Title: Higher spin symmetry in gravity and string

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Abstract: Higher spin symmetries are gauge symmetries sourced by massless particles with spin greater than two. When coupled with diffeomorphism, they give rise to higher spin gravity. After a review on higher spin gravity, I will discuss its holography and its embedding in the string theory. Finally I will talk about some applications of higher spin symmetry, both in string theory and in QFT.

Higher spin symmetry in gravity and string

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Based on work with *Datta, Eberhardt, Gaberdiel, Lin, Longhi, Gopakumar, Peng, Theisen, Wang, and Zhang*

Higher spin symmetry

Spin	Particles	symmetry
1	photon	$\mathfrak{u}(1)$
$3/2$	gravitino	supersymmetry
2	graviton	diffeomorphism
> 2	higher spin particles	higher spin gauge symmetry

Massless higher spin particles $\Longrightarrow$ higher spin gauge symmetry

1. Can couple to gravity $\Longrightarrow$ higher spin gravity
   $\Longrightarrow$ higher spin holography
   (simple, weak-weak duality, no need for susy...)
2. Subsector of stringy symmetry
   $\Longrightarrow$ help understand string theory
3. Symmetry of interesting QFT systems (from string theory)
   topological string, AGT correspondence, BPS algebras, $\cdots$

Outline

Introduction

Higher spin gravity and holography

Stringy symmetry

Plane partitions

Summary

Brief history of Higher-spin theories

Massless higher spin particles in flat space

- Free spin-s field: *Fronsdal '78*

  totally-symmetric, double traceless, rank-s tensor $\phi_{\mu_1\mu_2\ldots\mu_s}$
  with gauge symmetry $\delta\phi_{\mu_1\mu_2\ldots\mu_s} = \partial_{\{\mu_1}\zeta_{\mu_2\ldots\mu_s\}}$ (ζ: traceless)

$$\text{DOF} = \frac{(D-4+2s)(D-5+s)!}{(D-4)!s!} = \begin{cases} D-2 & \text{for } s=1 \\ \frac{D(D-3)}{2} & \text{for } s=2 \\ 0 & \text{for } D=3 \\ 2 & \text{for } D=4 \end{cases}$$

- Interacting, non-minimal coupling to gravity

  Bengtsson Bengtsson Brink '83, Metsaev '91

No-go against coupling HS to gravity (minimally) in flat space

Theorem (Coleman-Mandula '67)

S-matrix of interacting theory in flat space cannot have higher-spin symmetry

Vasiliev's higher-spin theory

Vasiliev: Go to AdS or dS *Fradkin Vasiliev '87*

- No S-matrix in (A)dS
- $\ell_{\rm (A)dS}$ serves as expansion parameter to control higher derivatives expansions. (non-local $\longrightarrow$ quasi-local.)

Vasiliev system

1. lives in AdS_d or dS_d
2. one field per spin from $s = 2, 3, \ldots, \infty$
   Vasiliev '91,Sundborg '01, Witten '01, Mikhailov '02, Klebanov-Polyakov '02
3. Extension of Einstein gravity
   - graviton couples to an infinite tower of massless higher-spin particles
   - diffeomorphism coupled with higher-spin gauge symmetry
   - Extension of Riemannian geometry
4. No lagrangian, a system of equation of motions.

Simplest higher spin gravity

3D Einstein gravity + negative c.c. $\Leftrightarrow$ $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ Chern-Simon

From pure Einstein gravity to pure higher spin gravity:

$$\boxed{\mathfrak{sl}(2) \longrightarrow \mathfrak{sl}(N)}$$

Action: $S = S_{\text{CS}}[A] - S_{\text{CS}}[\tilde{A}]$

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr}[A dA + \frac{2}{3} A^3] \qquad \text{with} \quad A, \tilde{A} \in \mathfrak{sl}(N, \mathbb{R})$$

