Title: How much geometry is in a truncated spectral triple?

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Abstract: A spectral triple consists of an algebra, a Hilbert space and a Dirac operator, and if these three fulfill certain relations to each other they contain the entire information of a compact Riemannian manifold.
Using the language of spectral triples makes it possible to generalize the concept of a manifold to include non-commutativity.

While it is possible to write down finite spectral triples, often categorized as fuzzy spaces, that describe discretized geometries, classical geometries are encoded in infinite dimensional spectral triples. However working in numerical systems (and maybe ultimately in physical systems), only a finite part of this information can be encoded, which opens the question; If we know a part of the spectrum, how clearly can we characterise a geometry.
In this talk I will present first steps towards answering this question.

Spacetime Foam

and the Cosmological Constant

Steven Carlip

U.C. Davis

Emmy Noether Workshop:
   The Structure of Quantum Space Time
Perimeter Institute, Waterloo, Canada
November 2019

The problem of quantum gravity

General Relativity + Quantum Mechanics =?

► Many approaches:

Non-commutative geometry

Causal Set theory

Causal Dynamical Triangulations   **Tensor models**

Spinfoam   **Group Field Theory**

Asymptotic Safety   **Holography**

Emergent Gravity   **String theory**

Loop quantum gravity

How can we understand them?

How can we construct solutions?

How can we extract predictions?

What is the space of geometries?

The path integral of Quantum Gravity

$$\langle f \rangle = \frac{\int f(g)\, e^{i\mathcal{S}(g)}\, \mathcal{D}[g]}{\int e^{i\mathcal{S}(g)}\, \mathcal{D}[g]}$$

Ingredients:

▶ Geometry g and measure $\mathcal{D}[g]$

▶ Functions of geometry f

▶ Action $\mathcal{S}$

So what do I do?

MC simulations can measure $\langle f(g) \rangle$, but what are good $f(g)$?

Should be

- ► completely covariant
- ► space independent
- ► efficient to measure
- ► connect to physics?
- ► help us understand phase space

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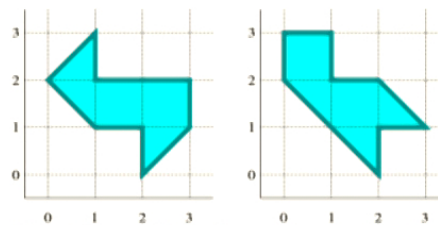
Motivation: Can we hear the shape of a drum?

Eigenvalue problem:

For a membrane Ω held fixed along bdry Γ the eigenvalue problem can be stated as:

$$\frac{1}{2}\nabla^2\psi_n(x) + \lambda_n\psi_n(x) = 0$$

$$\psi_n(x) = 0 \text{ on } \Gamma$$

If two membranes Ω_1, Ω_2 (boundaries Γ_1, Γ_2) lead to the same spectrum λ_n, are they the same (up to symmetry transformations)?

(M. Kac, The American Mathematical Monthly 73, 1-23 (1966))

Geometry as a spectral triple

$(\mathcal{A}, \mathcal{H}, D)$

- ▶ an Algebra $\mathcal{A}$ with action on $\mathcal{H}$
- ▶ a Hilbert space $\mathcal{H}$
- ▶ a Dirac operator D acting on $\mathcal{H}$

Axioms of non-commutative geometry [a]

- ▶ $\exists$ a faithfull action $\mathcal{A}$ in $\mathcal{H}$
- ▶ $\mathcal{H}$ is a bimodule over $\mathcal{A}$ (there is a left and a right action of $\mathcal{A}$ in $\mathcal{H}$)
- ▶ First order condition $[[D, a\triangleright], \triangleleft b] = 0$ for $a, b \in \mathcal{A}$

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[a]Abridged version

(A. Connes, Int.J.Geom.Meth.Mod.Phys. 5, 1215-1242 (2008))

(more detail e.g. A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

A simple geometry as a spectral triple

The circle as an algebra with a unitary operator U acting on $\mathcal{H} = L^2(\mathbb{S}^1)$

$$UU^* = 1 \qquad\qquad D = D^*$$
$$U^*[D, U] = 1 \qquad \leftrightarrow \qquad U^* D U = D + 1$$
$$D e_n = \lambda_n e_n \qquad \leftrightarrow \qquad D U e_n = (\lambda_n + 1) U e_n$$

