Title: Semiclassical Einstein equations in cosmological spacetimes

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Abstract: During this talk we shall discuss the backreaction of quantum matter fields on classical backgrounds by means of the semiclassical Einstein equation.
We shall see that self consistent solutions of this coupled system exist in the case of cosmological spacetimes.
Furthermore, Einstein equations governing the backreaction will transfer quantum matter fluctuations to the metric.
In particular, we will see how the singular structure of quantum matter will affect the spectrum of metric perturbations

Semiclassical Einstein equations in cosmological spacetimes

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Motivations

- **Quantum matter - gravity interplay** is fully described only within:

  quantum gravity.

- However, in certain regimes, it is possible to analyse this interplay within semiclassical approximation:
  - QFT on curved spacetime
  - Backreaction

$$G_{ab} = \langle T_{ab} \rangle_\omega$$

- Effects
  - In cosmology (particle creation, some model of inflation)
  - Black Hole Physics (Hawking radiation, evaporation)
- It equates **classical** quantities with **probabilistic** ones.
- The state is **not** a local object, the equation is **not** local.
- Is it **meaningful** to search recursively for a solutions of the Semiclassical Einstein equations?

Plan of the talk

- Quantum field theory on curved spacetime: the trace anomaly.
- Semiclassical Einstein equation in cosmology.
- Existence and uniqueness of its solutions in cosmology.

This talk is based on

- H. Gottschalk, P. Meda, NP, D. Siemssen, (in preparation).
- NP, D. Siemssen, JMP **56** 022303 (2015) .
- NP, D. Siemssen, CMP **334** 171-191 (2015).
- NP, CMP **305** 563-604 (2011).

Semiclassical Einstein equation in cosmology.

- Cosmological spacetimes

$$(M, g)\ , \qquad M = I \times \Sigma\ .$$

- For flat cosmological spacetimes

$$g = -dt \otimes dt + a(t)^2 dx^i \otimes dx^i\ ,$$

  - t the **cosmological time**.
  - a is the **scale factor**.
  - $H = \frac{d}{dt}\log(a)$ is the **Hubble parameter**.
  - $d\tau = a^{-1}dt$ is the **conformal time** $g = a(\tau)^2\left[-d\tau \otimes d\tau + dx^i \otimes dx^i\right]$.
- One DOF hence: simpler equation

$$-R = \langle T\rangle_\omega\ , \qquad \nabla_a T^{ab} = 0\ , \qquad \rho(\tau_0) = \langle T_{00}\rangle_\omega = H^2(\tau_0)$$

- We look for existence and uniqueness of solutions of that system.

Matter fields

- Massive scalar quantum field conformally coupled to gravity.

$$K\varphi = -\Box\varphi + \frac{1}{6}R\varphi + m^2\varphi = 0$$

- **Canonical quantization** is very well under control on every globally hyperbolic spacetime.
- Assign to every spacetime a $*-$algebra of observables

$$M \mapsto \mathcal{A}(M)$$

- $\mathcal{A}(M)$ generated by linear fields $\varphi(f)$, $f \in C_0^\infty(M)$ implementing:

$$\varphi^*(f) = \varphi(\overline{f}) \ , \qquad \varphi(Kf) = 0 \ , \qquad [\varphi(f), \varphi(h)] = i\Delta(f,h).$$

- Where $\Delta(f,h) = \Delta_R(f,h) - \Delta_A(f,h)$
(retarded minus advanced fundamental solutions of $K\varphi = 0$)

States

- A **state** ω is a positive normalized linear functional over $\mathcal{A}$
$$\omega : \mathcal{A} \to \mathbb{C}$$
- Once a state is chosen by **GNS theorem** we can represent $\mathcal{A}(M)$ as operators over some Hilbert $\mathfrak{H}$ space and ω as a normalized vector in $\mathfrak{H}$.
- Different states on $\mathcal{A}(M)$ give rise to inequivalent representations, since $\mathcal{A}(M)$ is chosen before the state we can still compare expectation values.
- $\mathcal{A}(M)$ is generated by $\varphi(f)$, $f \in C_0^\infty(M)$ states are characterised by $n-$point functions
$$\omega_n(f_1, \dots, f_n) := \omega(\varphi(f_1) \dots \varphi(f_n))$$
$\omega_n \in \mathcal{D}(M^n)$.