Translate to metric-like formalism

1. Dreibein $e = \frac{A - \tilde{A}}{2}$ and spin connection $\omega = \frac{A + \tilde{A}}{2}$
2. metric and spin-3 field

$$G_{\mu\nu} = \text{Tr}[e_\mu e_\nu] \qquad \varphi^{(3)}_{\mu\nu\rho} = \text{Tr}[e_{\{\mu} e_\nu e_{\rho\}}] \qquad \ldots$$

$\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Chern-Simons theory — Spectrum

Spectrum of $\mathfrak{sl}(N)$ Chern-Simons theory

1. Choose an $\mathfrak{sl}(2)$ subalgebra that corresponds to spin-2:

$$\text{spin-2}: \qquad \{L_1, \quad L_0, \quad L_{-1}\}$$

2. Decompose $\mathfrak{sl}(N)$ in terms of irreps of the gravitonal $\mathfrak{sl}(2)$

$$\text{spin-s} \;:\qquad \{W^{(s)}_m\} \qquad m = -s+1, \ldots, s-1$$

Principal embedding: 1 spin-s field for each $s = 2, ..., N$

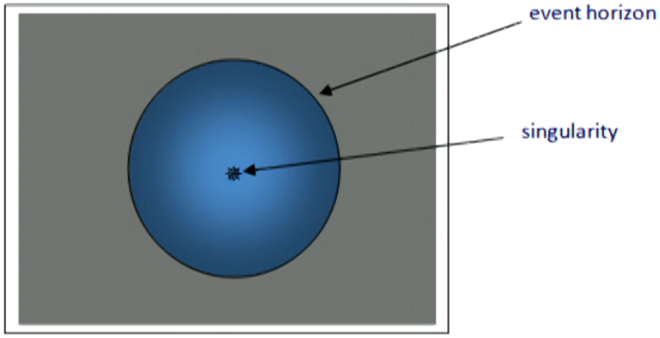
$$\begin{array}{ccccccc} W^{(4)}_3 & W^{(4)}_2 & W^{(4)}_1 & W^{(4)}_0 & W^{(4)}_{-1} & W^{(4)}_{-2} & W^{(4)}_{-3} \\ & W^{(3)}_2 & W^{(3)}_1 & W^{(3)}_0 & W^{(3)}_{-1} & W^{(3)}_{-2} & \\ & & L_1 & L_0 & L_{-1} & & \end{array}$$

$\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Chern-Simons theory — Classical solutions

1. No propagating degrees of freedom
2. Non-trivial classical solutions such as black holes

Black hole in Einstein gravity:

singlularity behind horizon
(cosmic censorship)



Model of a black hole

3. Problem of defining black holes in higher spin gravity:

   Singularity and horizon are not gauge-invariant concepts.

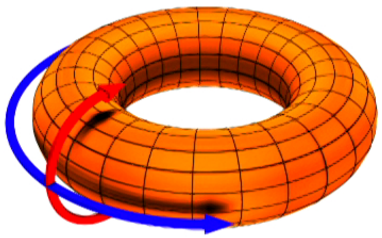
   (metric is coupled to higher-spin fields)

4. Question: how to define black hole in higher spin gravity"

Higher-spin black hole and conical surplus

Cosmic censorship = smoothness condition

gauge theory: $\mathrm{Hol}_{\mathcal{C}}(A) \equiv \mathcal{P}e^{\oint_{\mathcal{C}} A}$ is trivial around a shrinkable cycle.



- Shrinkable cycle is time $\Longrightarrow$ higher spin black holes

  Gutperle Kraus '11
- Shrinkable cycle is space $\Longrightarrow$ Conical surplus *Castro et.al. '11*

  (higer spin version of thermal AdS_3)

- $SL(2, \mathbb{Z})$-family of solutions

  WL Lin Wang '13

Vasiliev higher spin gravity

Vasiliev theory in AdS$_3$:

start from $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Chern-Simons

1. $\mathfrak{sl}(N) \longrightarrow \mathfrak{hs}[\lambda]$ ($\lambda \in \mathbb{R}$)
   - hs$[\lambda]$: an ∞-dim Lie algebra with generators of spin $2, \ldots, \infty$
   - hs$[\lambda = N]$ truncates to $\mathfrak{sl}(N)$
2. couple with a complex massive scalar $\Longrightarrow$ propagating DOF

Vasiliev theory in AdS$_{d\geq 4}$

1. Scalar cannot be decoupled
2. Based on frame-like fields $(e^{a_1 a_2 \ldots a_{s-1}}, w^{a_1 a_2 \ldots a_{s-1}, b_1 b_2 \ldots b_t})$
3. For $d \geq 4$, no Lagrangian yet.

