Title: Non-invertible anomalies and Topological orders

Speakers: Wenjie Ji

Series: Condensed Matter

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Abstract:

It has been realized that anomalies can be classified by topological phases in one higher dimension. Previous studies focus on â€™t Hooft anomalies of a theory with a global symmetry that correspond to invertible topological orders and/or symmetry protected topological orders in one higher dimension. In this talk, I will introduce an anomaly that appears on the boundaries of (non-invertible) topological order with anyonic excitations [1]. The anomalous boundary theory is no longer invariant under a re-parametrization of the same spacetime manifold. The anomaly is matched by simple universal topological data in the bulk, essentially the statistics of anyons. The study of non-invertible anomalies opens a systematic way to determine all gapped and gapless boundaries of topological orders, by solving simple eigenvector problems. As an example, we find all conformal field theories (CFT) of so-called ``minimal modelsâ€™â€™, except four cases, can be the critical boundary theories of Z_2 topological order (toric code). The matching of non-invertible anomaly have wide applications. For example, we show that the gapless boundary of double-semion topological order must have central charge c_L=c_R >= 25/28. And the gapless boundary of the non-Abelian topological order described by S_3 topological quantum field theory can be three-state Potts CFT, su(2)_4 CFT, etc. [1] WJ, Xiao-Gang Wen, arXiv: 1905.13279, Phys. Rev. Research 1,033054

Non-invertible Anomalies and Topological Order

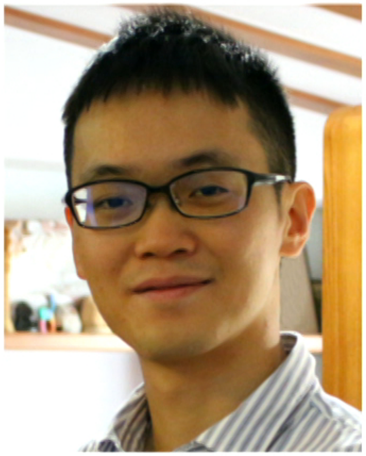
Wenjie Ji

Massachusetts Institute of Technology

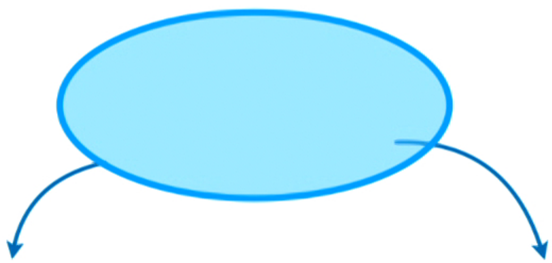
Perimeter Institute, Waterloo 2019

[**WJ**, Xiao-Gang Wen, Phys. Rev. Research 1, 033054]

Acknowledgment



Xiao-Gang Wen (MIT)   Shu-Heng Shao (IAS)



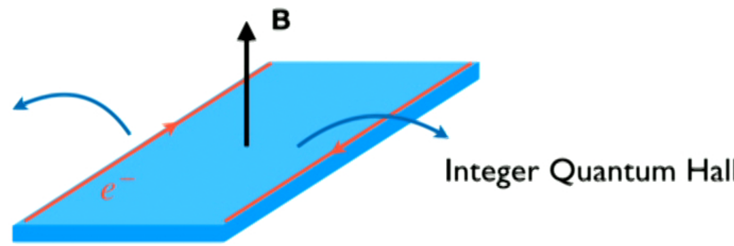
d-dimensional $\leftrightarrow$ $d+1$-dimensional