U generates the algebra

$$a = \sum_{\mathbb{Z}} a_n U^n$$

$$a_n \in \mathbb{C}$$

for any algebra element a

Spectral triple

$$(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), -i\partial_\phi)$$

A simple geometry as a spectral triple

For a commutative torus take two S^1 generators U, V

$$U^* U = V^* V = 1$$

We can make the torus non-commutative by introducing

$$UV = \vartheta\, VU \qquad\qquad \vartheta = e^{2\pi i \theta}$$

U, V generate the algebra

$$a = \sum_{\mathbb{Z}^2} a_{n,m} U^n V^m$$

for any algebra element a

Fuzzy space (p, q)

$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$

► The algebra are matrices:
$\mathcal{A}$ is a $*-$ algebra $M(n, \mathbb{C})$

► Acting on a Hilbert space:
$\mathcal{H} = V \otimes M(n, \mathbb{C})$
where V is a (p, q)-Clifford module

Extra ingredients to make it a real spectral triple

► KO-dimension;
$s = (q - p) \bmod 8$

► Chirality;
$\Gamma(v \otimes m) = \gamma v \otimes m$ with γ the chirality operator on V

► Real structure;
$J(v \otimes m) = Cv \otimes m^*$ where C is charge conjugation on V
$J : \mathcal{H} \to \mathcal{H}$ with $\langle Ju, Jv \rangle = \langle u, v \rangle$

Dirac operator : Form

Conditions on $\mathcal{D}$ for a real spetral triple

$$\mathcal{D} = \mathcal{D}^\dagger \qquad\qquad \mathcal{D}\Gamma = \pm\Gamma\mathcal{D}$$

$$\mathcal{D}J = \pm J\mathcal{D} \qquad [[\mathcal{D}, \rho(a)\triangleright], \triangleleft\rho(b)] = 0$$

Can be translated for a fuzzy space to:

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes (\overbrace{K_i m}^{\text{left action}} + \epsilon' \overbrace{m K_i^*}^{\text{right action}})$$

(J. Barrett, J.Math.Phys. 56, 082301 (2015).)

Explore path integral over fuzzy spaces

$$\langle f \rangle = \frac{\int f(D) e^{-\mathcal{S}(D)} \mathrm{d}D}{\int e^{-\mathcal{S}(D)} \mathrm{d}D}$$

$$= \frac{\int f(D(K_i)) e^{-\mathcal{S}(D(K_i))} \prod_i \mathrm{d}K_i}{\int e^{-\mathcal{S}(D(K_i))} \prod_i \mathrm{d}K_i}$$

The simplest action

$$\mathcal{S} = g_2 \,\mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

(J. Barrett, LG J.Phys. A49, 245001 (2016))

What do we want from an action?

- ▶ physical motivation
  $\Rightarrow$ lowest order when expanding a heat kernel
- ▶ bounded from below
  $\Rightarrow$ for some g_2
- ▶ rises fast to infinity
  $\Rightarrow$ to make simulations possible

More work on fuzzy spaces

Done:

▶ **First tests of model** (J. Barrett, LG J.Phys. A49, 245001 (2016))

▶ **Spectral dimension**
(J. Barrett, P. Druce, LG, J. Phys. A52 275203 (2019))

Work pending:

