice



[***Wei,*** *Haghnegahdar & Raussendorf, PRA '14*
Wei *& Raussendorf, PRA '15*]

Universality for QC in AKLT family

- Spin-2 AKLT state on square lattice is universal



- **Emerging (partial) picture for AKLT family**:

  AKLT states involving spin-2 and other lower spin entities are universal if they reside on a 2D frustration-free lattice (e.g. w/o triangles) with any combination of spin-2, spin-3/2, spin-1 and spin-1/2

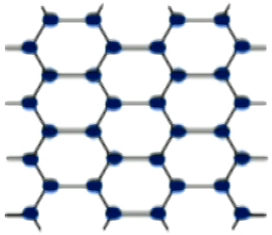
2D AKLT states for quantum computation

- On various lattices

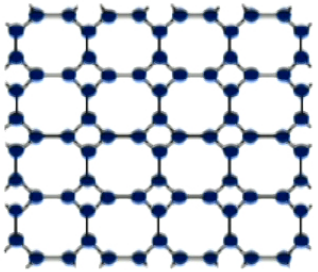
Miyake '11; ***Wei***, *Affleck & Raussendorf*, PRL '11
Wei, PRA '13, ***Wei***, *Haghnegahdar& Raussendorf, PRA '14*
Wei *& Raussendorf, PRA '15*

spin-3/2:

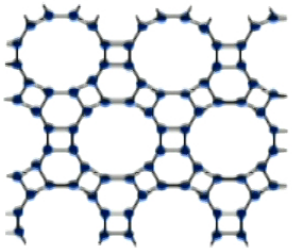
honeycomb



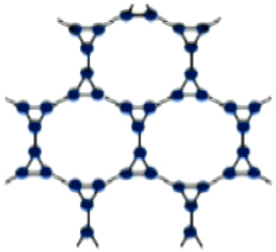
square-octagon



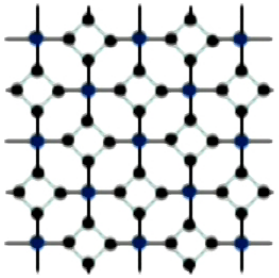
'cross'



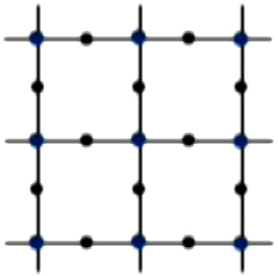
star



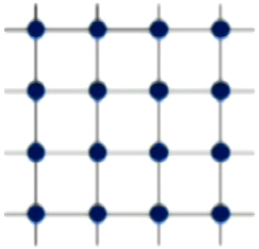
square-hexagon
(spin-2 spin-3/2 mixture)



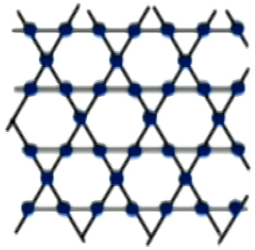
decorated-square
(spin-2 spin-1 mixture)



square
(spin-2)



Kagome
(spin-2)



Outline

I. Introduction

II. AKLT models and states for universal quantum computation (in MBQC framework)

III. Nonzero gap for some 2D AKLT models

IV. Summary

AKLT Hamiltonians and gap(?)

- On honeycomb lattice [AKLT '88]

$$H = \sum_{\text{edge } \langle i,j \rangle} \hat{P}_{i,j}^{(S=3)} = \sum_{\text{edge } \langle i,j \rangle} \left[\vec{S}_i \cdot \vec{S}_j + \frac{116}{243} (\vec{S}_i \cdot \vec{S}_j)^2 + \frac{16}{243} (\vec{S}_i \cdot \vec{S}_j)^3 + \frac{55}{108} \right]$$

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- Exponential decay of correlation functions (on hexagonal and square lattices): [AKLT'88, KLT '88]

$$0 \leqslant (-1)^{|i-j|} \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle \leqslant C \exp(-|i-j|/\xi)$$ C, ξ const. >0

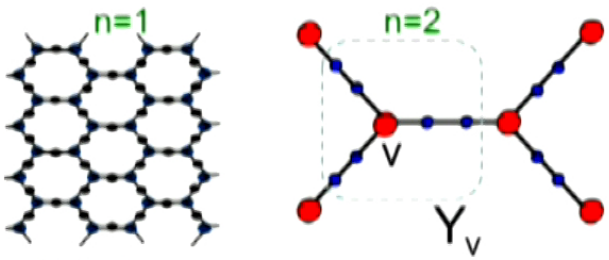
==> strongly suggests nonzero gap

"Gap established"!: Lemm, Sandvik &Wang, 1910.11810
and Pomata & Wei, 1911.01410

Progress in proving nonzero gap

- Decorating lattice Λ into $\Lambda^{(n)}$ by adding n spin-1 sites to each edge

$$H^{\mathrm{AKLT}}_{\Lambda^{(n)}} = \sum_{e \in \mathcal{E}_{\Lambda^{(n)}}} P_e^{(z(e)/2)}$$



- Abdul-Rahman, Lemm, Luica, Nachtegaele & Young (ALLNY), arXiv:1901.09297

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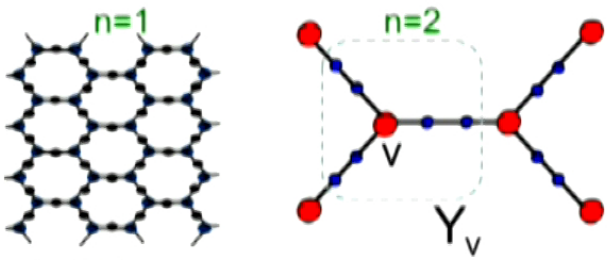
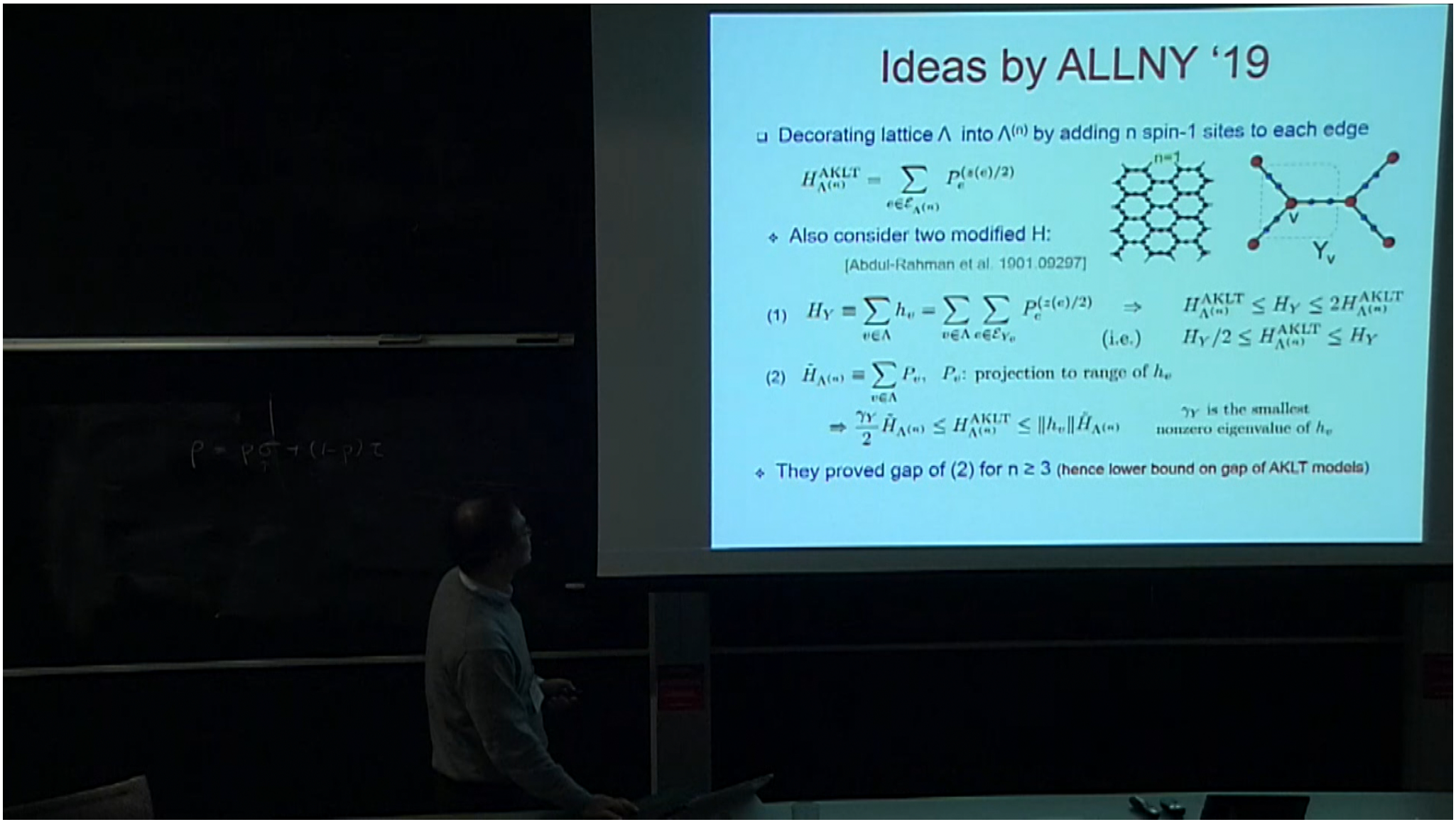


