Title: A multivariable approach to renormalisation: Meromorphic germs with linear poles and the geometry of cones

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Series: Colloquium

Date: November 13, 2019 - 2:00 PM

URL: http://pirsa.org/19110114

Abstract: Analytic renormalisation "Ã la Speer" using a multivariable approach typically leads to meromorphic germs in several variables whose poles are linear. In particular, Feynman integrals, multizeta functions and their generalisations, namely discrete sums on cones and discrete sums associated with trees give rise to meromorphic germs at zero with linear poles. We shall present a multivariable renormalisation scheme which amounts to a minimal subtraction scheme in several variables. It preserves locality in so far as the evaluation at poles is expected to factor on functions with independent sets of variables. Inspired by Speer, we shall discuss a class of generalised evaluators that do the job. Using a theory of Laurent expansions on meromorphic germs with linear poles at zero, we shall relate these generalised evaluators to the geometry of cones.

This talk is based on joint work with Pierre Clavier, Li Guo and Bin Zhang.

Multivariable renormalisation:

Meromorphic germs with linear poles and the geometry of cones

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joint work with Pierre Clavier, Li Guo and Bin Zhang

Perimeter Institute, November 13th 2019

Divergent sums and integrals

Occurence of singularities/ divergences

- at $s = 1$ in the **Riemann ζ-function** $\zeta(s) := \sum_{k=1}^{\infty} k^{-s}$;
- at ∞ in the ill-defined **Feynman integral**
  "$\int_{\mathbb{R}^4} \frac{1}{|k|^2+m^2}\, dk = \mathrm{Vol}(S^3) \int_0^{\infty} \frac{r^3}{r^2+m^2}\, dr$".

Extracting divergences

- ζ-function: $\zeta(s) - \underbrace{\frac{1}{s-1}}_{\text{counterterm}} = \gamma + O(|s-1|) \underset{s \to 1}{\longrightarrow} \gamma =: \zeta^{\mathrm{reg}}(1)$

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Extracting divergences

- ζ-function: $\zeta(s) - \underbrace{\frac{1}{s-1}}_{\text{counterterm}} = \gamma + \mathcal{O}(|s-1|) \underset{s\to 1}{\longrightarrow} \gamma =: \zeta^{\mathrm{reg}}(1)$
- Feynman integrals: $\int_0^R \frac{r^3}{r^2+m^2}\, dr - \underbrace{\left(\frac{R^2}{2} + m^2 \log R\right)}_{\text{counterterm}}$

$$\underset{R\to\infty}{\longrightarrow} m^2 \log m =: ⨍_0^\infty \frac{r^3}{r^2+m^2}\, dr.$$

Higher order singularities/ divergences

Divergent products of sums and integrals

$$(\zeta(s))^2 - \text{counterterms} \xrightarrow[s\to 1]{?} \zeta^{\text{reg}}(1)^2 = \gamma^2;$$

$$\left(\int_0^R \frac{r^3}{r^2+m^2}\, dr\right)^2 - \text{counterterms} \xrightarrow[R\to\infty]{?} \left(\oint_0^\infty \frac{r^3}{r^2+m^2}\, dr\right)^2.$$

Divergent counterterms might combine with convergent terms to contribute to finite terms.

Sums and integrals associated with higher algebraic structures

- multiple integrals associated with Feynman diagrams.
- multizeta functions (nested sums) that generalise to
  - conical zeta functions associated with cones;
  - branched zeta functions associated with trees.

Dealing with higher divergences 1

A first naive approach

- $f_i(z) = a_i z^{-1} + h_i(z) \in \mathcal{M}$, the set of meromorphic germs in one variable with a simple pole at zero;
- Subtract the pole and evaluate the holomorphic part at the zero pole: $f_i^{\text{reg}}(0) = \lim_{z \to 0} \left(f_i(z) - \underbrace{a_i z^{-1}}_{\text{counterterms}} \right) := h_i(0)$.
- Loss of multiplicativity : $(f_1(z) f_2(z) - \text{counterterms}) \underset{z \to 0}{\longrightarrow}$ $(f_1 f_2)^{\text{reg}}(0) :=$ $f_1^{\text{reg}}(0) f_2^{\text{reg}}(0) + \underbrace{a_1 \cdot h_2'(0) + a_2 \cdot h_1'(0)}_{\text{extra terms}} \neq f_1^{\text{reg}}(0) f_2^{\text{reg}}(0)$.