How to quantize the theory?

Higher-spin holography

Understand quantum higher spin gravity via its holographic dual

$$\text{gravity with higher spin gauge symmetry in AdS}_{d+1}$$
$$\Updownarrow$$
$$\text{CFT}_d \text{ with high spin currents}$$

Examples of higher spin holography

- Vasiliev in AdS_4 = O(N) model *Klebanov Polyakov '02* *Sezgin Sundell '02*
- Vasiliev in AdS_3 = $\mathcal{W}_{N,k}$ minimal model $(=\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}})$ *Gaberdiel Gopakumar '10*

Features of higher-spin holography

1. Weak/Weak duality

$$\left(\frac{R}{\ell_p}\right)^4 \sim N \qquad \left(\frac{R}{\ell_s}\right)^4 \sim \lambda = (g_{YM})^2 N$$

Traditional Gauge/Gravity correspondence: Strong/Weak duality

$$R \gg \ell_s \gg \ell_p \quad \Longrightarrow \quad \boxed{N \to \infty \qquad \lambda \gg 1}$$

Higher-Spin theory: Weak/Weak duality

$$\ell_s \gg R \gg \ell_p \quad \Longrightarrow \quad \boxed{N \to \infty \qquad \lambda \ll 1}$$

2. Supersymmetry is not required.

3. CFT dual is always free *Maldacena-Zhiboedov '11, Stanev '13*
   - Except for AdS_3/CFT_2
   - AdS version of Coleman-Mandula

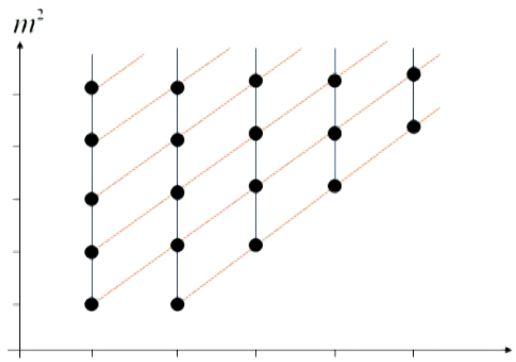
Potential problems and embedding in string theory

1. Not clear how to quantize Vasiliev gravity (e.g. no Lagrangian)
2. Still some unresolved questions (light states, classical limit, non-locality)
3. If one can embed Vasiliev's gravity into string theory (a UV-complete theory), can understand it from string theory

tensionless limit and stringy symmetry

String theory has infinite number of massive higher spin particles

$$m \sim \frac{1}{\ell_s}$$



Tensionless limit: $\frac{\ell_s}{\ell_{\mathrm{AdS}}} \to \infty$ (quantum: 1)

massive higher spin particle $\Longrightarrow$ massless $\Longrightarrow$ stringy symmetry

- subalgebra: Vasiliev higher spin symmetry (one per spin)

(from Leading Regge trajectory)

Vasiliev '91, Sundborg '01, Witten '01, Mikhailov '02, Klebanov-Polyakov '02

Tensionless string in AdS_3

- Symmetry enhancement from bulk AdS_3 to boundary CFT_2

	Bulk Isometry		Bndy global symmetry
Einstein	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	$\xrightarrow{\text{Brown Henneaux '86}}$	Virasoro $\oplus$ Virasoro
	$\downarrow$		$\downarrow$
Higher-Spin	$\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$	$\xrightarrow[\text{Henneaux Rey '10}]{\text{Campoleoni et.al. '10}}$	$\mathcal{W}_\infty[\lambda] \oplus \mathcal{W}_\infty[\lambda]$