▶ Larger matrix sizes

▶ Recognize geometry

▶ What is matter / include matter

▶ Analytic results

$\mathrm{Cov}(\lambda_i^2, \lambda_j^2)$ Type $(1,3)$

$N = 5$

$g_2 = -3.35$ $g_2 = -3.7$ $g_2 = -4.0$
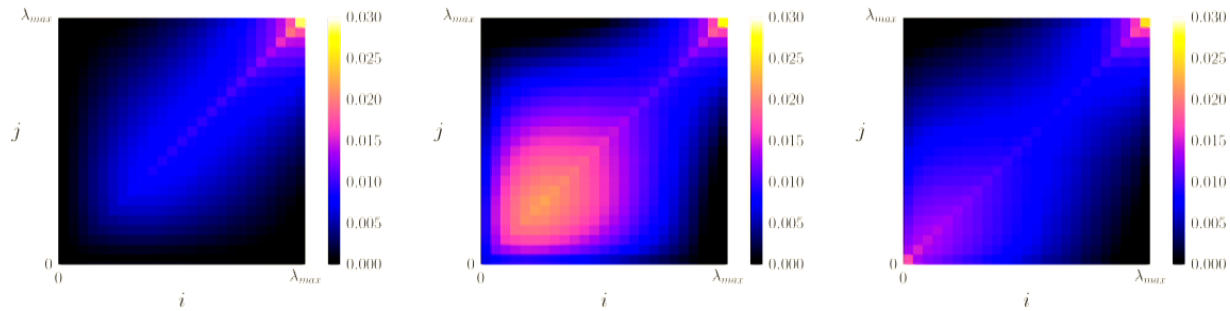
$N = 10$
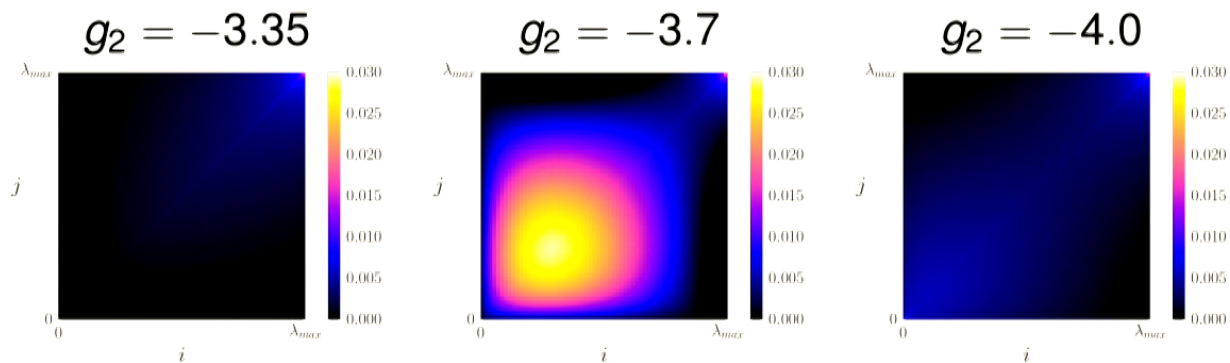
Features   (LG J.Phys.A50, 275201 (2017))

► Diagonals away from PT

► Blob at PT, stronger correlation for larger N

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Truncating a spectral triple

Describing a smooth manifold as a spectral triple leads to infinite dimensional $\mathcal{A}, \mathcal{H}, D \ldots$

Truncate D

Replace the infinite D by a $n \times n$ matrix

$$D \to P_n D P_n$$

with P_n a projector on the n smallest eigenvalues.

We assume that the finite D is a truncation of the infinite one, and that there are no small eigenvalues that we don't see.

Conditions on geometry

The one sided Heisenberg relation

$$\langle Y[D, Y]^d \rangle = \gamma$$

Where γ is a chirality and
$Y \in \mathcal{A} \otimes C_k$, $Y = \sum_i \Gamma_i Y^i$ with $\Gamma_i \in C_k$, and $Y^2 = \sum_i Y^i Y^i = 1$

Y is idempotent and Y^i are embedding maps for the sphere

Quanta of geometry

If D satisfies this equation & the axioms above the spectral triple is a union of non-commutative d-spheres (for infinite spectra).