Extended algebra of Wick polynomials

- We need to extend $\mathcal{A}(M)$ to include objects like

$$\varphi^n(f), \qquad T_{ab}(f)$$

However, these are divergent quantities

$$\omega(\varphi^2(x)) = \lim_{y\to x} \omega_2(y,x) = \infty$$

- We need a regularization prescription which implements some **normal ordering**.
- In a **Hadamard state** the singular structure is universal:

$$\omega_2 = \mathcal{H} + W, \qquad \mathcal{H} = \frac{U}{\sigma_\epsilon} + V \log\left(\frac{\sigma_\epsilon}{\lambda^2}\right) + W$$

Equivalent to **Microlocal Spectrum Condition** (remnant of the spectrum condition).

- Point splitting regularization

$$:\varphi^2(x):_{\mathcal{H}} = \lim_{y\to x} \varphi(x)\varphi(y) - \mathcal{H}(x,y)$$

- We extend $\mathcal{A}(M)$ to include normal ordered Wick powers. ▸ Details.

Point splitting regularization - backreaction

- **Stress-Energy Tensor:**

$$T_{ab} := \partial_a\varphi\partial_b\varphi - \frac{1}{6}g_{ab}\left(\partial_c\varphi\partial^c\varphi + m^2\varphi^2\right) - \frac{1}{6}\nabla_{(a}\partial_{b)}\varphi^2 + \frac{1}{6}\left(R_{ab} - \frac{R}{6}g_{ab}\right)\varphi^2.$$

Hence

$$T_{ab}(x) = \lim_{y\to x} D_{ab}\varphi(x)\varphi(y)$$

- **Expectation values:** obtained subtracting the Hadamard singularity $\mathcal{H}$ from ω_2

$$\langle T_{ab}\rangle_\omega := \omega(:T_{ab}:_\mathcal{H}) = \lim_{y\to x} D_{ab}\left[\omega_2(x,y) - \mathcal{H}(x,y)\right]$$

- We need a **rule** to prescribe a state ω on $\mathcal{A}(M)$ for every FRW spacetime M.
- Use the semiclassical equation to select M on which $G_{ab} = \langle T_{ab}\rangle_\omega$.

Components of $\langle T \rangle$

- The trace of the stress tensor T can be decomposed in the following three contributions
$$\langle T \rangle_\omega = T_{anomaly} + T_{ren.freedom} + T_{state}$$
- **Anomalous term** is there because $\mathcal{H}$ is not a solution of the equation of motion. We have enough freedom to require $\nabla^j T_{ji} = 0$
$$T_{anomaly} = \frac{1}{2880\pi^2}\left(C_{ijkl}C^{ijkl} + R_{ij}R^{ij} - \frac{R^2}{3} + \Box R\right) + \frac{m^4}{4}$$
- **Renormalization freedom**
$$T_{ren.freedom} = \alpha m^4 + \beta m^2 R + \gamma \Box R$$
  - α expresses a renorm. of the cosmological constant $\Longrightarrow$ adding αm^4 to $\mathcal{L}$
  - β expresses a renorm. of the Newton constant $\Longrightarrow$ adding $\beta m^2 R$ to $\mathcal{L}$
  - γ is a pure quantum freedom. $\Longrightarrow$ adding γR^2 or $\gamma R_{ij}R^{ij}$ to $\mathcal{L}$
- **State dependent contribution**
$$T_{state} = m^2[W]$$

$W(x,y) = \omega_2(x,y) - \mathcal{H}(x,y)$ the smooth part in ω_2. $[W](x) = W(x,x)$.

Effect of the anomaly

Assume T_{state} negligible, fix α, β we get an equation that can be solved.

- $T_\mu{}^\nu = (-\rho, P, P, P)$

$$\rho = c\frac{H^4}{4} + \frac{d}{a^4}, \qquad P = -c\left(\frac{H^2\dot{H}}{3} + \frac{H^4}{4}\right) + \frac{d}{3a^4}$$