Virasoro

spin-2 $\quad \ldots \quad L_{-4} \quad L_{-3} \quad L_{-2} \quad L_{-1} \quad L_0 \quad L_1 \quad L_2 \quad L_3 \quad L_4 \quad ..$

Tensionless string in AdS_3

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Higher-Spin	$\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$	$\xrightarrow[\text{Henneaux Rey '10}]{\text{Campoleoni et.al. '10}}$	$\mathcal{W}_\infty[\lambda] \oplus \mathcal{W}_\infty[\lambda]$

$$\mathfrak{hs}[\lambda]$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
spin-5		X_{-4}	X_{-3}	X_{-2}	X_{-1}	X_0	X_1	X_2	X_3	X_4
spin-4			U_{-3}	U_{-2}	U_{-1}	U_0	U_1	U_2	U_3	
spin-3				W_{-2}	W_{-1}	W_0	W_1	W_2		
spin-2					L_{-1}	L_0	L_1			

Tensionless string in AdS_3

- Symmetry enhancement from bulk AdS_3 to boundary CFT_2

	Bulk Isometry		Bndy global symmetry
Einstein	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	→ *Brown Henneaux '86*	Virasoro $\oplus$ Virasoro
	↓		↓
Higher-Spin	$\mathfrak{hs}[\lambda] \oplus \mathfrak{hs}[\lambda]$	→ *Campoleoni et.al. '10*, *Henneaux Rey '10*	$\mathcal{W}_\infty[\lambda] \oplus \mathcal{W}_\infty[\lambda]$

- In tensionless limit, the worldsheet theory of string in $\mathrm{AdS}_3 \times S^3 \times T^4$ is a symmetric orbifold of T^4.

Gaberdiel Gopakumar '18, Eberhardt Gaberdiel Gopakumar '19

- Stringy symmetry at AdS_3 $\sim$ chiral algebra of $\mathrm{Sym}^N(T^4)$
  - What is the mathematical description?

Stringy symmetry $\gg$ higher-spin symmetry $\gg$ conformal symmetry

Size of stringy symmetry

Virasoro ($\sim$ partition) *Euler*

$$\prod_{k=1}^{\infty}\frac{1}{1-q^k}=\sum_{n=0}p(n)q^n=1+q+2\,q^2+3\,q^3+5\,q^4+7\,q^5+11\,q^6+\cdots$$

$$p(n)\sim\frac{1}{n}\cdot\exp\left(\sqrt{\frac{2}{3}}\,\pi\,\sqrt{n}\right)$$ *Hardy Ramanujan '18*

higher spin symmetry ($\sim$ plane partition) *MacMahon*

$$\prod_{k=1}^{\infty}\frac{1}{(1-q^k)^k}=\sum_{n=0}M(n)q^n=1+q+3\,q^2+6\,q^3+13\,q^4+24\,q^5+48\,q^6+\cdots$$

$$M(n)\sim n^{-\frac{25}{36}}\cdot\exp\left(\frac{3\zeta(3)^{\frac{1}{3}}}{2^{\frac{2}{3}}}n^{\frac{2}{3}}\right)$$ *Wright '31*

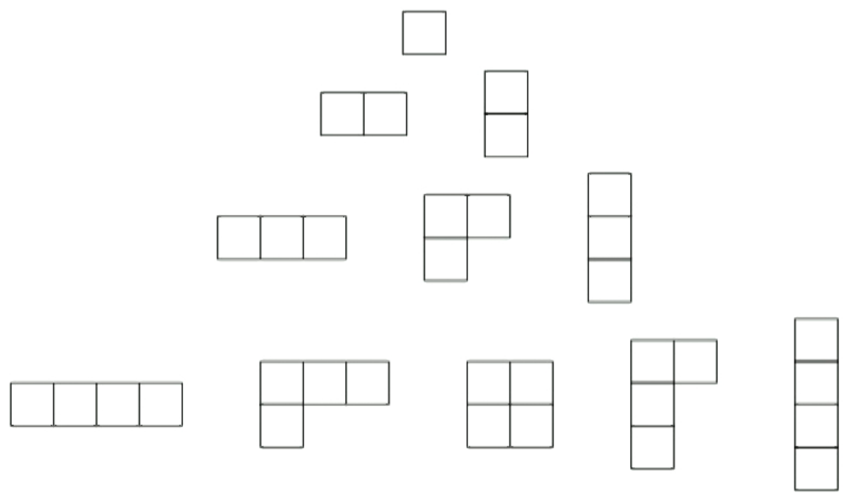
stringy symmetry ($\sim$ double partition)