(A.H. Chamseddine, A. Connes, V. Mukhanov, Phys.Rev.Lett. 114, 091302 (2015))

Heisenberg relations as a constraint

Can turn the one sided Heisenberg relation into a constraint for computer simulations

$$\left\| \langle Y[D, Y]^d \rangle - \gamma \right\|_{HS}^2$$

with $\left\| a_{ij} \right\|_{HS}$ the Hilbert-Schmidt norm (element wise norm)

Motivation:

Using this as an action $\mathcal{S}$ in Monte Carlo simulations should force the spectral triples probed by the algorithm to be close to d-spheres

(A.H. Chamseddine, A. Connes, V. Mukhanov, Phys.Rev.Lett. 114, 091302 (2015))

(LG, A. Stern, W. van Suijlekom work in progress)

This is a (truncated) sphere

$Y[D,Y][D,Y] - \gamma$ real part

$Y[D,Y][D,Y] - \gamma$ imaginary part
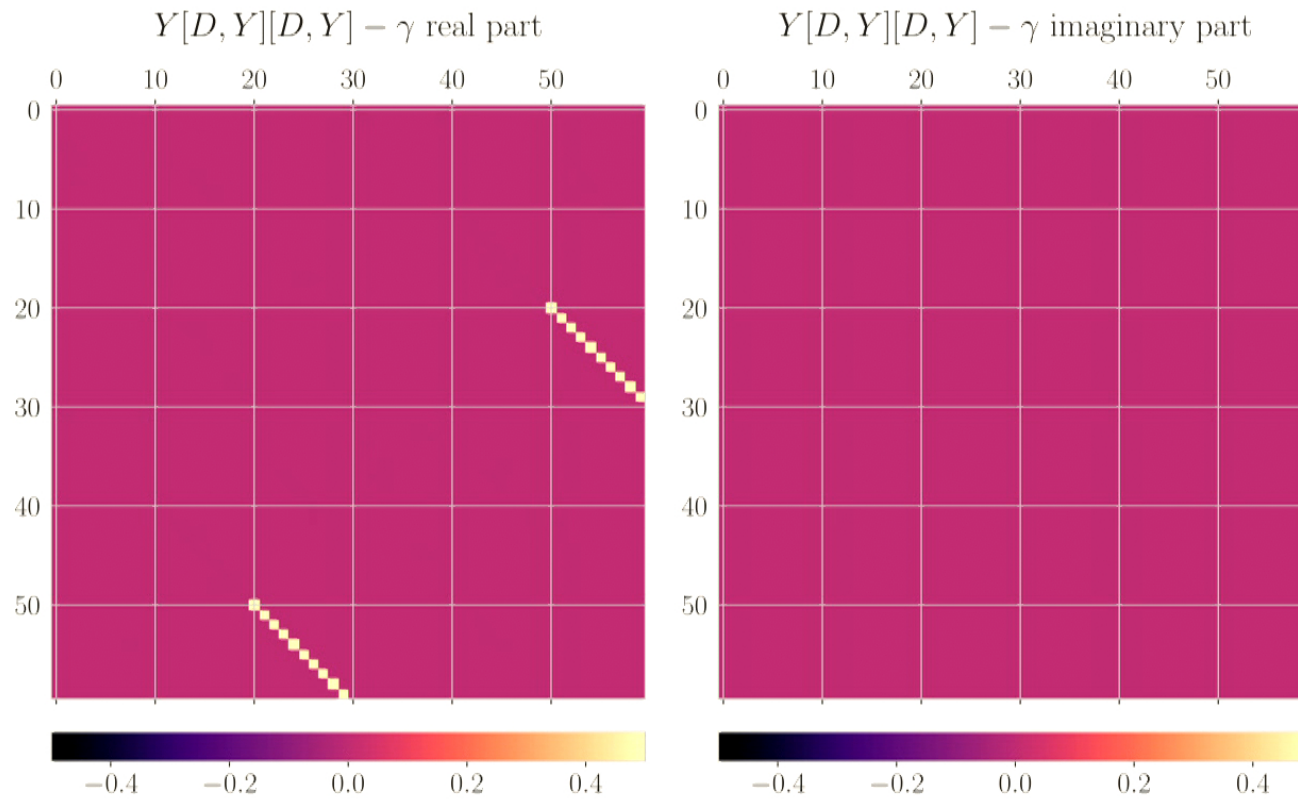
This is not a sphere!
(but minimizes the Heisenberg equation constraint)