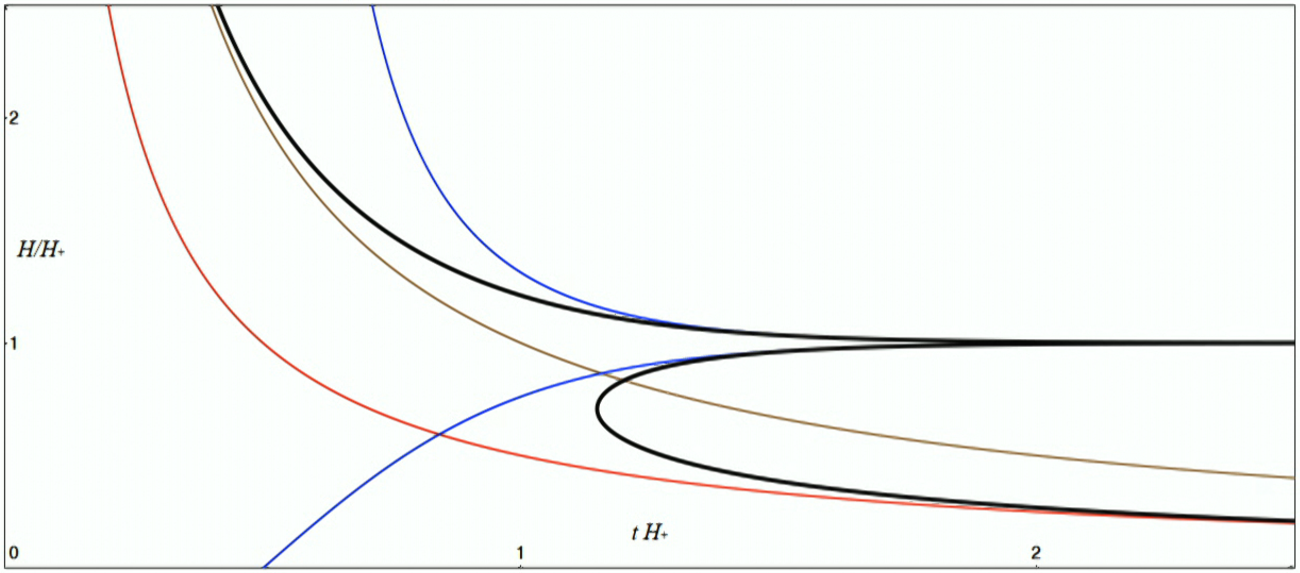
(d=0) two fixed points

$$H^2 = \frac{H^4}{C} + \Lambda$$

$$H_\pm^2 = \frac{C}{2}\left(1 \pm \sqrt{1 - \frac{4\Lambda}{C}}\right), \qquad H_+^2 \simeq C, \qquad H_-^2 \simeq \Lambda$$

Near these fixed points the solutions can be obtained.

With some choice of γ and β $H = \Lambda$ and $H = H_+$ are stable solutions.



- ($m = 0$) a length scale is introduced (proportional to G).
  Two fixed points instead of one *[Wald 80, Starobinsky 80, Vilenkin 85]*.
- Quantum effects are **not negligible** at least in the past.
- The upper branch is not physically acceptable.

State dependent contributions

- Some hypothesis (state as close as possible to a "vacuum")
  - Gaussian (only ω_2 matters)
  - pure, homogeneous, isotropic
- The **pure**, **homogeneous** and **isotropic** Gaussian state

$$\omega_2(x,y) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\overline{\chi_k}(x_0)}{a(x_0)}\frac{\chi_k(y_0)}{a(y_0)} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} d\mathbf{k} \,,$$

$$\chi_k''(\tau) + (m^2 a(\tau)^2 + k^2)\chi_k(\tau) = 0,$$
$$\overline{\chi_k}\frac{d}{d\tau}\chi_k - \frac{d}{d\tau}\overline{\chi_k}\chi_k = i \,.$$

Initial conditions for χ

- At $\tau = \tau_0$ we fix the state to be as close as possible to the vacuum.
- In order to be close to the vacuum at $\tau = \tau_0$ we fix

$$\chi_k(\tau_0) = \frac{1}{\sqrt{2k_0}}\,, \qquad \chi_k'(\tau_0) = -i\sqrt{\frac{k_0}{2}}\,, \qquad k_0 = \sqrt{k^2 + m^2 a_0^2}$$

- It is an **adiabatic state** of order 0 at $\tau = 0$.
  (Although it is not Hadamard it is sufficiently regular to construct T) *[Parker, Lüders Roberts]*
- If we can do this for $\tau_0 \to -\infty$ and if it corresponds to a null surface we get an **Hadamard state**. (Bunch Davies in de Sitter)
- No analytical solution for χ, however, treating $V = m^2(a(\tau)^2 - a_0^2)$ as a perturbation potential χ can be constructed by a convergent **Dyson series** around

$$\chi_k^0(\tau) = \frac{e^{-ik_0\tau}}{\sqrt{2k_0}}$$

T for various states

- Consider states constructed in such a way that

$$\tilde{W}(x,y) - W(x,y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\left(\frac{\overline{\chi_k}(x_0)}{a(x_0)}\frac{\chi_k(y_0)}{a(y_0)} + \frac{\overline{\chi_k}(y_0)}{a(y_0)}\frac{\chi_k(x_0)}{a(x_0)}\right) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} F(\mathbf{k}) d\mathbf{k}\ ,$$

  where $F(\mathbf{k})$ is rapidly decreasing.