Boundary states ? topological phases

Any additional properties = Anomaly

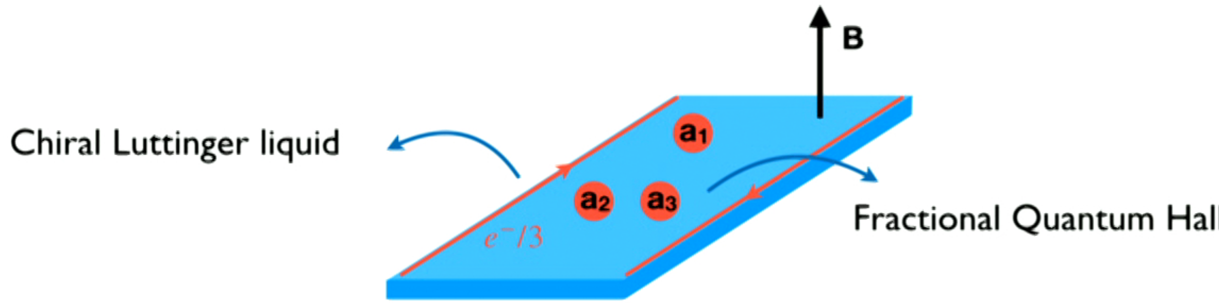
electron moving in a
single direction

$$H_{\text{bdy}} = i \int dx \psi^\dagger \partial_x \psi$$

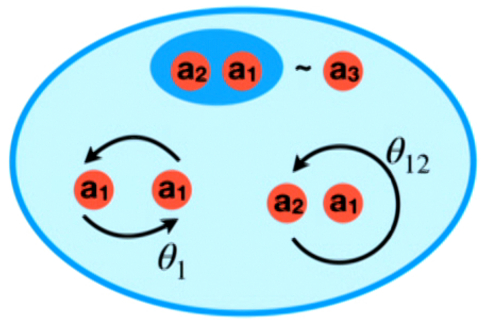


Bulk Topological orders without anyons

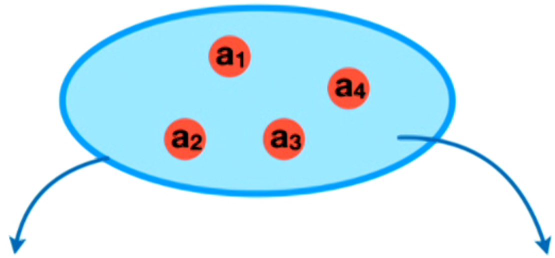
Boundary e.x. Integer quantized electric Hall conductance



Bulk Topological orders with anyons



Boundary e.x. Fractional quantized electric Hall conductance
thermal Hall conductance

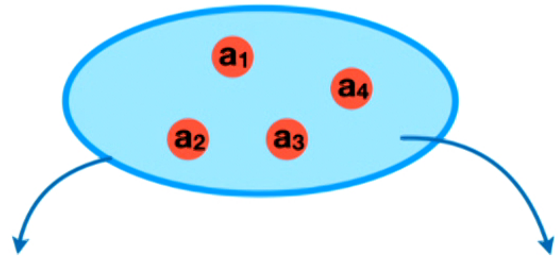


Boundary ↔ **Bulk**

No thermal Hall conductance — non-chiral topological order

anything special ? — with anyons

Application



Boundary	$\leftrightarrow$	**Bulk**
Gapped_1, Gapped_2, $\cdots$ CFT_1, CFT_2, $\cdots$		Anyon models
two gapped boundaries all unitary CFTs with central charge $c_L = c_R < 1$, except four cases		Toric code

Application

Boundary of anyon model $\leftrightarrow$ **Purely 1D**

Stability? Different most relevant perturbation

Ising CFT	Transverse Ising model
on boundary of toric code	at critical point
Majorana mass term	spin operator
scal. dim. $= 1$	scal. dim. $= \frac{1}{8}$

Purely 1D system

Low energy description?

Gapped Count states in the ground states

Gapless/critical Conformal field theory (CFT)

spin-$\frac{1}{2}$ Heisenberg model $\rightarrow SU(2)$ CFT $c = 1$

Transverse Ising model $\rightarrow$ Ising CFT $\mathcal{M}(3,4)$ $c = \frac{1}{2}$

predict specific heat $c_T = \frac{1}{2}$ in certain unit

Tricritical Ising model $\rightarrow$ Tricritical Ising CFT $\mathcal{M}(4,5)$ $c = \frac{7}{10}$

CFT predicts specially discrete values of specific heat for 1d critical models.

1D gapless/ critical system

Universal study of 1d critical system

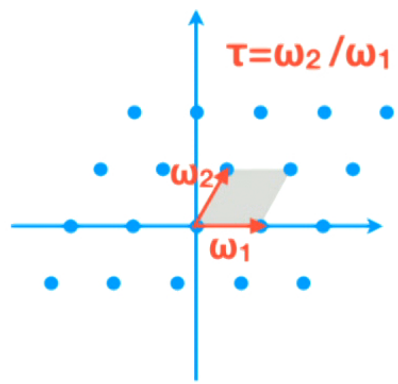
Minimal models $\mathcal{M}(p, p+1)$

p	c	lattice model
3	$\frac{1}{2}$	Ising
4	$\frac{7}{10}$	Tricritical Ising
5	$\frac{4}{5}$	Tetracritical Ising, 3-state Potts
$\vdots$		

How are they determined?

1+1 d gapless/ critical system

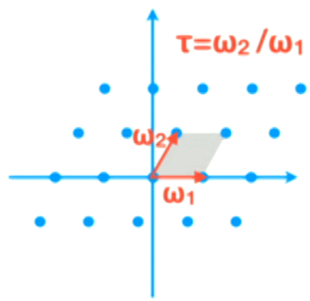
Power Complete spectrum solved, given by partition function on a torus.

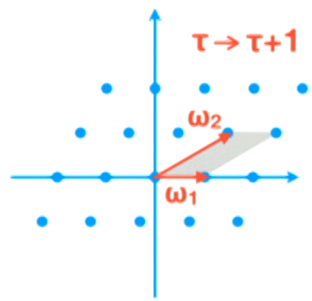


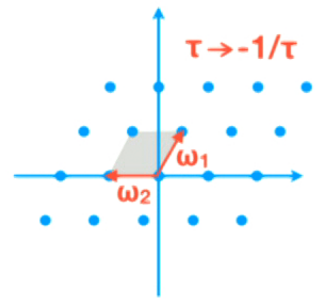
$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\ \mathrm{e}^{-(Im\tau H - \mathrm{i}\ \mathrm{Re}\,\tau P)} = \sum_{|\phi_i\rangle} \langle\phi_i|\mathrm{e}^{-(Im\tau\epsilon_i - \mathrm{i}\ \mathrm{Re}\,\tau p_i)}|\phi_i\rangle$$