- Abdul-Rahman, Lemm, Luica, Nachtegaele & Young (ALLNY), arXiv:1901.09297

Theorem 2.2. *The spectral gap above the ground state of the AKLT model on the edge-decorated honeycomb lattice with $n \geq 3$ has a strictly positive lower bound uniformly for all finite volumes with periodic boundary conditions.*

✓ First analytic proof of nonzero gap for some 2D AKLT models ☺
(but still not the undecorated honeycomb model)

- Nothing can be said about n=1 & 2 cases regarding spectral gap
  What about other lattices? Undecorated lattices?
  ➔ Later


Ideas by ALLNY '19
Decorating lattice Λ into Λ(n) by adding n spin-1 sites to each edge
$H^{\mathrm{AKLT}}_{\Lambda^{(n)}} = \sum_{e \in \mathcal{E}_{\Lambda^{(n)}}} P_e^{(s(e)/2)}$
n=1
v
Y_v
Also consider two modified H:
[Abdul-Rahman et al. 1901.09297]
(1) $H_Y \equiv \sum_{v \in \Lambda} h_v = \sum_{v \in \Lambda} \sum_{e \in \mathcal{E}_{Y_v}} P_e^{(z(e)/2)} \Rightarrow H^{\mathrm{AKLT}}_{\Lambda^{(n)}} \leq H_Y \leq 2 H^{\mathrm{AKLT}}_{\Lambda^{(n)}}$
(i.e.) $H_Y/2 \leq H^{\mathrm{AKLT}}_{\Lambda^{(n)}} \leq H_Y$
(2) $\tilde{H}_{\Lambda^{(n)}} \equiv \sum_{v \in \Lambda} P_v$, P_v: projection to range of h_v
$\Rightarrow \frac{\gamma_Y}{2} \tilde{H}_{\Lambda^{(n)}} \leq H^{\mathrm{AKLT}}_{\Lambda^{(n)}} \leq \|h_v\| \tilde{H}_{\Lambda^{(n)}}$
γ_Y is the smallest nonzero eigenvalue of h_v
They proved gap of (2) for n ≥ 3 (hence lower bound on gap of AKLT models)

How to prove nonzero gap?

[AKLT '88,Knabe '88, Fannes, Nachtergaele & Werner '92, …., Gosset & Mozgunov '16, Lemm & Mozgunov '19, Anshu '19, Abdul-Rahman et al. 1901.09297]

- Squaring H:

$$(\tilde{H}_{\Lambda^{(n)}})^2 = \tilde{H}_{\Lambda^{(n)}} + \frac{1}{2}\sum_{v\neq w}(P_v P_w + P_w P_v) \quad [\text{Note } P_v^2 = P_v]$$

- Throwing out non-overlapping $P_v P_w \geq 0$

$$\geq \tilde{H}_{\Lambda^{(n)}} + \sum_{(v,w)\in\mathcal{E}_\Lambda}(P_v P_w + P_w P_v)$$

- Overlapping $P_v P_w$ can be non-positive.

But if we have: $P_v P_w + P_w P_v \geq -\varepsilon(P_v + P_w)$

Want ε>0 as small as possible

then we have

$$(\tilde{H}_{\Lambda^{(n)}})^2 = \tilde{H}_{\Lambda^{(n)}} - \varepsilon \sum_{(v,w)\in\mathcal{E}_\Lambda}(P_v + P_w)$$

$$\geq (1 - z\varepsilon_n)\tilde{H}_{\Lambda^{(n)}} = \gamma\tilde{H}_{\Lambda^{(n)}}$$

[z: coordination #]

How to prove nonzero gap?

[AKLT '88,Knabe '88, Fannes, Nachtergaele & Werner '92, ..., Gosset & Mozgunov '16, Lemm & Mozgunov '19, Anshu '19, Abdul-Rahman et al. 1901.09297]

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$$\geq (1 - z\varepsilon_n)\tilde{H}_{\Lambda^{(n)}} = \gamma\tilde{H}_{\Lambda^{(n)}} \quad [z\text{: coordination \#}]$$

- If $\gamma = (1-z\varepsilon) > 0$, then there is a nonzero gap

A useful lemma

- [Fannes, Nachtergaele, Werner '92]:

  For two projectors E & F (also works for 1-E & 1-F with same ε):

  $$EF + FE \geq -\varepsilon(E+F)$$

  $$\varepsilon = ||EF - E \wedge F||$$

  E ∧ F : projection onto ran(E)= E$\mathcal{H}$ & ran(F)=F$\mathcal{H}$

  - Proof discussed later

- **(1-zε)** > 0 implies there is a nonzero gap
  - Want ε<1/z (z=3 for honeycomb)

[Abdul-Rahman et al. (ALLNY) 1901.09297]