- Estimate

$$\rho := \langle T_{00}\rangle_{\tilde{\omega}} - \langle T_{00}\rangle_{\omega}$$

  Fix ρ at τ_0 then for $\tau < \tau_0$ it holds that

$$\frac{C}{a^4} \leq \rho \leq \frac{C}{a^2 a_0^2}$$

- Obtained noticing that the energy per mode $h = |\chi_k'|^2 + (k^2 + m^2 a^2)|\chi_k|^2$ can only decrease in an expanding universe.
- States close to thermal equilibrium.

$$\rho \sim \frac{C}{a^3}$$

- Difference of energy density in two states behaves like ordinary matter.

- After fixing the renormalization freedom (α, β and $\gamma = -1/360$) we may rewrite the equation as a **Volterra-functional equation**.
- For simplicity, in this talk, we will discard the anomalous part and we will assume that $\Lambda = 0$.

$$a'(\tau) = a'_0 + m^2 \int_{\tau_0}^{\tau} [W] a^3 \, d\eta$$

$$[W] = \frac{1}{2\pi^2 a^2} \int_0^\infty \left[\overline{\chi}_k \chi_k - \frac{1}{2\sqrt{k^2 + m^2 a^2}} \right] k^2 dk$$

$$\chi_k''(\tau) + (m^2 a(\tau)^2 + k^2)\chi_k(\tau) = 0,$$

- We fix the state so that it looks like the vacuum at $\tau = \tau_0$.
- Construct χ with a convergent **Dyson series** around χ^0 treating $v = (a^2 - a(0)^2)m^2$ as a perturbation potential
- We can control $\langle \varphi^2 \rangle_\omega = [W]$ w.r.to a' and its (first-)functional derivative on $C^0(\tau_0, \tau)$. ▸ const.

$$\|\langle \varphi^2 \rangle_\omega\|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0) \;, \quad \|D\langle \varphi^2 \rangle_\omega\|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0)\|\delta a'\|_\infty$$

- We get an estimate valid on every spacetime ($\forall a' \in C^0[\tau_0, \tau]$)

Local existence

The Volterra like equation seen before is thus a fixed point equation

$$a = F(a)$$

we may then find a applying recursively F on a_1.

$$a_n = F(a_{n-1}), \qquad a = \lim_{n\to\infty} a_n$$

Proposition

Fix a_0 and the state at τ_0. An **unique** *solution a_I* **exists** *in $I = [\tau_0, \tau_1)$ for some $\tau_1 > \tau_0$.*

Proved applying the Banach fixed point theorem to the Volterra like equation. The estimates permit to construct a contraction map.

- After fixing the renormalization freedom (α, β and $\gamma = -1/360$) we may rewrite the equation as a **Volterra-functional equation**.
- For simplicity, in this talk, we will discard the anomalous part and we will assume that $\Lambda = 0$.

$$a'(\tau) = a'_0 + m^2 \int_{\tau_0}^{\tau} [W] a^3 \, d\eta$$

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- We fix the state so that it looks like the vacuum at $\tau = \tau_0$.
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- We can control $\langle \varphi^2 \rangle_\omega = [W]$ w.r.to a' and its (first-)functional derivative on $C^0(\tau_0, \tau)$. ▸ const.

$$\| \langle \varphi^2 \rangle_\omega \|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0) \ , \quad \| D \langle \varphi^2 \rangle_\omega \|_\infty \leq c(\|a'\|_\infty, \tau - \tau_0) \| \delta a' \|_\infty$$

- We get an estimate valid on every spacetime ($\forall a' \in C^0[\tau_0, \tau]$)

Global solution

Proposition

Let a_I in $I = [\tau_0, \tau_1)$ be a solution then, if $a'(\tau_1)$ do not diverge and $a(\tau_1) > 0$ the solution can be extended further in a **unique** *way to a_J with $I \subset J$.*

- We can order all the solutions a_I. $a_I \leq a_J$ is $I \subset J \Longrightarrow$ a maximal solution exists

Proposition

The maximal solutions is unique because of the unique extension.

Summarizing: fixing the initial condition, either the solution exists till infinity or a singularity is encountered. ($a = 0$, $a' = \infty$)

Other initial values

- Changing the initial values for χ correspond to change the state.

$$\chi_{k,1} = A\chi_k + B\overline{\chi}_k$$

- If the state are sufficiently close to ω (B suff. reg.) we can still find solutions.
- The obtained solution is at least C^2.
- The employed estimates for $[W]$ do not permit to control the global behaviour from the initial condition.
- Numerical methods can be applied.