$$\prod_{k=1}^{\infty}\frac{1}{(1-q^k)^{p(k)}}=\sum_{n=0}D(n)q^n=1+q+3\,q^2+6\,q^3+14\,q^4+27\,q^5+58\,q^6+\cdots$$

$$D(n)\sim n\cdot\exp\left(\frac{\pi^2}{6}n\right)$$ *Kaneiwa '79*

Partition of integer from Young diagrams

Partition of n can be represented by Young diagrams with n boxes.

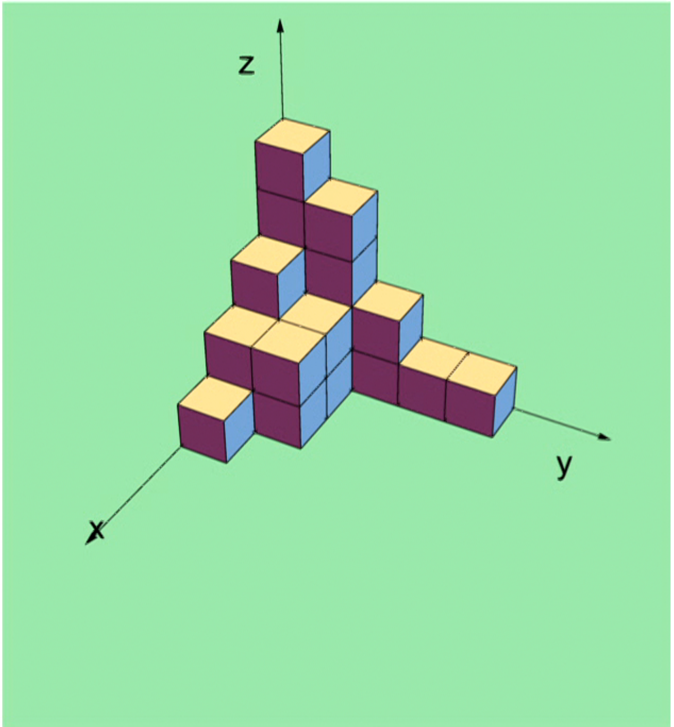


n	Young diagrams with n boxes	$p(n)$
0	.	1
1		1
2		2
3		3
4		5

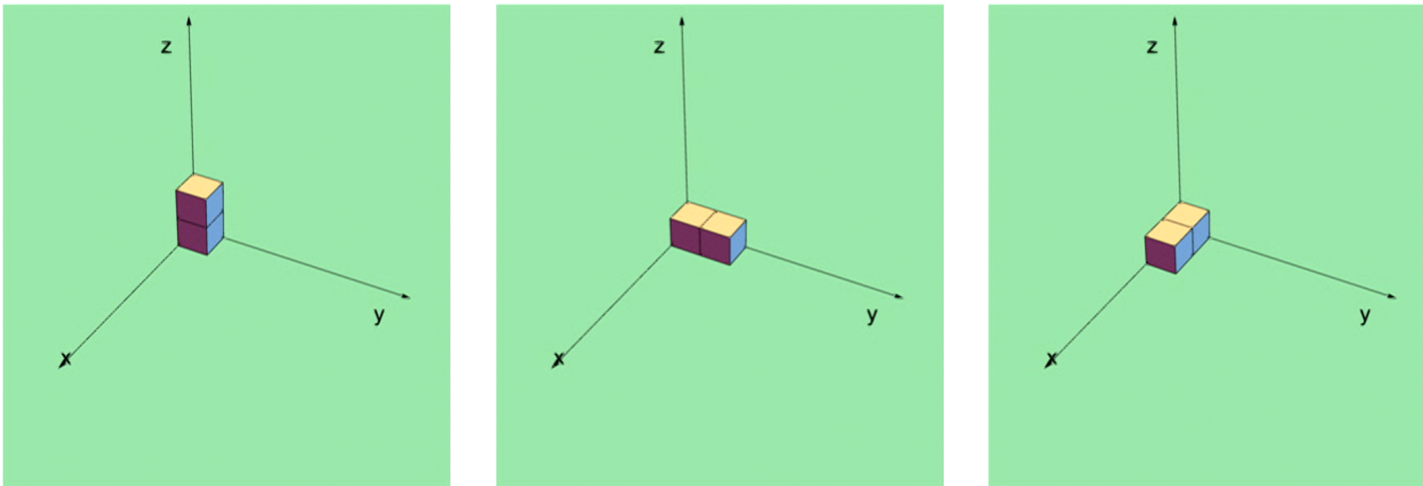
$$\sum_{n=0} p(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + \cdots$$