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Example

$$\left(f_1(z) = z \wedge f_2(z) = \tfrac{1}{z}\right) \Longrightarrow f_1^{reg}(0)\, f_2^{reg}(0) = 0 \neq 1 = (f_1\, f_2)^{reg}(0).$$

Dealing with higher divergences 2

Alternative approach: a multivariable point of view

- multivariable meromorphic germ:
$$f_1(z_1)\, f_2(z_2) = \underbrace{\frac{a_1\, a_2}{z_1\, z_2} + a_1\, h_2'(0)\, \frac{z_2}{z_1} + a_2\, h_1'(0)\, \frac{z_1}{z_2}}_{\text{counterterms}} + h_1(z_1)\, h_2(z_2);$$
- independence/ locality/ orthogonality relation: $\frac{1}{z_1} \perp z_2$; $\frac{1}{z_2} \perp z_1$;
- $(f_1(z_1)\, f_2(z_2) - \text{counterterms}) \underset{z_i \to 0}{\longrightarrow} h_1(0)\, h_2(0) =: (f_1\, f_1)^{\text{reg}}(0)$.

Partial multiplicativity in a locality set up

Multiplicativity holds for independent functions:

$$f_1 \perp f_2 \Longrightarrow f_1^{\text{reg}}(0)\, f_2^{\text{reg}}(0) = (f_1\, f_2)^{\text{reg}}(0).$$

The need for meromorphic germs in several variables

Multiple variable (analytic) renormalisation

has various assets

- it avoids "fake" finite terms;
- and hence preserves locality;
- it amounts to a "minimal subtraction scheme" in several variables;
- it applies to any theory that gives rise to meromorphic germs with linear poles.

The price to pay

is the introduction of

- multivariable meromorphic germs;
- locality/independence in order to keep the variables separate.

II. Meromorphic germs

Meromorphic germs in several variables

Meromorphic germs with linear poles

- $\mathcal{M}(\mathbb{C}^k) \ni f = \frac{h(\ell_1,\cdots,\ell_n)}{L_1^{s_1}\cdots L_n^{s_n}}$, h holomorphic germ, $s_i \in \mathbb{Z}_{\geq 0}$,
- $\ell_i : \mathbb{C}^k \to \mathbb{C}$, $L_j : \mathbb{C}^k \to \mathbb{C}$ linear forms with real coefficients (lie in $\mathcal{L}(\mathbb{C}^k)$).
- **Example:** $(z_1, z_2) \longmapsto \frac{z_1 - z_2}{z_1 + z_2}$.

Independence of meromorphic germs

- **Dependence** set $\mathrm{Dep}(f) := \langle \ell_1, \cdots, \ell_m, L_1, \cdots, L_n \rangle$.
- An inner product Q on $\mathbb{R}^k$ induces one on $\mathcal{L}(\mathbb{C}^k)$ and we set

$$f_1 \perp^Q f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2).$$

- $(z_1 - z_2) \perp^Q (z_1 + z_2)$ with Q: canonical inner product on $\mathbb{R}^2$.

Polar germs and cones

Polar germs

A Q-**polar germ** in $\mathcal{M}(\mathbb{C}^k)$: $S := \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}$, such that

- h is holomorphic at zero i.e. $h \in \mathcal{M}_+(\mathbb{C}^k)$;
- $\ell_1, \cdots, \ell_m$, $L_1, \cdots, L_n$ are linearly independent and $\langle \ell_1, \cdots, \ell_m \rangle \perp^Q \langle L_1, \cdots, L_n \rangle$.

Polar germs generate the subspace $\mathcal{M}_-(\mathbb{C}^k) \subset \mathcal{M}(\mathbb{C}^k)$.

Supporting cones

- **supporting cone** in $\mathbb{R}^k$ of the germ S: $C(S) := \Sigma_{i=1}^m \mathbb{R}_+ L_i$;
- A family of cones is **properly positioned** if the cones meet along faces and their union does not contain any nontrivial subspace;
- A family $S_j, j \in J$ of polar germs whose supporting cones form a family of **properly positioned** cones is called **properly positioned**.