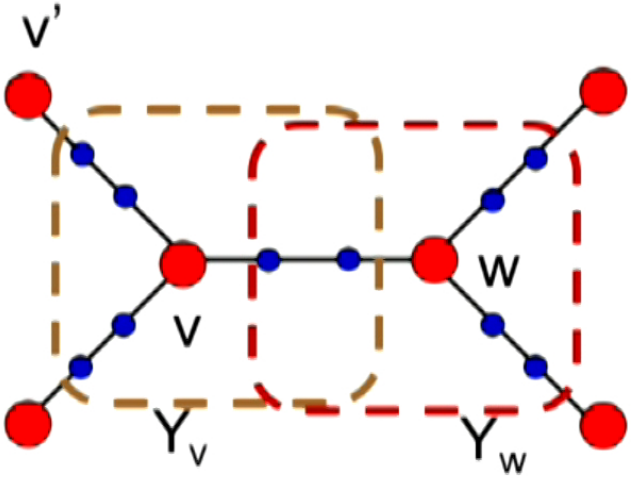
Proposition 2.1. *Let*

$$A_n = \frac{4}{3^n\left(1 - \frac{8(1+3^{-2n-1})}{3^n(1-3^{-2n})}\right)}.$$

Then, for all $n \geq 3$, the quantity ε_n defined in (2.7) satisfies

$$\varepsilon_n \leq A_n + A_n^2\left(1 + \frac{8(1+3^{-2n-1})^2}{3^n(1-3^{-2n})^2}\right) < 1/3. \tag{2.10}$$

Key point in upper bounding ε



- Use $E=I-P_v$ (projection to local ground space supported on Y_v), $F=I-P_w$ (projection to local ground space supported on Y_w)
- $E \wedge F$: projection to Ran(E) ∩ Ran(F), i.e. local ground space supported on $Y_v \cup Y_w$

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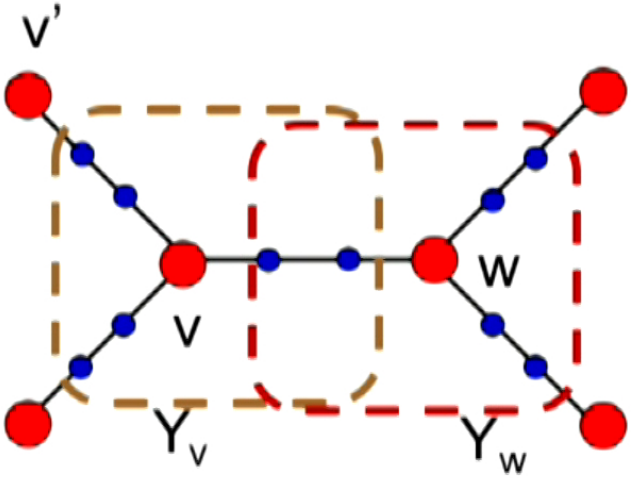

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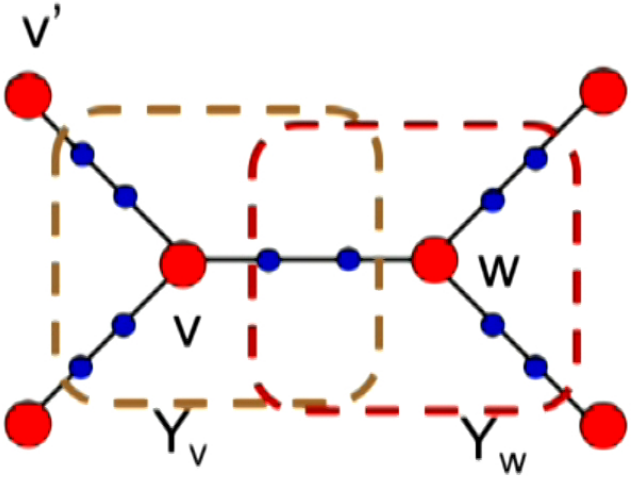
$$\varepsilon = ||EF - E \wedge F|| = \sup \frac{|\langle\phi|EF - E\wedge F|\psi\rangle|}{||\phi||\,||\psi||}$$

- ALLNY 1901.09297 used tensor-network approaches (e.g. MPS) to give an upper bound on ε [No time for details here] < 1/3 for n ≥ 3

Key point in upper bounding ε



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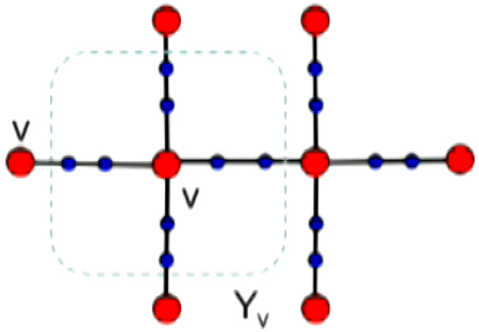
- n=1 case: EF - E ∧ F is operator roughly on size of 12 qubits, unfortunately ε≈0.4778 > 1/3 (gap inconclusive);
  n=2 operator on ~ 20 qubits (not shown in ALLNY); n=5 → ~43.6 qubits

Our main results I

[Pomata & Wei: PRB '19]

- Analytically prove AKLT models on decorated square lattice (spin-2 + spin-1 decoration) are gapped for n ≥ 4
- Prove AKLT models on decorated mixed degree 3 & 4 lattices are gapped for n ≥ 4
  - Proof extends to lattices with same local structure: e.g. decorated square lattices gapped ↔ decorated kagome lattices gapped ↔ decorated diamond lattices gapped



- Reduce the effective size to obtain ε by exact diagonalization

n	deg. 3, e.g. honeycomb	deg. 4, e.g. square	mixed deg. 3&4	deg. 6
1	0.4778328889	0.5234369088	0.5001917602	0.6027622993
2	0.1183378500	0.1218467396	0.1200794787	0.1285855428
3	0.0384373228	0.0389033280	0.0386700977	
4	0.0124460198	0.0124961718	0.0124710706	
5	0.0041321990			

gapped

Useful lemma to upper bound η

- [Fannes, Nachtergaele, Werner '92]:

  For two projectors E & F:

$$EF + FE \geq -\varepsilon(E+F) \qquad (\varepsilon \geq \eta \ \text{ in our case})$$

$$\varepsilon = ||EF - E \wedge F|| \qquad E \wedge F : \text{projection onto ran(E)= E}\mathcal{H} \ \& \ \text{ran(F)=F}\mathcal{H}$$

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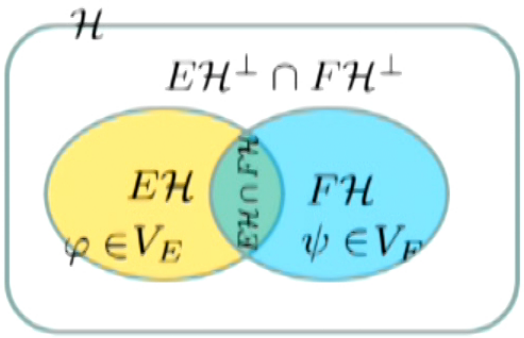
Hilbert space and two projectors



E & F are projectors;
$V_E \equiv E\mathcal{H} \cap (E\mathcal{H} \cap F\mathcal{H})^\perp$
and similarly V_F do not include intersection

$$EF + FE \geq -\varepsilon(E+F)$$

$$\varepsilon = ||EF - E \wedge F||$$

- Consider eigenvalue equation α in [-1,1]:

$$(E+F)\Upsilon = (1-\alpha)\Upsilon$$

  - If α= -1, $\Upsilon \in E\mathcal{H} \cap F\mathcal{H}$
  - If α= 1, $\Upsilon \in E\mathcal{H}^\perp \cap F\mathcal{H}^\perp$