1+1d CFT on a torus τ

Re-parametrize the same torus, pick a different spacetime unit cell







$Z(\tau, \bar{\tau})$

$\mathcal{T} : Z(\tau, \bar{\tau}) \to Z(\tau + 1, \bar{\tau} + 1)$

$\mathcal{S} : Z(\tau, \bar{\tau}) \to Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right)$

$$Z(\tau + 1, \bar{\tau} + 1) = Z(\tau, \bar{\tau}) \quad Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau})$$

$\Rightarrow$ Modular invariant

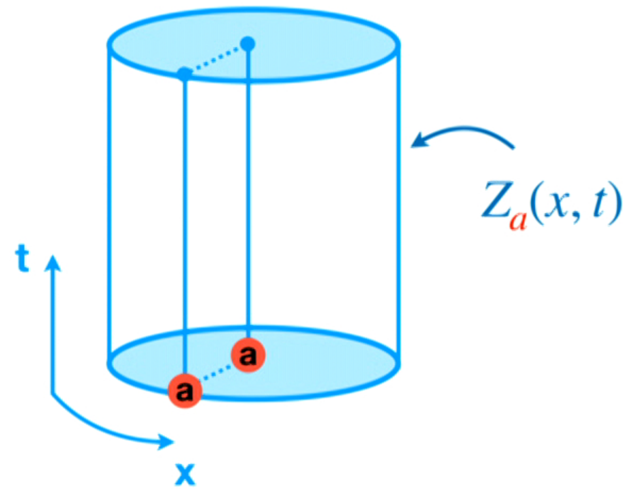
What modular invariance can do?

Minimal models $\mathcal{M}(p, p+1)$

p	c	lattice model
3	$\frac{1}{2}$	Ising
4	$\frac{7}{10}$	Tricritical Ising
5	$\frac{4}{5}$	Tetracritical Ising, 3-state Potts
$\vdots$		

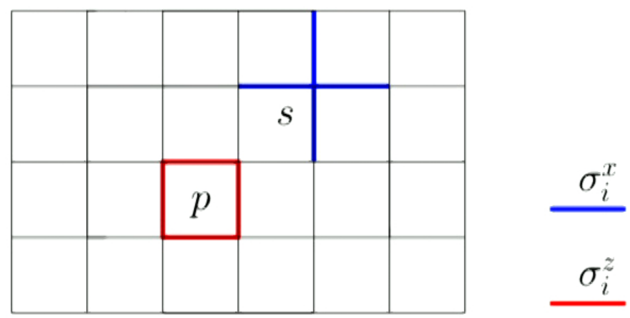
Boundary of anyon models

Proper description : Vector of partition functions



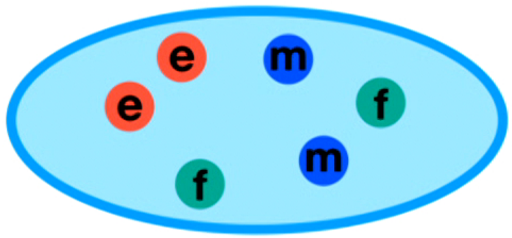
Bulk	**Boundary**
Different anyon	Different excitations

$\mathbb{Z}_2$ topological order on lattice – Toric code



$$H = -\sum_p g_p \, \boxed{p} \, - \sum_s g_s \, \underset{s}{+}$$

$\mathbb{Z}_2$ topological order / Toric code



Bulk simple topological data

anyon $i = \mathbf{1} \quad e \quad m \quad f$

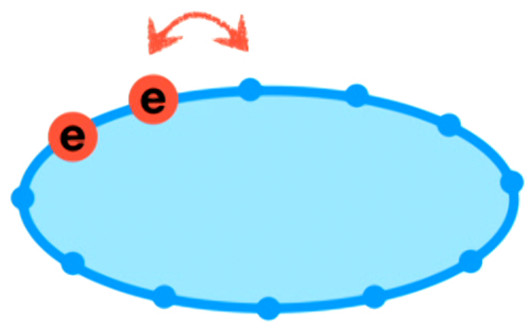
self & mutual statistics $T^{\text{top}}_{ij} = \delta_{ij}$  $\quad S^{\text{top}}_{ij} =$ 

$$T^{\mathbb{Z}_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad S^{\mathbb{Z}_2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Effective boundary Hamiltonian of toric code

Toric code boundary Hamiltonian = Transverse Ising model

Bulk: vacuum



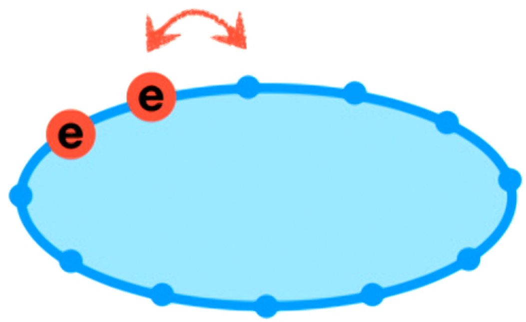
Boundary: Start with m-condensed boundary $e \sim f$