Decomposition of meromorphic germs

Theorem

(L. Guo, S.-P., B. Zhang 2017)Given a meromorphic germ $f \in \mathcal{M}(\mathbb{C}^k)$, there exists a finite set of polar germs

$$\mathcal{M}_-(\mathbb{C}^k) \ni \left\{ S_j = \frac{h_j}{L_{j1}^{s_{j1}} \cdots L_{jn_j}^{s_{jn_j}}} \right\}_{j \in J}$$

- *that are properly positioned,*
- *whose denominators are pairwise not proportional,*
- *and a holomorphic germ h,*

such that $$f = \left[\sum_{j\in J} S_j\right] + h.$$

The holomorphic germ h is unique yet the decomposition is not

An example of Laurent expansion

Definition

The decomposition $f = \sum S_i + h =: \mathfrak{L}_{\mathcal{C}}(f)$*is a* $\mathcal{C}$*-Laurent expansion of* f *with* $\mathcal{C} = \{(C(S_j)), j \in J\}$, *a properly positioned family of simplicial cones.*

Example

$$f(z_1, z_2) := \frac{z_1 - z_2}{z_1\, z_2\, (z_1 + z_2)} \quad \text{non simplicial}$$

$$= \frac{1}{z_1\, z_2} - \frac{2}{z_1\, (z_1 + z_2)} \quad \text{non properly positioned}$$

$$= \frac{1}{z_2\, (z_1 + z_2)} - \frac{1}{z_1\, (z_1 + z_2)} \quad \text{properly positioned.}$$

$\mathcal{C} := \{\langle e_1, e_1 + e_2\rangle, \langle e_2, e_1 + e_2\rangle\}$; $\quad \mathfrak{L}_{\mathcal{C}}(f) = \frac{1}{z_2\,(z_1+z_2)} - \frac{1}{z_1\,(z_1+z_2)}$.

III. Where the linear poles come from

Linear poles arise from constraints

Multiple sums and integrals with certain types of constraints give rise to meromorphic germs with linear poles.

The constraints come from the underlying algebraic structure:

- Graphs: Feynman integrals associated with Feynman graphs in quantum field theory;
- Cones: Integrals and sums associated with cones in equivariant geometry and string theory;
- Trees: Integrals and sums associated with trees in number theory.

One wants to **evaluate** at zero such meromorphic germs, which is possible via generalised evaluators once we know that they have linear poles.

Multiple sums and integrals with constraints: prototypes

Feynman integrals (N.-V. Dang, B. Zhang 2017)

- Multiple integrals associated with Feynman diagrams;
- Constraints given by conservation of momentum;linear constraints in the integration variables (internal momenta) k_ℓ and parameters (external momenta) p_j.

Multizeta functions (D. Manchon, S.-P. 2010)

- Nested multiple sums $\zeta(s_1, \cdots, s_k) := \sum_{0<n_k<\cdots<n_1} n_1^{-s_1} \cdots n_k^{-s_k}$
- Conical constraints: $0 < x_k < \cdots < x_1$ given by Chen cones.

Multiple sums and integrals with constraints II

Multizeta functions generalise to

Conical zeta functions (L. Guo, S.-P., B. Zhang 2017)

- Discrete sums associated with general (strongly) convex polyhedral cones obtained as "moments" of generating exponential sums (Laplace transforms) on the cone;
- with general conical constraints given by intersecting half spaces;

Branched zeta functions (P. Clavier, L. Guo, S. P., B. Zhang 2019)

- Discrete sums associated with trees;
- with tree-like constraints which generalise the nested constraints on ladder trees.

IV. Evaluating germs at zero

Extending the evaluation at zero beyond holomorphic germs

Generalised evaluators (inspired by Speer)

A generalised (resp. holomorphic) **evaluator** on $\mathcal{M}(\mathbb{C}^k)$ is a **linear form** (resp. **map)** $\mathcal{E}:\mathcal{M}(\mathbb{C}^k)\longrightarrow\mathbb{C}$, (resp. $E:\mathcal{M}(\mathbb{C}^k)\longrightarrow\mathcal{M}_+(\mathbb{C}^k)$) such that $\mathcal{E}|_{\mathcal{M}_+(\mathbb{C}^k)}=\mathrm{ev}_0$, resp. $E|_{\mathcal{M}_+(\mathbb{C}^k)}=Id$.