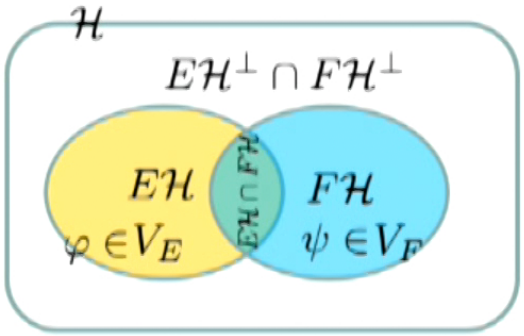
- If α in (-1,1), unique decomposition

$$\Upsilon = \varphi + \psi \quad (\varphi \in V_E \,\&\, \psi \in V_F)$$

and $F\varphi = -\alpha\psi, \; E\psi = -\alpha\varphi$ (can prove this)

hence $(EF + FE)\Upsilon = -\alpha(1-\alpha)\Upsilon$

Proving $\varepsilon = ||EF - E \wedge F||$



E & F are projectors;
V_E and V_F do not include intersection

- $E \wedge F$ projects onto $E\mathcal{H} \cap F\mathcal{H}$
- If α in (-1,1),

  $(E+F)\Upsilon = (1-\alpha)\Upsilon$

  has unique decomposition

  $\Upsilon = \varphi + \psi \quad F\varphi = -\alpha\psi, \; E\psi = -\alpha\varphi$
- Then $(EF - E \wedge F)\psi = EF\psi = -\alpha\varphi$

  (can show φ & ψ have same norm)

$||EF - E \wedge F|| \geq |\alpha|$

hence $\varepsilon = \max_{\text{eigen } |\alpha| \neq 1} \alpha$

$= ||EF - E \wedge F||$

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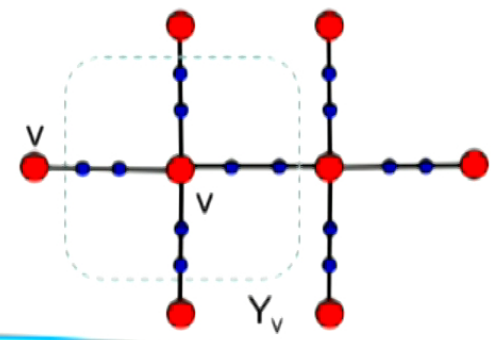

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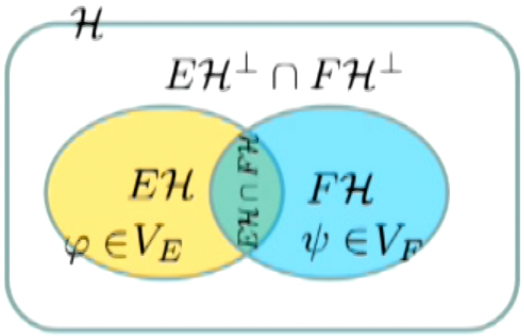

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  $(E+F)\Upsilon = (1-\alpha)\Upsilon$

  has unique decomposition

  $\Upsilon = \varphi + \psi \quad F\varphi = -\alpha\psi, \; E\psi = -\alpha\varphi$

- Then $(EF - E \wedge F)\psi = EF\psi = -\alpha\varphi$

  (can show φ & ψ have same norm)

$||EF - E \wedge F|| \geq |\alpha|$

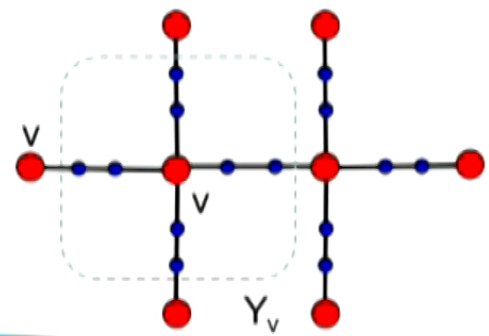
hence $\varepsilon = \max_{\text{eigen } |\alpha| \neq 1} \alpha$

$= ||EF - E \wedge F||$

Our main results I

[Pomata & Wei: PRB '19]

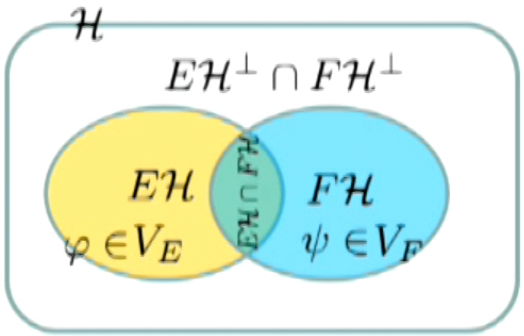
- Analytically prove AKLT models on decorated square lattice (spin-2 + spin-1 decoration) are gapped for n ≥ 4
- Prove AKLT models on decorated mixed degree 3 & 4 lattices are gapped for n ≥ 4
  - Proof extends to lattices with same local structure: e.g. decorated square lattices gapped ↔ decorated kagome lattices gapped



- Reduce the effective size to obtain ε by exact diagonalization

n	deg. 3, e.g. honeycomb	deg. 4, e.g. square	mixed deg. 3&4	deg. 6
1	0.4778328889	0.5234369088	0.5001917602	0.6027622993
2	0.1183378500	0.1218467396	0.1200794787	0.1285855428
3	0.0384373228	0.0389033280	0.0386700977	
4	0.0124460198	0.0124961718	0.0124710706	
5	0.0041321990			