Effective Hamiltonian of e energy gap U + hop around

$$H = -\frac{U}{2}\sum_j \sigma_j^z - J\sum_j \sigma_j^x \sigma_{j+1}^x - \epsilon_0 L \qquad \sigma_i^z = \begin{cases} 1 & \text{empty} \\ -1 & \text{occupied by e} \end{cases}$$

Effective boundary Hamiltonian of toric code

Bulk: vacuum



Boundary: Start with m-condensed boundary

Effective Hamiltonian of e energy gap U + hop around

$$H = -\frac{U}{2}\sum_j \sigma_j^z - J\sum_j \sigma_j^x \sigma_{j+1}^x - \epsilon_0 L \qquad \sigma_i^z = \begin{cases} 1 & \text{empty} \\ -1 & \text{occupied by e} \end{cases}$$

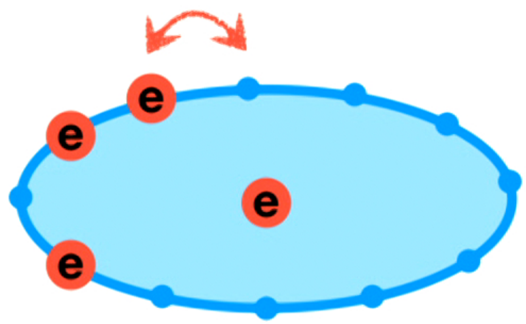
Global constraint

Total number of e is even $\prod_j \sigma_j^z = 1$

Boundary condition $\sigma_{N+1}^x = \sigma_1^x$

Effective boundary Hamiltonian of toric code

Bulk: e-sector



Boundary:

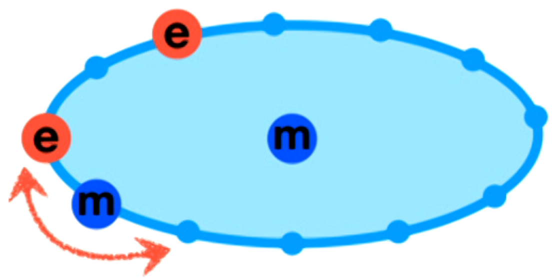
Global constraint

Total number of e is odd $\qquad \prod_j \sigma_j^z = -1$

Boundary condition $\qquad \sigma_{N+1}^x = \sigma_1^x$

Effective boundary Hamiltonian of toric code

Bulk: m-sector



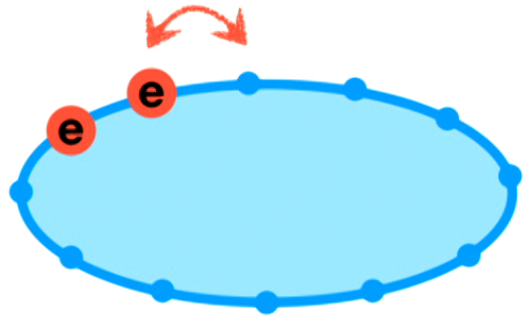
Boundary:

Global constraint

Number of e is even $\prod_j \sigma_j^z = 1$

Boundary condition $\sigma_{N+1}^x = -\sigma_1^x$

Boundary of $\mathbb{Z}_2$ topological order



$$H_{\text{bdy}} = -\frac{U}{2}\sum_j \sigma_j^z - J\sum_j \sigma_i^x \sigma_{i+1}^x$$

Bulk	**Boundary** constraint
1	Periodic b.c. $\mathbb{Z}_2$ even
e	Periodic b.c. $\mathbb{Z}_2$ odd
m	Anti-Periodic b.c. $\mathbb{Z}_2$ even
f	Anti-Periodic b.c. $\mathbb{Z}_2$ odd

$e = (1, -1) \quad m = (-1, 1)$

Bulk Anyon = ($\mathbf{Z}_2$ flux , $\mathbf{Z}_2$ charge)

Boundary states = (Bdy condition, charge)

Boundary: vector of partition function

Low energy partition function

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a}\, \mathrm{e}^{-\beta H_a}$$

Low temperature limit $\beta \to \infty$ with fixed $\frac{\beta}{L}$

$|\phi\rangle$ is gapped $\Longrightarrow \mathrm{e}^{-\beta E_{|\phi\rangle}} \to 0$

$|\phi\rangle$ is gapless $\Longrightarrow \mathrm{e}^{-\frac{\beta}{L}\epsilon_{|\phi\rangle}}$

Boundary partition function of $\mathbb{Z}_2$ topological order

$$H_{\text{bdy}} = -\frac{U}{2}\sum_j \sigma_j^z - J\sum_j \sigma_i^x\sigma_{i+1}^x - \epsilon_0 L$$

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a}\, e^{-\beta H_a}$$

Gapped m-condensed boundary $|J| < \frac{U}{2}$

Bulk a	Boundary Z_a	states
1	1	single ground state $\epsilon_0 = 0$
e	0	gapped excitation
m	1	condensed $\epsilon_m = \epsilon_0$
f	0	gapped excitation

$$Z^{\text{m-condensed}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

"Smooth boundary"