Q-Generalised evaluators

A Q-generalised (resp. holomorphic) evaluator on $\mathcal{M}(\mathbb{C}^k)$ is a generalised (resp. holomorphic) evaluator on $\mathcal{M}(\mathbb{C}^k)$ which is **partially multiplicative** (locality):

$$(f\perp^Q g)\Longrightarrow\left(\mathcal{E}^Q(f\,g)=\mathcal{E}^Q(f)\,\mathcal{E}^Q(g)\right),$$

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Examples of generalised evaluators

Minimal subtraction scheme in several variables

- (L. Guo, S.-P., B. Zhang/ N. Berline, M. Vergne 2015) Q-orthogonal splitting: $\mathcal{M}(\mathbb{C}^k) = \mathcal{M}_+(\mathbb{C}^k) \oplus^{\perp^Q} \mathcal{M}_-(\mathbb{C}^k)$;
- $\Longrightarrow {\pi_+}^Q : \mathcal{M}(\mathbb{C}^k) \to \mathcal{M}_+(\mathbb{C}^k)$ is a Q-holomorphic evaluator (locality morphism);
- $\Longrightarrow \mathcal{E}^Q = \mathrm{ev}_0 \circ {\pi_+}^Q$ is a Q-generalised evaluator.

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Evaluators à la Speer

$\mathcal{E}^{\mathrm{Speer}} := \mathrm{ev}^{\mathrm{reg}}_{z_k=0} \circ \cdots \circ \mathrm{ev}^{\mathrm{reg}}_{z_1=0}$

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$\mathcal{E}^{\mathrm{Speer}} := \mathrm{ev}^{\mathrm{reg}}_{z_k=0} \circ \cdots \circ \mathrm{ev}^{\mathrm{reg}}_{z_1=0}$ is not a Q-generalised evaluators since (for the canonical inner product)

$$\mathcal{E}^{\mathrm{Speer}}\left(\frac{z_1-z_2}{z_1+z_2}\right) = -1 \neq 0 = \mathcal{E}^Q\left(\frac{z_1-z_2}{z_1+z_2}\right).$$

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Interlude: Cones versus fractions

From cones to fractions

The Laplace transform of a simplicial cone $C(v_1, \cdots, v_k) = \sum_{i=1}^k \mathbb{R}_{\geq 0} v_i$ generated by a basis $\{v_1, \cdots, v_k\}$

$$\int_{C(v_1,\cdots,v_k)} e^{-\langle x, \epsilon \rangle} dx = \prod_{i=1}^{k} \int_0^\infty e^{-\langle t_i \langle v_i, \epsilon \rangle \rangle} dt_i$$

$$= \frac{1}{\prod_{i=1}^k \langle v_i, \epsilon \rangle} = \frac{1}{\prod_{i=1}^k L_i(\epsilon)}.$$

From fractions to decorated cones

- $\frac{1}{\prod_{i=1}^k L_i(\epsilon)} \longleftrightarrow C(v_1, \cdots, v_k)$ (for simplicial cones);
- $\frac{1}{\prod_{i=1}^k L_i{}^{s_i}(\epsilon)} \longleftrightarrow C((v_1, s_1); \cdots ; (v_k, s_k))$ (decorated cones).

Q-Generalised evaluators

On polar germs

$S = \frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}} \Longrightarrow$ (using partial multiplicativity)

$$\mathcal{E}^Q(S) = \mathcal{E}^Q(h(\ell_1, \cdots, \ell_m))\, \mathcal{E}^Q\left(\frac{1}{L_1^{s_1} \cdots L_n^{s_n}}\right) = h(0)\, \mathcal{E}^Q\left(\frac{1}{L_1^{s_1} \cdots L_n^{s_n}}\right)$$

So Q-generalised evaluators on polar germs are determined by their values on the (decorated) supporting cones.

On general meromorphic germs

$$\mathcal{E}^Q(f) = \sum_{j \in J} \mathcal{E}^Q(S_j) + \mathcal{E}(h) = \sum_{j \in J} h_j(0)\, \mathcal{E}^Q\left(\frac{1}{L_{j1}^{s_1} \cdots L_{jn}^{s_n}}\right) + h(0).$$

Q-Generalised evaluators

Theorem (P. Clavier, L. Guo, S. P., B. Zhang 2019)

Q-generalised evaluators on general meromorphic germs are determined by their values on the family of (decorated) supporting cones.

In progress

- This can be generalised replacing the orthogonality relation $\perp^Q$ by a more general regular locality relation
  orthogonality relation $\perp$ $\longleftrightarrow$ orthogonal complement $X \to X^\perp$
  regular locality relation $\top$ $\longleftrightarrow$ complement map $X \to X^\top$
- The Galois group of automorphisms of $\mathcal{M}(\mathbb{C}^k)$ that are the identity on $\mathcal{M}_+(\mathbb{C}^k)$ acts on generalised evaluators.
- The general linear group acts transitively on generalised evaluators determined by inner products.

THANK YOU FOR YOUR ATTENTION!