Reducing Hilbert space size

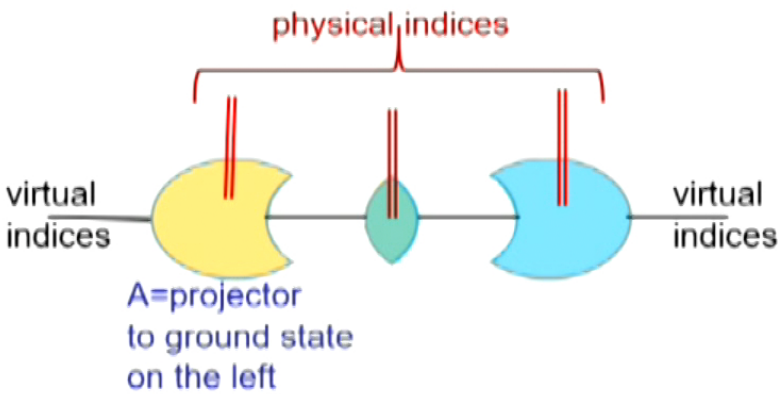


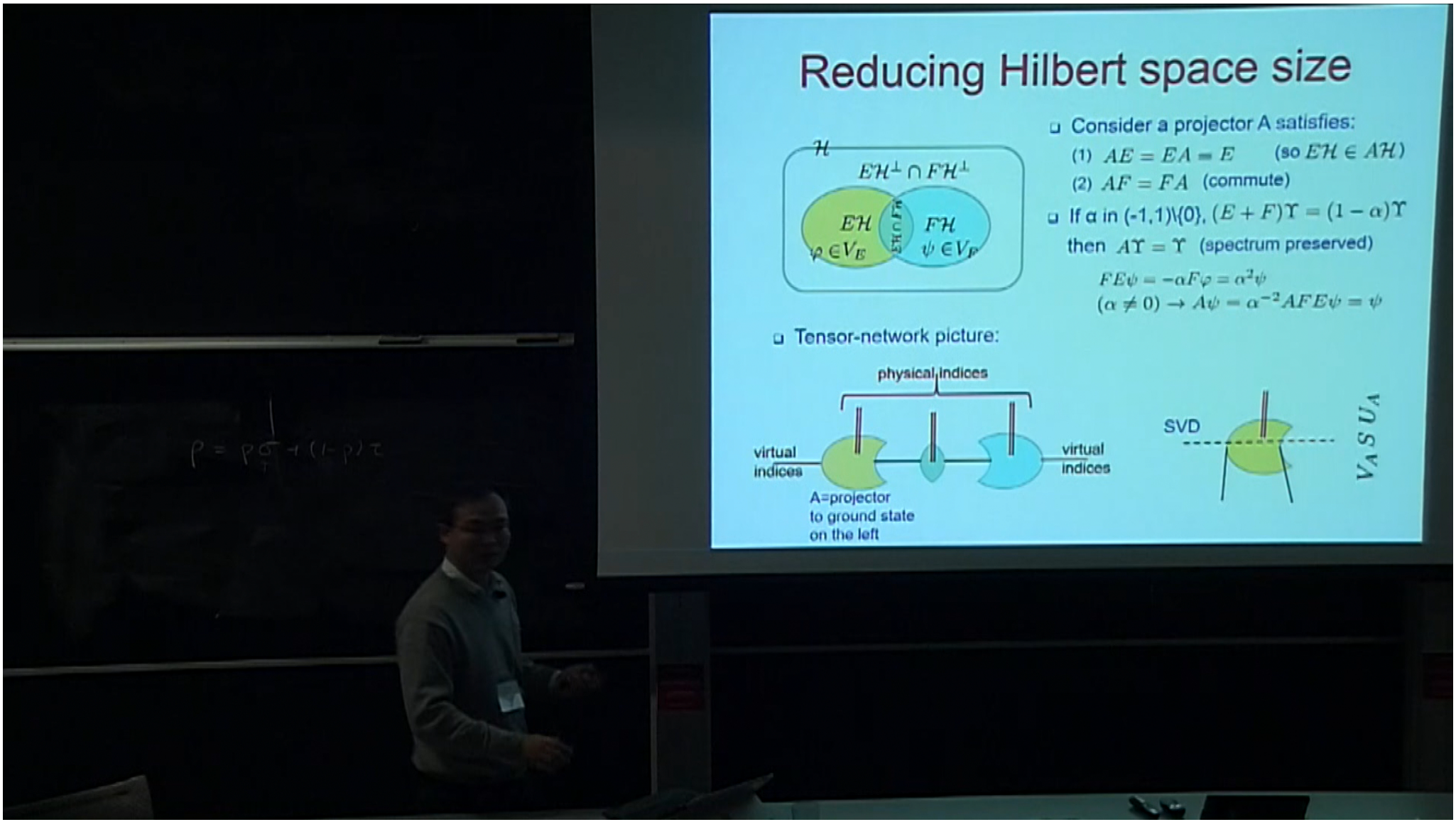
- Consider a projector A satisfies:
  - (1) $AE = EA = E$ (so $E\mathcal{H} \in A\mathcal{H}$)
  - (2) $AF = FA$ (commute)
- If α in (-1,1)\{0}, $(E+F)\Upsilon = (1-\alpha)\Upsilon$

  then $A\Upsilon = \Upsilon$ (spectrum preserved)

$$FE\psi = -\alpha F\varphi = \alpha^2\psi$$
$$(\alpha \neq 0) \rightarrow A\psi = \alpha^{-2}AFE\psi = \psi$$

- Tensor-network picture:

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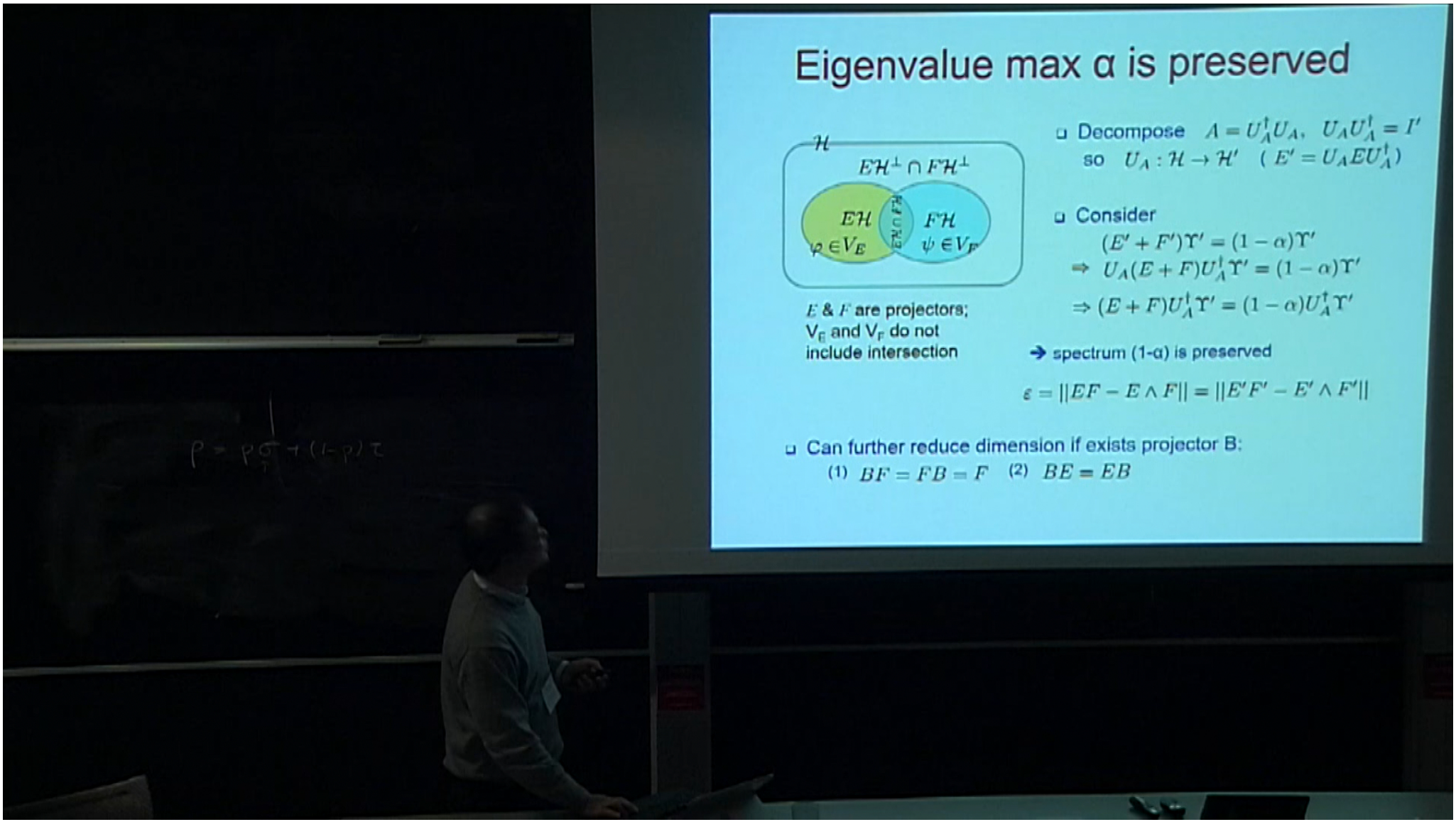
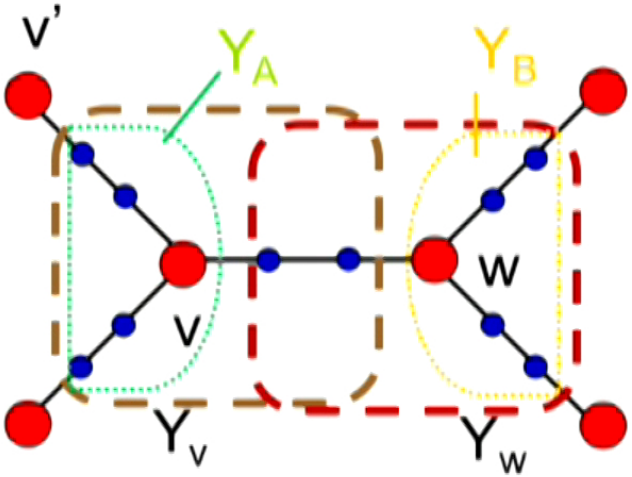
$\mathcal{H}$
$E\mathcal{H}^\perp \cap F\mathcal{H}^\perp$
$E\mathcal{H}$
$\varphi \in V_E$
$E\mathcal{H} \cap F\mathcal{H}$
$F\mathcal{H}$
$\psi \in V_F$
Tensor-network picture:
physical indices
virtual indices
virtual indices
A=projector to ground state on the left
SVD
$V_A S U_A$