$Y[D,Y][D,Y] - \gamma$ real part

$Y[D,Y][D,Y] - \gamma$ imaginary part
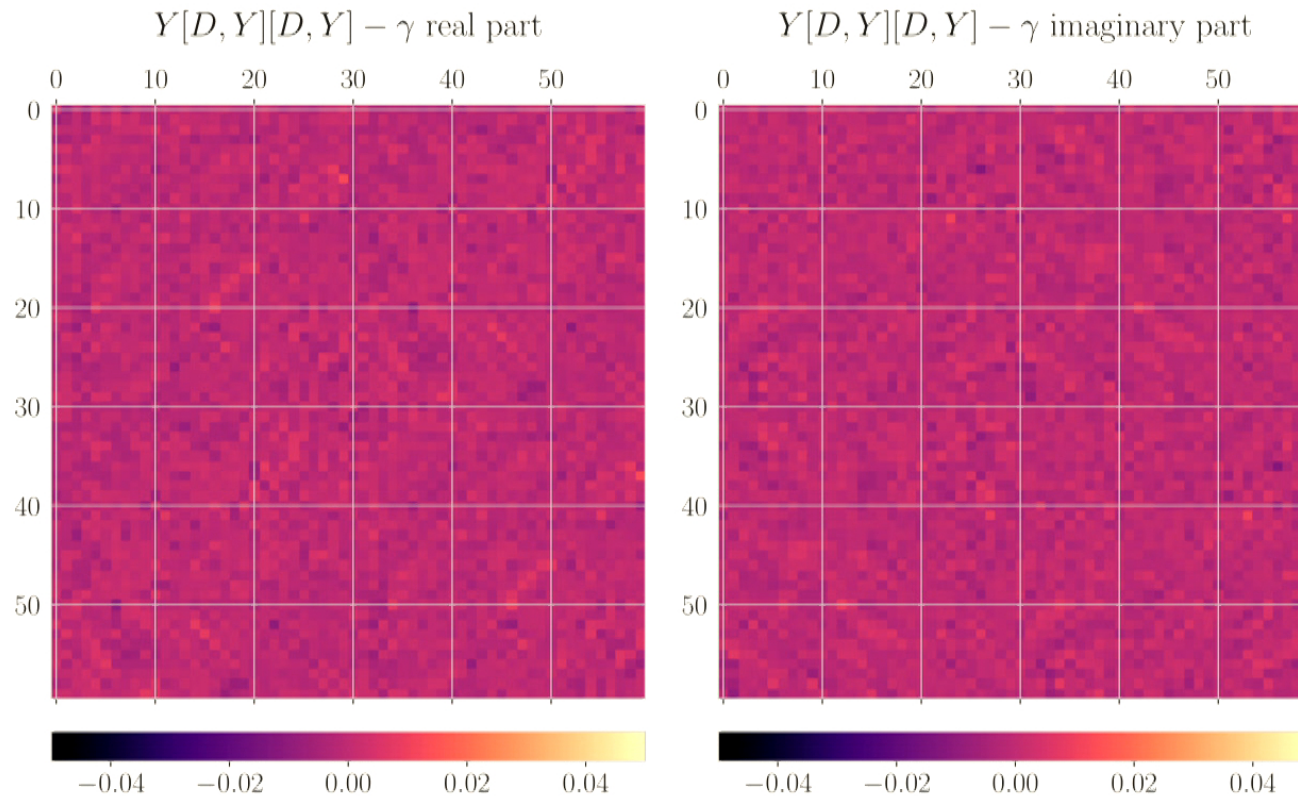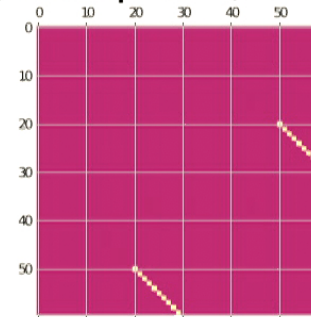
Analytic confirmation

Solution:

All operators of the form $D + cB$, where $B = sin(\pi D)$ and $c \in \mathbb{C}$ satisfy the Heisenberg relation in the infinite case.