Generic coupling to the curvature

(Work in progress with H. Gottschalk, P.Meda, D. Siemssen)

$$K\varphi = -\Box\varphi + \xi R\varphi + m^2\varphi = 0$$

- Modify the equation studied above accordingly

$$\langle\varphi^2\rangle_\omega = [W] = \frac{1}{2\pi^2 a^2}\int_0^\infty \left[\overline{\chi}_k\chi_k - \frac{1}{2k_0} + \frac{V}{4k_0^3}\right] k^2 dk$$

where

$$V = (6\xi - 1)\frac{a''}{a} + m^2(a^2 - a_0^2) \qquad \chi_k''(\tau) + (k_0^2 + V)\chi_k(\tau) = 0,$$

Generic coupling to the curvature

- A careful analysis gives that the semiclassical Einstein equation has the form

$$V = \mathcal{L}(V) + \mathit{reminder}, \qquad \mathcal{L}(V) = \int_{\tau_0}^{\tau} V'(\eta) \log(\tau - \eta) d\eta$$

  - where $V = \alpha \frac{a''}{a} + \mathit{corrections}$
  - $\mathcal{L}(V)$ can be controlled only with V'. ($\|\mathcal{L}(V)\|_\infty \leq c\|V'\|_\infty$).
  - $\mathcal{L}^{-1}$ exists and it is continuous (wrt to L^2 and L^∞ norms)
  - New fixed point equation

$$V = \mathcal{L}^{-1}(V) - \mathcal{L}^{-1}(\mathit{reminder}) = F(V)$$

    $F(V)$ can be proved to be a contraction map.

  - Numerical methods can be now used to find a solution.

▸ Spherically symmetric.

Negative energy flux from conformal anomaly

A simplified model

$$ds^2 = -2A(U,V)dUdV + R(U,V)^2\left(d\theta^2 + \sin\theta^2 d\varphi^2\right)$$

Conservation of the stress tensor implies that

$$(\nabla T)_V = -\frac{1}{AR^2}\partial_U\left(T_{VV}R^2\right) - \frac{1}{R^2}\partial_V\left(\frac{T_{UV}}{A}R^2\right) - 2T_\theta{}^\theta\frac{\partial_V R}{R} = 0$$

$T_{VV}R^2$ measures the flux of ingoing energy. Assuming the semiclassical Einstein equation for a conformal massless scalar field

$$T_\theta{}^\theta = \frac{1}{8\pi}G_\theta{}^\theta, \qquad T_a{}^a = -2\frac{T_{UV}}{A} + 2T_\theta{}^\theta$$

The trace anomaly

$$T_a{}^a = \frac{1}{2880\pi^2}\left(C_{abcd}C^{abcd} + R_a{}^b R_b{}^a - \frac{1}{3}R^2\right)$$

Vaidya metric

In advanced Eddington-Finkelstein coordinates

$$g = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega$$

$$G_\theta{}^\theta = 0.$$

The trace anomaly of a massless conformal invariant scalar field is

$$T_a{}^a = 48\alpha \frac{M(v)^2}{r^6}$$

Assuming $T_{vv} r^2 \to 0$ for $r \to \infty$, from $\nabla^a T_{ab} = 0$ we get

$$T_{VV} r^2 = 48\alpha \left(\frac{2}{3} \frac{M\dot{M}}{r^3} - \left(\frac{1}{4} - \frac{1}{5} \frac{2M}{r} \right) \frac{4M^2}{r^4} \right)$$

If $\dot{M}$ is small or $\dot{M} < 0$ the flux of energy at $r = 2M$ is negative.

Dynamical mass change along the apparent horizon?

The semiclassical Einstein equation $G_{vv} r^2 = T_{vv} r^2$

$$2\dot{M}(v) = 48\alpha \left(\frac{2}{3} \frac{M\dot{M}}{r^3} - \left(\frac{1}{4} - \frac{1}{5} \frac{2M}{r} \right) \frac{4M^2}{r^4} \right)$$

On the **apparent horizon** ($r = 2M$) is

$$\left(1 - \frac{2\alpha}{M^2} \right) \dot{M}(v) = -\frac{3}{10} \frac{\alpha}{M^2}$$

$M(v)$ tends to decrease along $\mathcal{H}$.

Question:
Is the full solution compatible with this observation?

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Summary

- Semiclassical Einstein equation can be used in cosmology.
- It is a well posed initial value problem.
- Numerical methods can safely applied provided the equation is written in a certain way.

Thanks a lot for your attention!