[Kitaev-Kong '12]

Boundary partition function of $\mathbb{Z}_2$ topological order

$$H_{\text{bdy}} = -\frac{U}{2}\sum_j \sigma_j^z - J\sum_j \sigma_i^x\sigma_{i+1}^x - \epsilon_0 L$$

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a}\, e^{-\beta H_a}$$

Gapped boundaries

$|J| < \frac{U}{2}$

$$Z^{\text{m-condensed}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

"Smooth edge"

$J > \frac{U}{2}$

$$Z^{\text{e-condensed}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

"Rough edge"

[Kitaev-Kong '12]

Boundary partition function of $\mathbb{Z}_2$ topological order

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a}\, \mathrm{e}^{-\beta H_a}$$

Gapless boundaries $J = \frac{U}{2}$

Rough answer: transverse Ising model at critical point = Ising CFT

What is Z_a?

Boundary partition function of $\mathbb{Z}_2$ topological order

Gapless boundaries

$$H_{\text{bdy}} = -\sum_j (\sigma_j^z + \sigma_i^x \sigma_{i+1}^x) - \epsilon_0 L$$

$$Z_{\text{anyon } a} = \text{Tr}_{\mathcal{H}_a}\, \mathrm{e}^{-\beta H_a}$$

Gapless excitations vacuum σ $\psi\bar{\psi}$ μ ψ $\bar{\psi}$

Bulk	Boundary constraint	states
1	P.b.c. $\mathbb{Z}_2$ even	vacuum "0", $\psi\bar{\psi}$
e	P.b.c. $\mathbb{Z}_2$ odd	σ
m	AP.b.c. $\mathbb{Z}_2$ even	μ
f	AP.b.c. $\mathbb{Z}_2$ odd	ψ, $\bar{\psi}$

[Levin-Wen '03, unpublised, Chen-Jian-Kong-You-Zheng '19]

Vector of partition functions

Bulk Toric code

Gapless boundaries Ising CFT

$$\begin{bmatrix} Z_{\mathbf{1}} \\ Z_e \\ Z_m \\ Z_f \end{bmatrix} = \begin{bmatrix} |\chi_{\mathbf{1}}|^2 + |\chi_\psi|^2 \\ |\chi_\sigma|^2 \\ |\chi_\mu|^2 \\ \chi_\psi \bar{\chi}_{\mathbf{1}} + \chi_{\mathbf{1}} \bar{\chi}_{\bar{\psi}} \end{bmatrix}$$

Gapped boundaries

$$Z^{e-cond} = [1\ 1\ 0\ 0]^T \qquad Z^{m-cond} = [1\ 0\ 1\ 0]^T$$

A vector of partition functions describe various boundaries of anyon model.

vector index = bulk anyon

Construct case by case?

Boundary	$\leftrightarrow$	**Bulk**
More CFTs if add 4-spin interactions?		Toric code
$U(1)$ CFT?		$K = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

Look for

(i) A schematic way, given a TO/CFT pair, check if there is a solution of vector of partition function.

(ii) Independent of particular microscopic construction

(iii) Something universal about boundary/anyonic bulk correspondence?

Hint from Ising CFT $\leftrightarrow$ Toric code

Under modular transformation?

$Z_a(\tau,\bar\tau) = \sum_{ij} \chi_i(\tau)\ M^a_{ij}\ \bar\chi_j(\bar\tau)$

$$\chi_i(\tau+1) = T^{\mathrm{CFT}}_{ij}\chi_j(\tau) \qquad T^{\mathrm{Is}} = e^{-i\frac{2\pi}{24}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i2\pi\frac{1}{2}} & 0 \\ 0 & 0 & e^{i2\pi\frac{1}{16}} \end{bmatrix}$$

$$Z_a(\tau+1,\bar\tau+1) = \sum_{ij} \bar\chi_i(\bar\tau)\ \widetilde{M}^a_{ij}\ \chi_j(\tau) \qquad \widetilde{M}^a_{ij} = T^{\mathrm{CFT}\dagger} M^a_{ij} T^{\mathrm{CFT}}$$

Hint from Ising CFT $\leftrightarrow$ Toric code

Under modular transformation?

$$Z^{\text{Ising}}{}_a(\tau+1,\bar{\tau}+1) = T^{\text{toric code}}{}_{ab} Z_b(\tau,\bar{\tau})$$

Hint from Ising CFT $\leftrightarrow$ Toric code

Under modular transformation?

$$Z^{\text{Ising}}{}_a(\tau+1,\bar{\tau}+1) = T^{\text{toric code}}{}_{ab} Z_b(\tau,\bar{\tau})$$

$$Z^{\text{Ising}}{}_a(-1/\tau,-1/\bar{\tau}) = S^{\text{toric code}}{}_{ab} Z_b(\tau,\bar{\tau})$$

Boundary	$\leftrightarrow$	**Bulk**
$Z_a(\tau)$ under modular transformation		universal anyon data