When we truncate both D^{S^2} and the Heisenberg equation

$c = \pm 1/2$ solve exactly[a]
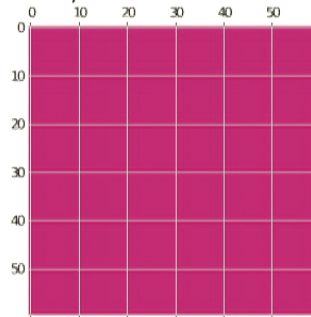
$c = 0$, the sphere, does not

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[a]if truncation is odd/ even

Analytic confirmation

First order condition

The reason the solutions with $c \neq 0$ are not relevant at infinite size is that they do not satisfy the first order constraint

$$[[f(D^{S^2}), a \triangleright], \triangleleft b] = 0 \qquad a, b \in \mathcal{A},$$

and hence do not correspond to a spectral triple.
However the defect at finite size is similar for both cases.

Non-commutative distance

Distance measure in non-commutative geometry

(A. Connes, Noncommutative Geometry. (Academic Press, 1994))

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{|\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1\}$$

Example:

Calculate distance between points x, y from function f
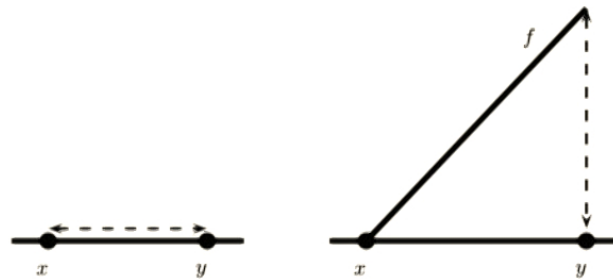
figure from

(W. van Suijlekom "Noncommutative Geometry and Particle Physics" Springer (2015))

Larger cutoff ⇒ more points

We could alternatively have phrased the third point in the lemma as follows: the maps ϕ_Λ and b (from 1) are asymptotically inverse to each other in the sense that $d(x, (b \circ \phi_\Lambda)(x)) = O(\Lambda^{-1})$ and $d((\phi_\Lambda \circ b)(\omega), \omega) \leq \sqrt{\delta(\omega)} + O(\Lambda^{-2})$.

In particular the previous lemma tells us how to scale the number of generated states with Λ:

Corollary

A sequence of equidistributed subsets $\{V_n\}_n$ of M, in the sense that $\min d|_{V_n \times V_n \setminus \Delta} = \Theta(|V_n|^{-1/n})$, will satisfy

$$\sup_{x,y \in V_n} \frac{|d(x,y) - d(\phi_{\Lambda_n}(x), \phi_{\Lambda_n}(y))|}{d(x,y)} = O(1)$$

as $\Lambda \to \infty$, whenever $|V_n| = \Theta(\text{rank } P_{\Lambda_n})$.

So how exactly do we define the states?

Localized states

We use the dispersion and the embedding maps Y_i from the Heisenberg relations

$$\delta(\omega_k) = \sum_i \langle\omega|Y_i^2|\omega\rangle - \langle\omega|Y_i|\omega\rangle^2 + \sum_{j<k} \frac{c}{\delta(\omega_j, \omega_k)}$$

Now find a set of coherent states ω that minimizes this and plug them into distance equation. The repulsive potential is to ensure even distribution of points.
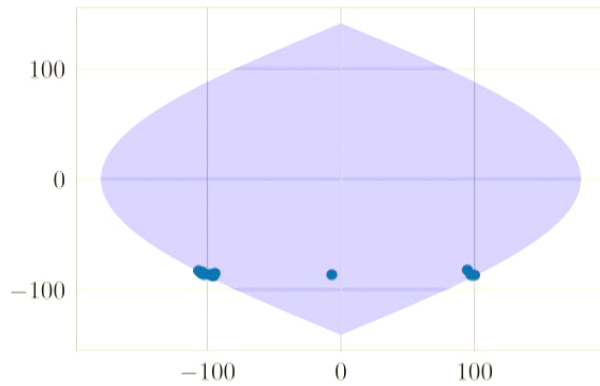
Advantage:

We can use it to plot the states and the generated geometry using the Y_i as embedding coordinates, for illustration purposes.