Eigenvalue max α is preserved
$\mathcal{H}$
$E\mathcal{H}^\perp \cap F\mathcal{H}^\perp$
$E\mathcal{H}$
$\varphi \in V_E$
$E\mathcal{H} \cap F\mathcal{H}$
$F\mathcal{H}$
$\psi \in V_F$
E & F are projectors;
V_E and V_F do not
include intersection
Decompose $A = U_A^\dagger U_A, \; U_A U_A^\dagger = I'$
so $U_A : \mathcal{H} \to \mathcal{H}'$ $(E' = U_A E U_A^\dagger)$
Consider
$(E' + F')\Upsilon' = (1-\alpha)\Upsilon'$
$\Rightarrow U_A(E+F)U_A^\dagger \Upsilon' = (1-\alpha)\Upsilon'$
$\Rightarrow (E+F)U_A^\dagger \Upsilon' = (1-\alpha)U_A^\dagger \Upsilon'$
➔ spectrum (1-α) is preserved
$\varepsilon = ||EF - E \wedge F|| = ||E'F' - E' \wedge F'||$
Can further reduce dimension if exists projector B:
(1) $BF = FB = F$ (2) $BE = EB$

Numerical procedure



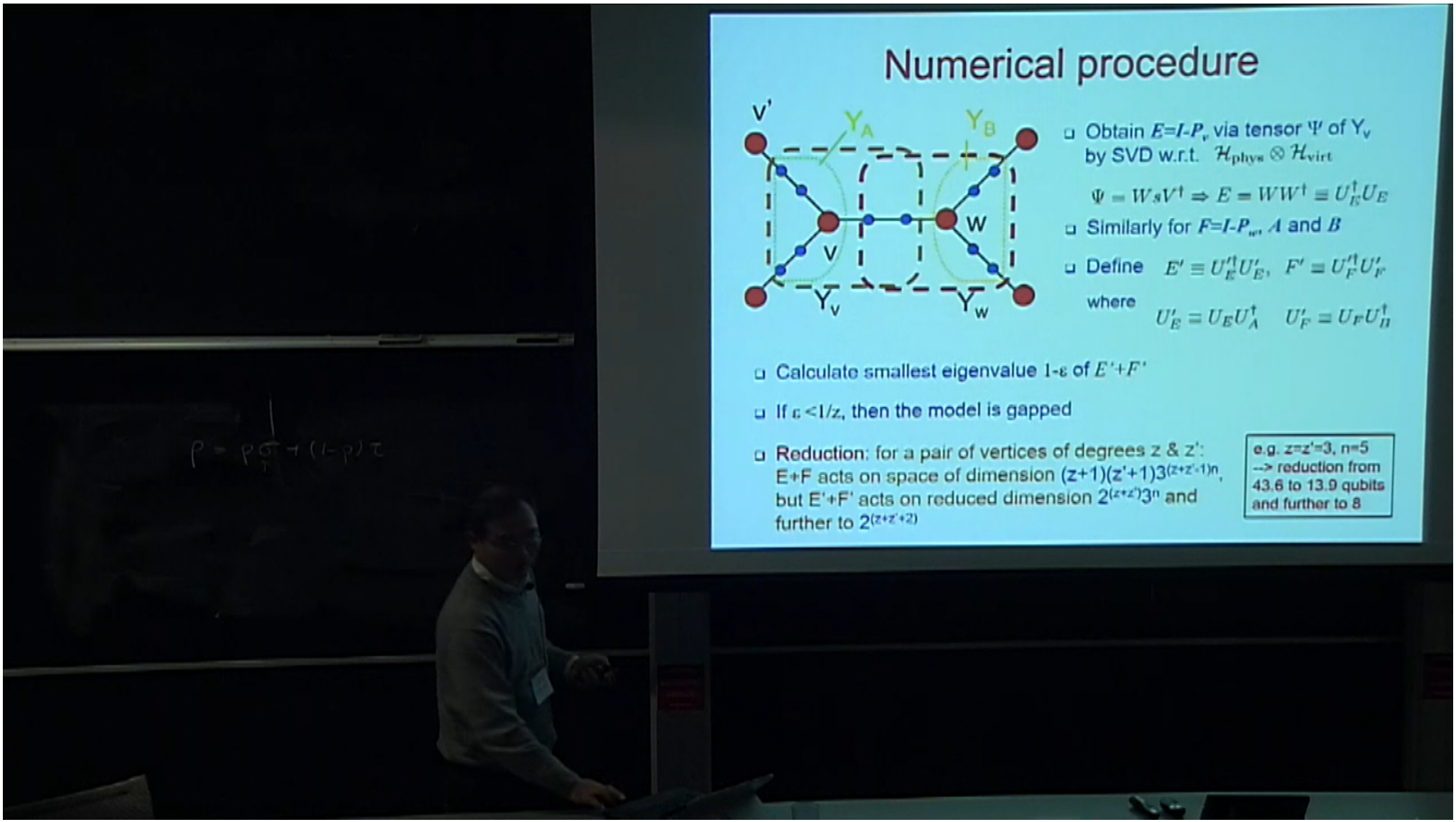
- Obtain $E=I-P_v$ via tensor Ψ of Y_v by SVD w.r.t. $\mathcal{H}_{\rm phys} \otimes \mathcal{H}_{\rm virt}$

$$\Psi = W s V^\dagger \Rightarrow E = W W^\dagger \equiv U_E^\dagger U_E$$

- Similarly for $F=I-P_w$, A and B
- Define $E' \equiv U_E'^\dagger U_E', \quad F' \equiv U_F'^\dagger U_F'$

  where $U_E' \equiv U_E U_A^\dagger \quad U_F' \equiv U_F U_B^\dagger$

- Calculate smallest eigenvalue $1-\varepsilon$ of $E'+F'$
- If $\varepsilon < 1/z$, then the model is gapped