How general? Gapped boundaries

Gapped boundaries

$Z^{e-cond} = [1\ 1\ 0\ 0]^T \quad Z^{m-cond} = [1\ 0\ 1\ 0]^T$

$$T^{\text{toric code}}{}_{ab} Z_b = Z_a$$

$$S^{\text{toric code}}{}_{ab} Z_b = Z_a$$

Only two independent eigenvectors

How general? A Luttinger liquid boundary

Bulk Toric code/$\mathbb{Z}_2$ topological order

described by $U(1)$ Chern-Simons theory $K = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

Boundary <u>$U(1)$ CFT at level 4</u> $c = 1$ primaries $l = 0, 1, 2, 3$

$$\begin{bmatrix} Z_{\mathbf{1}} \\ Z_e \\ Z_m \\ Z_f \end{bmatrix} = \begin{bmatrix} |\chi_0|^2 + |\chi_2|^2 \\ |\chi_1|^2 + |\chi_3|^2 \\ \chi_1\bar{\chi}_3 + \chi_3\bar{\chi}_1 \\ \chi_0\bar{\chi}_2 + \chi_2\bar{\chi}_0 \end{bmatrix}$$

$$Z^{U(1)}{}_a(\tau + 1, \bar{\tau} + 1) = T^{\text{toric code}}{}_{ab} Z_b(\tau, \bar{\tau})$$
$$Z^{U(1)}{}_a(-1/\tau, -1/\bar{\tau}) = S^{\text{toric code}}{}_{ab} Z_b(\tau, \bar{\tau})$$

A 1 + 1d theory with *non-invertible anomaly*

1. defined on a space-time torus τ , it has a multi-component partition function

$$Z_a(\tau, \bar{\tau})$$

2. under torus re-parametrization $\mathcal{T}: \ \tau \to \tau + 1, \quad \mathcal{S}: \ \tau \to -\frac{1}{\tau}$

   $Z_a(\tau, \bar{\tau})$ transform *covariantly* according to 2+1d topological data

   $\{\text{anyon } a\}$ anyon self-statistics T^{top} mutual statistics S^{top}

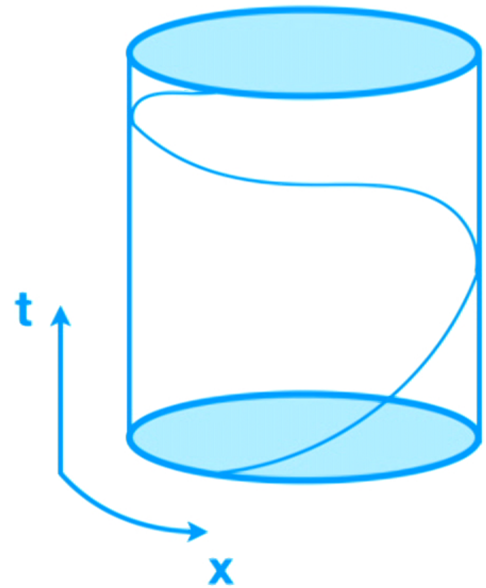
$$T^{\text{top}}_{ab} Z_b(\tau, \bar{\tau}) = Z_b(\tau + 1, \bar{\tau} + 1)$$

$$S^{\text{top}}_{ab} Z_b(\tau, \bar{\tau}) = Z_b\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right)$$

Non-invertible anomaly = canceled by the 2 + 1d (non-invertible) topological order

An intuition about T transformation

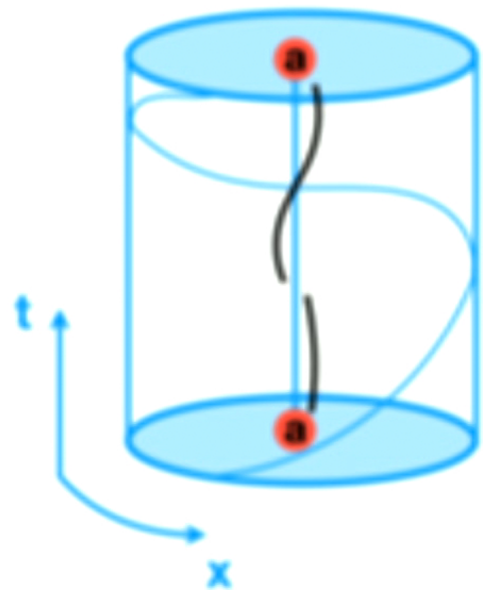
Rotate 2π and glue back



Purely 1D ring nothing change

An intuition about T transformation

Rotate 2π and glue back



A boundary ring anyon self-rotate and accumulate a phase

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Construct case by case?

Boundary	$\leftrightarrow$	Bulk
More CFTs if add 4-spin interactions?		Toric code
$U(1)$ CFT?		$K = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

Look for

(i) A schematic way, given a TO/CFT pair, check if there is a solution of vector of partition function.

(ii) Independent of particular microscopic construction

(iii) Something universal about boundary/anyonic bulk correspondence?