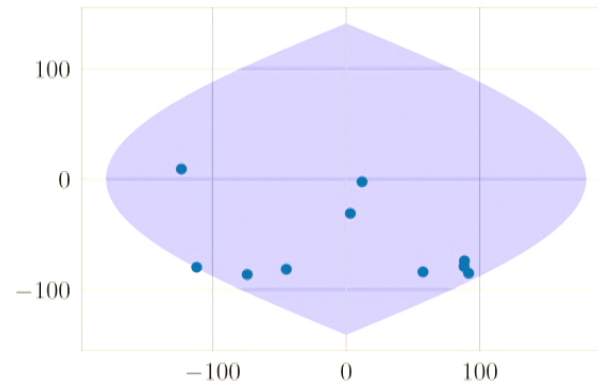# How does the state size change with the cutoff?

State for Λ = 4

State for Λ = 10

State for Λ = 16
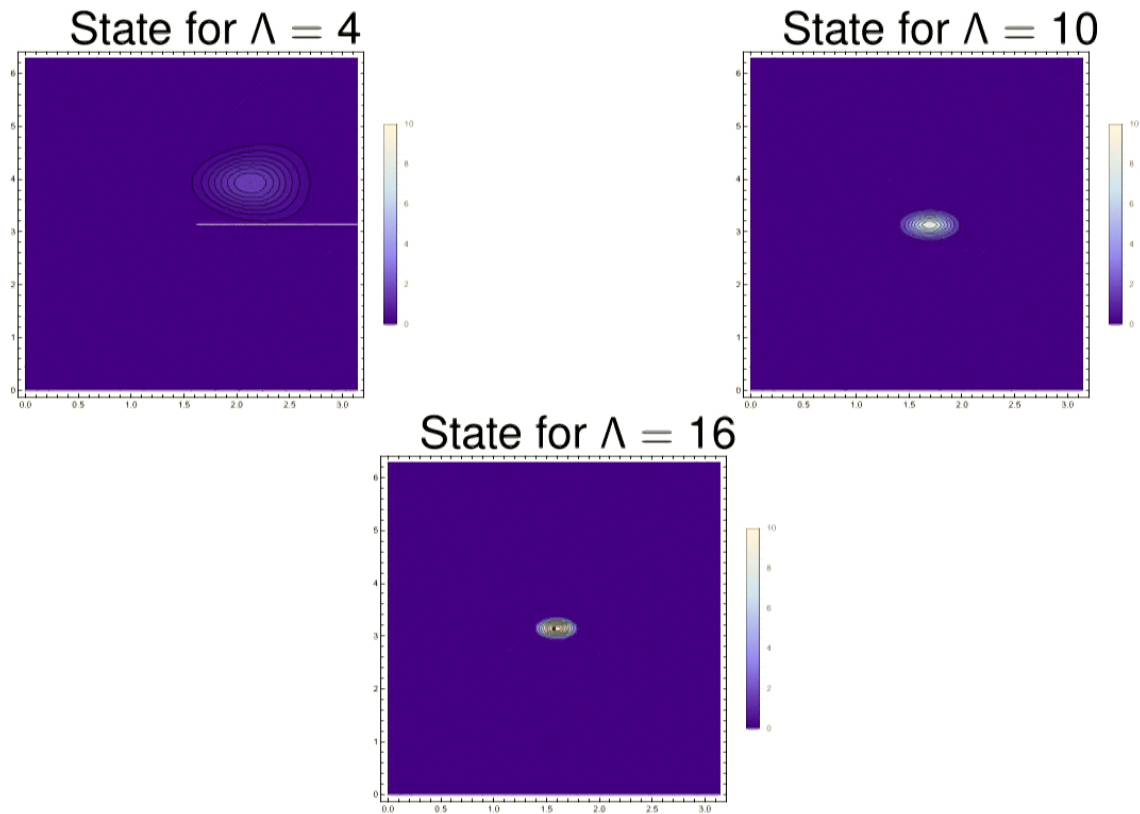
What effect does the repulsive potential have
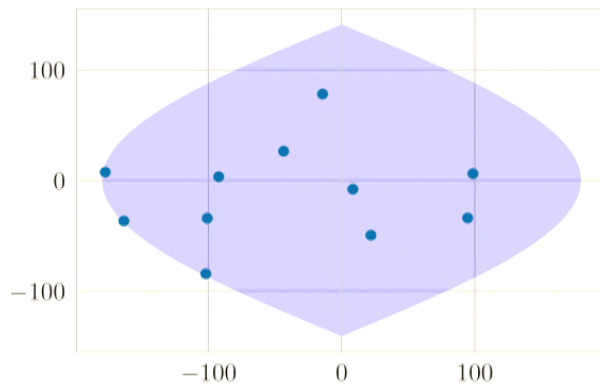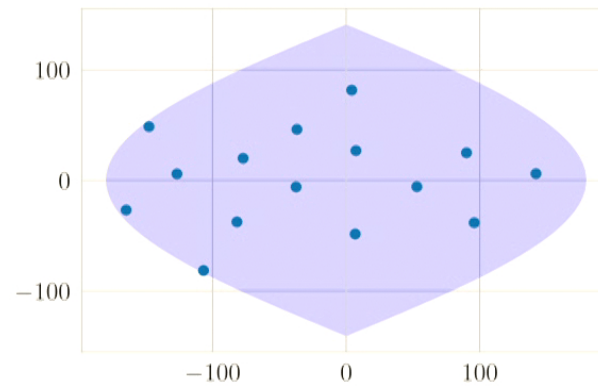
A picture of geometry