Numerical procedure
V'
Y_A
Y_B
W
V
Y_V
Y_W
Obtain $E=I-P_v$ via tensor Ψ of Y_v by SVD w.r.t. $\mathcal{H}_{\rm phys} \otimes \mathcal{H}_{\rm virt}$
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Similarly for $F=I-P_w$, A and B
Define $E' \equiv U_E'^\dagger U_E'$, $F' \equiv U_F'^\dagger U_F'$
where $U_E' = U_E U_A^\dagger$ $U_F' = U_F U_B^\dagger$
Calculate smallest eigenvalue 1-ε of $E'+F'$
If ε <1/z, then the model is gapped
Reduction: for a pair of vertices of degrees z & z': E+F acts on space of dimension (z+1)(z'+1)$3^{(z+z'-1)n}$, but E'+F' acts on reduced dimension $2^{(z+z')}3^n$ and further to $2^{(z+z'+2)}$
e.g. z=z'=3, n=5 --> reduction from 43.6 to 13.9 qubits and further to 8

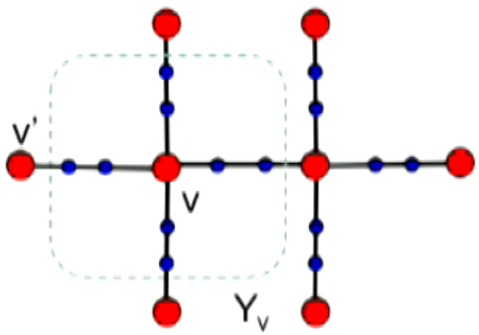
Improved lower bound on gap

- Consider re-arrangement of H:

$$H^{\mathrm{AKLT}}_{\Lambda^{(n)}} = \sum_{v\in\Lambda} h'_{Y;v}$$

$$h'_{Y;v} = \sum_{e\in\mathcal{E}_{Y_v}\setminus\mathcal{E}_v} \frac{1}{2} P_e^{(z(e)/2)} + \sum_{e\in\mathcal{E}_v} P_e^{(z(e)/2)}$$

$$\Rightarrow \Delta_Y \tilde{H}_{\Lambda^{(n)}} \leq H^{\mathrm{AKLT}}_{\Lambda^{(n)}} \leq \|h'_{Y;v}\| \tilde{H}_{\Lambda^{(n)}}$$



$\mathcal{E}_v$: the set of edges incident on v

$\Delta_Y(n)$: smallest nonzero eigenvalue of $h'_{Y;v}$

$$\Rightarrow \mathrm{gap}(H^{\mathrm{AKLT}}_{\Lambda^{(n)}}) \geq \gamma(n) \equiv \Delta_Y(n)(1 - z\varepsilon_n),$$

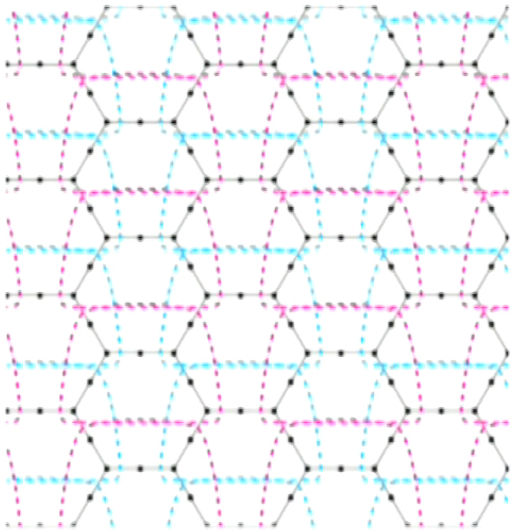
n	$\Delta_Y(n)$ for deg. 3	gap lower bound $\gamma(n)$	$\Delta_Y(n)$ for deg. 4	gap lower bound $\gamma(n)$
1	0.283484861		0.170646233	
2	0.239907874	0.154737328	0.197934811	0.101463966
3	0.207152231	0.183265099		

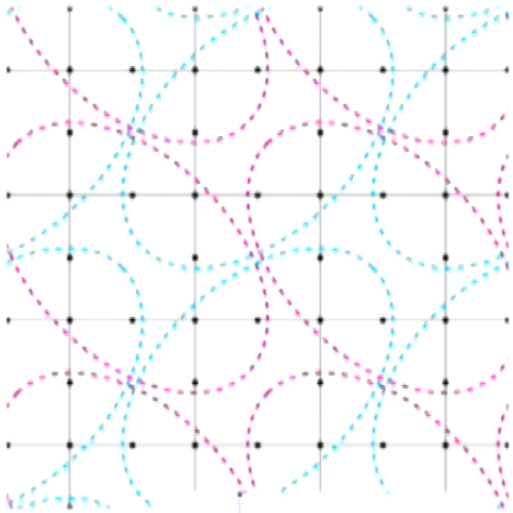
- Observation: naive extrapolation of lower bound from n=3 & n=2 linearly [1] to n=1: $\gamma(1)\approx 0.1262096$, [2] to n=0: $\gamma(0)\approx 0.097682$ cf. iPEPS: $\Delta=0.10$

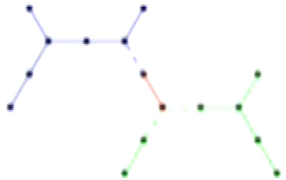
Main results II

- AKLT models on n=1 (singly) decorated honeycomb and square lattices are gapped ➔ by choosing bigger regions

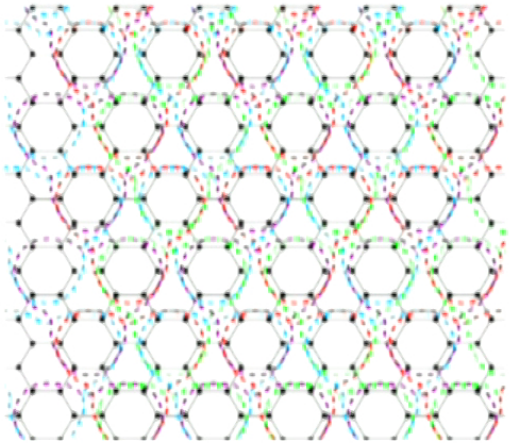
[Pomata & **Wei**, arXiv:1911.01410]







AKLT models on trivalent lattices are gapped!



[Pomata & **Wei**, arXiv:1911.01410]

	Method	ε	$\tilde{\xi}$	D_{tot}	γ	Δ
Square-	I	0.1062	2	2^{12}		
octagon	I	0.1590	4		0.05453	0.008281
Star	I	0.1111	4	2^{9}		
Cross	I	0.1998	3	2^{16}	0.08232	0.03299
Honeycomb	IIb	0.1446	6	2^{26}	0.04107	
					0.7784	0.004250
Hon., $n = 1$	IIa	0.1531	4	$2^{7}3^{4}$	0.2051	0.02743
Square, $n = 1$	IIa	0.2204	4	$2^{13}3^{4}$	0.1223	0.01450

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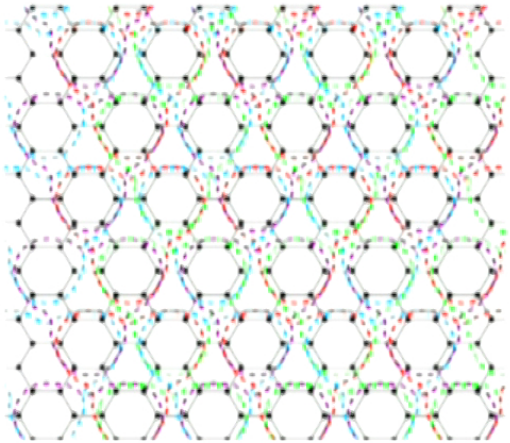