CFT/TO pair

Boundary $\leftrightarrow$	**Bulk**
Pick a CFT	Fix an anyon model

Solve an eigenvector problem for M_{ai}, a labels anyon, i labels primaries.

$$T^{\text{top}}_{ab}\,(T^{\text{CFT}})^*_{ij}\,M_{bj} = M_{ai}$$
$$S^{\text{top}}_{ab}\,(S^{\text{CFT}})^*_{ij}\,M_{bj} = M_{ai}$$

CFT/TO pair

Boundary $\leftrightarrow$	**Bulk**
Pick a CFT	Fix an anyon model

Solve an eigenvector problem for M_{ai}, a labels anyon, i labels primaries.

$$T^{\text{top}}_{ab}\,(T^{\text{CFT}})^*_{ij}\,M_{bj} = M_{ai}$$
$$S^{\text{top}}_{ab}\,(S^{\text{CFT}})^*_{ij}\,M_{bj} = M_{ai}$$

More critical boundaries of toric code

Tricritical Ising $\mathcal{M}(4,5)$ $\quad c_L = c_R = \frac{7}{10}$

I	$\mathbf{1}$	σ	σ'	ϵ	ϵ'	ϵ''
h_I	0	$\frac{3}{80}$	$\frac{7}{16}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{2}$

$$\begin{bmatrix} Z_{\mathbf{1}} \\ Z_e \\ Z_m \\ Z_f \end{bmatrix} = \begin{bmatrix} |\chi_0|^2 + |\chi_{\frac{1}{10}}|^2 + |\chi_{\frac{3}{5}}|^2 + |\chi_{\frac{3}{2}}|^2 \\ |\chi_{\frac{7}{16}}|^2 + |\chi_{\frac{3}{80}}|^2 \\ |\chi_{\frac{7}{16}}|^2 + |\chi_{\frac{3}{80}}|^2 \\ \chi_0\bar{\chi}_{\frac{3}{2}} + \chi_{\frac{1}{10}}\bar{\chi}_{\frac{3}{5}} + \chi_{\frac{3}{5}}\bar{\chi}_{\frac{1}{10}} + \chi_{\frac{3}{2}}\bar{\chi}_0 \end{bmatrix}$$

General critical boundaries of toric code

All minimal models, except 4 cases ($\mathcal{M}(p, p+1)$ with $p = 17, 18, 29, 30$) can be the gapless boundary of $\mathbb{Z}_2$ topological order.

Prediction

Stability: most relavant term in the vacuum sector $Z_{\mathbf{1}}$.

$$\begin{bmatrix} Z_{\mathbf{1}} \\ Z_e \\ Z_m \\ Z_f \end{bmatrix} = \begin{bmatrix} |\chi_{\mathbf{1}}|^2 + |\chi_\psi|^2 \\ |\chi_\sigma|^2 \\ |\chi_\mu|^2 \\ \chi_\psi \bar{\chi}_{\mathbf{1}} + \chi_{\mathbf{1}} \bar{\chi}_{\bar{\psi}} \end{bmatrix}$$

Compare with 1d Ising chain $\quad Z = |\chi_{\mathbf{1}}|^2 + |\chi_\sigma|^2 + |\chi_\psi|^2$

In general more stable than CFT in purely 1D, since some excitations are projected out.

Example: Double semion topological order

Bulk

i	$\mathbf{1}$	s	s^*	b
θ_i	1	$e^{i2\pi\frac{1}{4}}$	$e^{-i2\pi\frac{1}{4}}$	1

$$T^{\mathbb{Z}_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S^{\mathbb{Z}_2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Boundary

Gapped $Z^T = [1\ 0\ 0\ 1]$ boson condensed

Bulk

Effective theory $U(1)$ Chern-Simons theory with a $K = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ matrix

Boundary $\underline{u(1)_2}$ $c_L = c_R = 1$

I	0	1
h_I	0	$\frac{1}{4}$

$$\begin{bmatrix} Z_{\mathbf{1}} \\ Z_s \\ Z_{s^*} \\ Z_b \end{bmatrix} = \begin{bmatrix} |\chi_0|^2 \\ \chi_1\bar{\chi}_0 \\ \chi_0\bar{\chi}_1 \\ |\chi_1|^2 \end{bmatrix}$$

Prediction I No relevant perturbation, from $Z_1 = |\chi_0|^2$.
exists marginal perturbation in $|\chi_0|^2$

Boundaries of double semion topological order

For a RCFT to be the boundary theory of a topological order, it must have primary fields J with

$$\mathrm{e}^{\mathrm{i}\,2\pi(h_L^J - h_R^J)} = \theta_j$$

For double semion bulk, no minimal model solution for $p < 7$.