The truncated sphere at size 60

Embedded distance graph
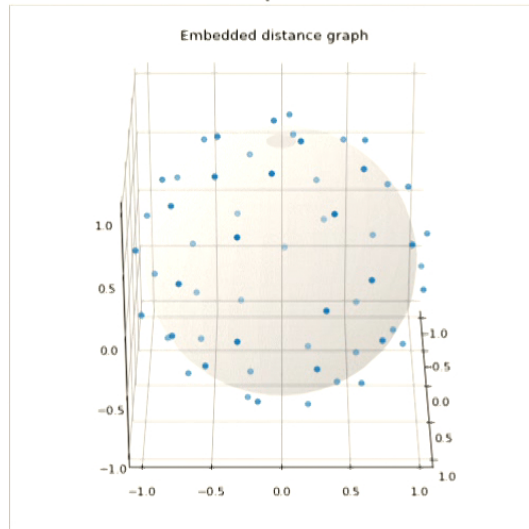
The analytic solution at size 60

Embedded distance graph with B
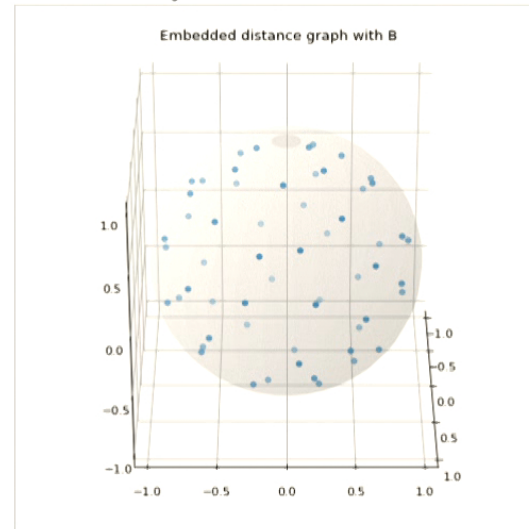
▶ generate states for a $n \times n$ matrix & calculate pairwise distances

▶ use graph embedding algorithm to find a locally isometric embedding

▶ wonder why the analytic solution is smaller

Summary

Todays story:

- ► Exploring NCG using computer simulations
- ► simulations in fuzzy spaces
- ► truncated NCGs as basis for simulations
- ► first numerical tests of one sided Heisenberg relation and Connes distance function

Immediate follow up:

- ► What is the difference between the two geometries?
- ► More simulations:
    - ► two-sided Heisenberg equation
    - ► path integral using Heisenberg equation as constraint
- ► More efficient imaging
    $\Rightarrow$ Use imaging on more states

Thanks for listening to my talk,

and this fantastic conference!

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