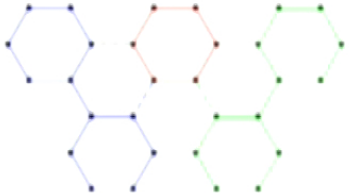

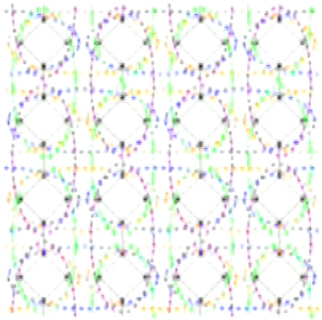

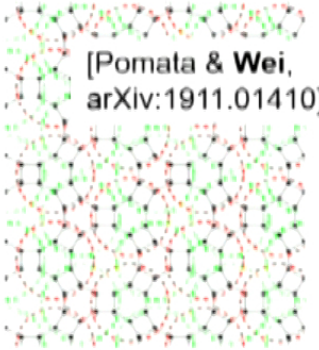

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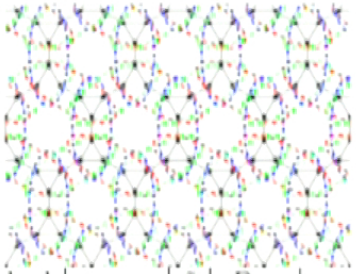

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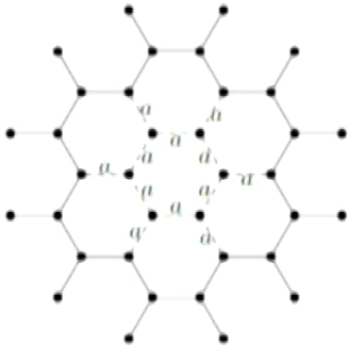
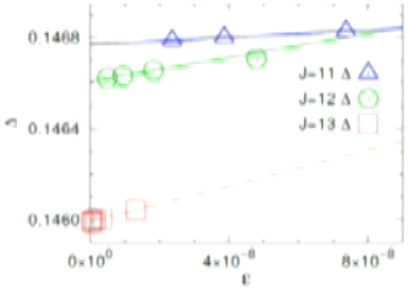
Approach in Lemm, Sandvik & Wang

[arxiv:1910.11810]

- Reduce gap criterion to the gap of the 36-site problem: Needs finite-size gap $\gamma_F(a = 1.4) > 0.1385$



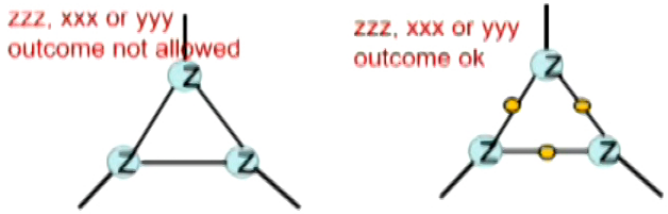


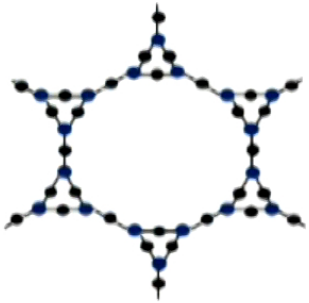
- Use DMRG to numerically verify this:

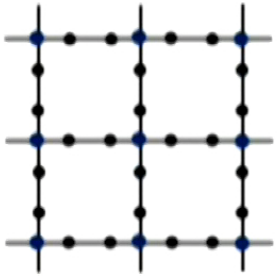
$$\Delta(J = 13) = 0.14599$$

Discussions

- Decoration of spin-1 sites make the AKLT state more likely to be universal
  - Short 1D AKLT wire between neighboring undecorated sites
- Decoration removes the frustration feature of measurement:

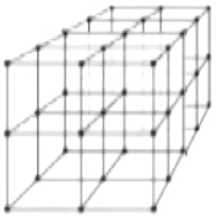






- Decoration weakens/removes Néel order: e.g. on 3D cubic lattice

[Parameswaran, Sondhi & Arovas '09]: AKLT state on cubic lattice is Néel ordered



- AKLT model gapless, but
  → adding decoration make the decorated model gapped (at least for n=2 sites per edge)
  → weakens tendency toward long-range order

Discussions: “deformation”

- Can consider deformed AKLT states and investigate phase diagrams

[Niggemann, Klümper& Zittartz ‘97,’00, Hieida,Okunishi& Akutsu ‘99, Darmawan, Brennen, Bartlett ‘12, Huang, Wagner, Wei’16, Huang,Pomata,Wei ‘18]

- Example on square lattice:

$$H(\vec{a}) \equiv \sum_{\langle i,j \rangle} D(\vec{a})_i^{-1} \otimes D(\vec{a})_j^{-1}\, h_{ij}^{(\mathrm{AKLT})}\, D(\vec{a})_i^{-1} \otimes D(\vec{a})_j^{-1}$$

- deformation:

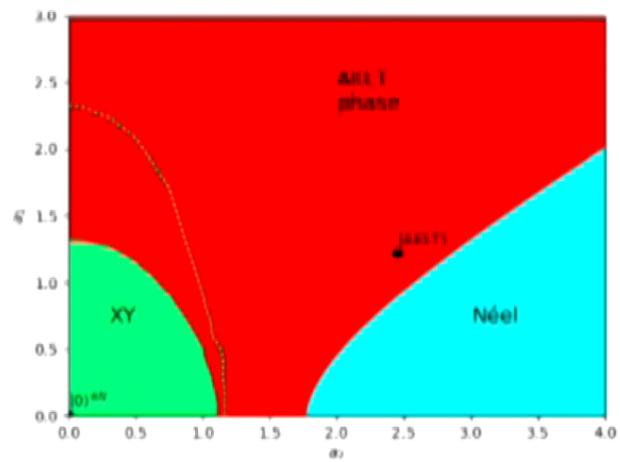
$$\begin{aligned} D(a_1, a_2) \;=\; & \frac{a_2}{\sqrt{6}}(|S_z = 2\rangle\langle S_z = 2| + |S_z = -2\rangle\langle S_z = -2|) \\ & + \frac{2a_1}{\sqrt{6}}(|S_z = 1\rangle\langle S_z = 1| + |S_z = -1\rangle\langle S_z = -1|) \\ & + |S_z = 0\rangle\langle S_z = 0| \end{aligned}$$

- ground state:

$$|\Psi(\vec{a})_{\mathrm{deformed}}\rangle \propto D(\vec{a})^{\otimes N} |\psi_{\mathrm{AKLT}}\rangle$$

$$|\Psi_{\mathrm{AKLT}}\rangle = |\Psi(a_1 = \sqrt{6}/2, a_2 = \sqrt{6})\rangle$$

[Huang,Pomata,Wei ‘18]



Summary and open questions

- Discussed AKLT family of states for universal measurement-based QC
- Discussed how to establish nonzero gap for AKLT models on decorated lattices & Archimedean trivalent lattices

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- Discussed AKLT family of states for universal measurement-based QC
- Discussed how to establish nonzero gap for AKLT models on decorated lattices & Archimedean trivalent lattices

- Universal MBQC using AKLT states with higher spins S>2?
- What is essential symmetry that stabilizes the AKLT phase? Can the entire phase be universal resource?
- Proving nonzero gap for AKLT models on the square lattice?