Gapless boundaries of 2+1D double semion topological order must have central charge

$$c \geq \frac{25}{28}$$

A basis transformation

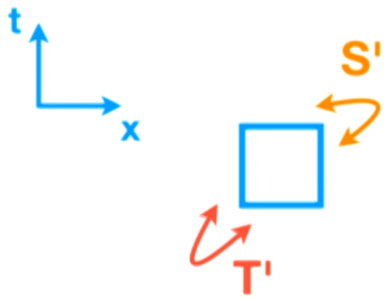
Example: $\mathbb{Z}_2$ topological order

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

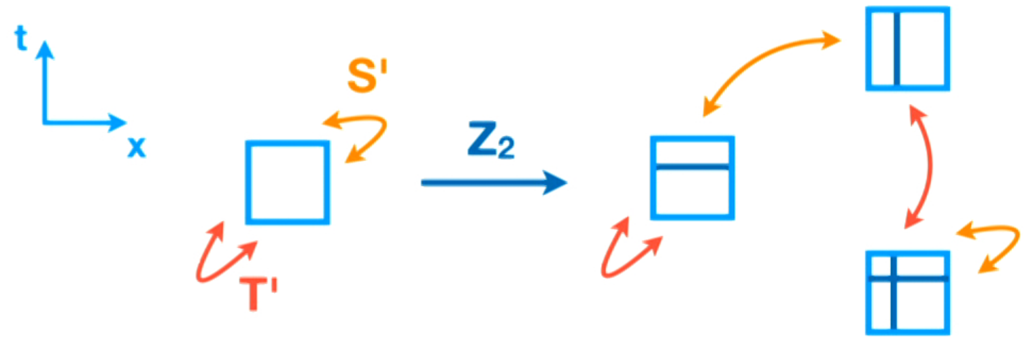
$$Z^{\mathrm{Is}} = \begin{bmatrix} |\chi_{\mathbf{1}}|^2 + |\chi_\psi|^2 \\ |\chi_\sigma|^2 \\ |\chi_\mu|^2 \\ \chi_\psi \bar{\chi}_{\mathbf{1}} + \chi_{\mathbf{1}} \bar{\chi}_{\bar{\psi}} \end{bmatrix} \xrightarrow{\quad M \quad} Z^{\mathrm{Is}\prime} = \begin{bmatrix} |\chi_{\mathbf{1}}|^2 + |\chi_\sigma|^2 + |\chi_\psi|^2 \\ |\chi_{\mathbf{1}}|^2 - |\chi_\sigma|^2 + |\chi_\psi|^2 \\ \chi_\psi \bar{\chi}_{\mathbf{1}} + \chi_{\mathbf{1}} \bar{\chi}_{\bar{\psi}} + |\chi_\mu|^2 \\ -\chi_\psi \bar{\chi}_{\mathbf{1}} - \chi_{\mathbf{1}} \bar{\chi}_{\bar{\psi}} + |\chi_\mu|^2 \end{bmatrix}$$

$$S' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

A CFT with an anomaly free $\mathbb{Z}_2$ symmetry

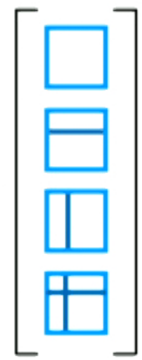


S', T' orbit



An inverse basis transformation

$$\begin{bmatrix} Z_{\mathbf{1}} \\ Z_e \\ Z_m \\ Z_f \end{bmatrix} = M^{-1}$$

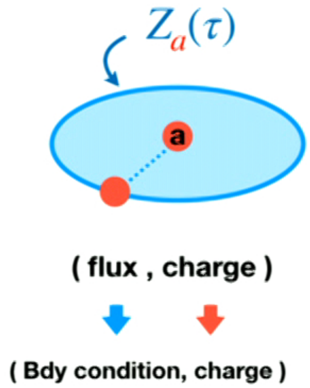


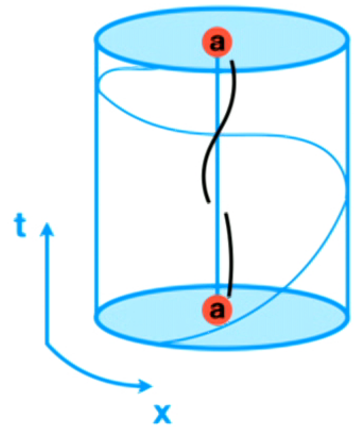
Take-away Given any modular invariant CFT with an anomaly free $\mathbb{Z}_2$ symmetry, there exists a modular covariant CFT as the boundary of $\mathbb{Z}_2$ topological order.

Minimal model $\mathcal{M}(p, p+1)$

- All minimal models, except 4 cases[1] can be the gapless boundary of $\mathbb{Z}_2$ topological order. The most stable one has central charge $c = \frac{1}{2}$.
- The critical 3-Potts model and tricritical 3-Potts model can be the gapless boundary of $\mathbb{Z}_3$ topological order and S_3 topological order.
- The gapless boundary of all other (untwisted) topological order with discrete gauge group G has central charge $c \geq 1$.

Non-invertible anomaly and Anyon models





$$Z_a(\tau+1) = T^{top}_{ab} Z_b$$

$$Z_a(-1/\tau) = S^{top}_{ab} Z_b$$

Non-invertible anomaly

Good for

schematic	TO/CFT pair through eigenvector equation
universal	Z_2 anomaly free CFT $\Rightarrow$ Toric code boundary
	Boundary of anyon models are more stable

Open remarks

- Examples are known yet that a faithful set of gapless boundary theories requires the mapping class group transformation on higher genus?
- Lattice construction for gapped and gapless boundaries ?
- Relation between invertible and non-invertible anomaly (to appear)