Plane partition via box stacking

Plane Partition of n can be represented by 3D stacking of n boxes



Stacking 2 boxes



$$1 + q + 3\,q^2 \cdots$$

plane partition

plane partition *MacMahon '12*

$$\sum_{n=0} M(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + \cdots$$

$$M(n) \sim n^{-\frac{25}{36}} \cdot \exp\left(\frac{3\zeta(3)^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}} \right)$$ *Wright '31*

partition, plane partition, solid partition

Partition

Euler

$$\sum_{n=0} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 + \cdots$$

$$p(n) \sim \tfrac{1}{n} \cdot \exp\left(\sqrt{\tfrac{2}{3}}\,\pi\,\sqrt{n}\right)$$

Hardy Ramanujan '18

plane partition

MacMahon '12

$$\sum_{n=0} M(n)q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + \cdots$$

$$M(n) \sim n^{-\frac{25}{36}} \cdot \exp\left(\frac{3\zeta(3)^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}}\right)$$

Wright '31

Solid partition

$$\sum_{n=0} S(n)q^n = \qquad = 1 + q + 4\,q^2 + 10\,q^3 + 26\,q^4 + 59\,q^5 + 140\,q^6 + \cdots$$

$$S(n) \sim \exp\left(c\,n^{\frac{3}{4}}\right)$$

Bhatia '97

MacMahon's conjecture on d-dim partition

$$\sum_{\Lambda_d} q^{|\Lambda_d|} = \sum_{n=0} P^{(d)}(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{\binom{k+d-3}{d-2}}}$$

partition $(d=2)$: $$\sum_{n=0} p(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} \qquad \checkmark$$

plane partition $(d=3)$: $$\sum_{n=0} M(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} \qquad \checkmark$$

solid partition $(d=4)$: $$\sum_{n=0} S(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{\frac{k(k+1)}{2}}}$$

$$= 1 + q + 4\,q^2 + 10\,q^3 + 26\,q^4 + 59\,q^5 + 141\,q^6 + \cdots$$

compute by hand

$$\sum_{n=0} S(n) q^n = 1 + q + 4\,q^2 + 10\,q^3 + 26\,q^4 + 59\,q^5 + 140\,q^6 + \cdots$$

Generating function for $d \geq 4$ dimensional partition is still unknown.

Asymptotics of d-dimensional partition counting

Partition

$$p(n) \sim \frac{1}{n} \cdot \exp\left(\sqrt{\frac{2}{3}}\,\pi\,\sqrt{n}\right)$$

Hardy Ramanujan '18

plane partition

$$M(n) \sim n^{-\frac{25}{36}} \cdot \exp\left(\frac{3\zeta(3)^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}}\right)$$

Wright '31

d-dimensional partition

$$P^{(d)}(n) \sim \exp\left(c\, n^{\frac{d-1}{d}}\right)$$

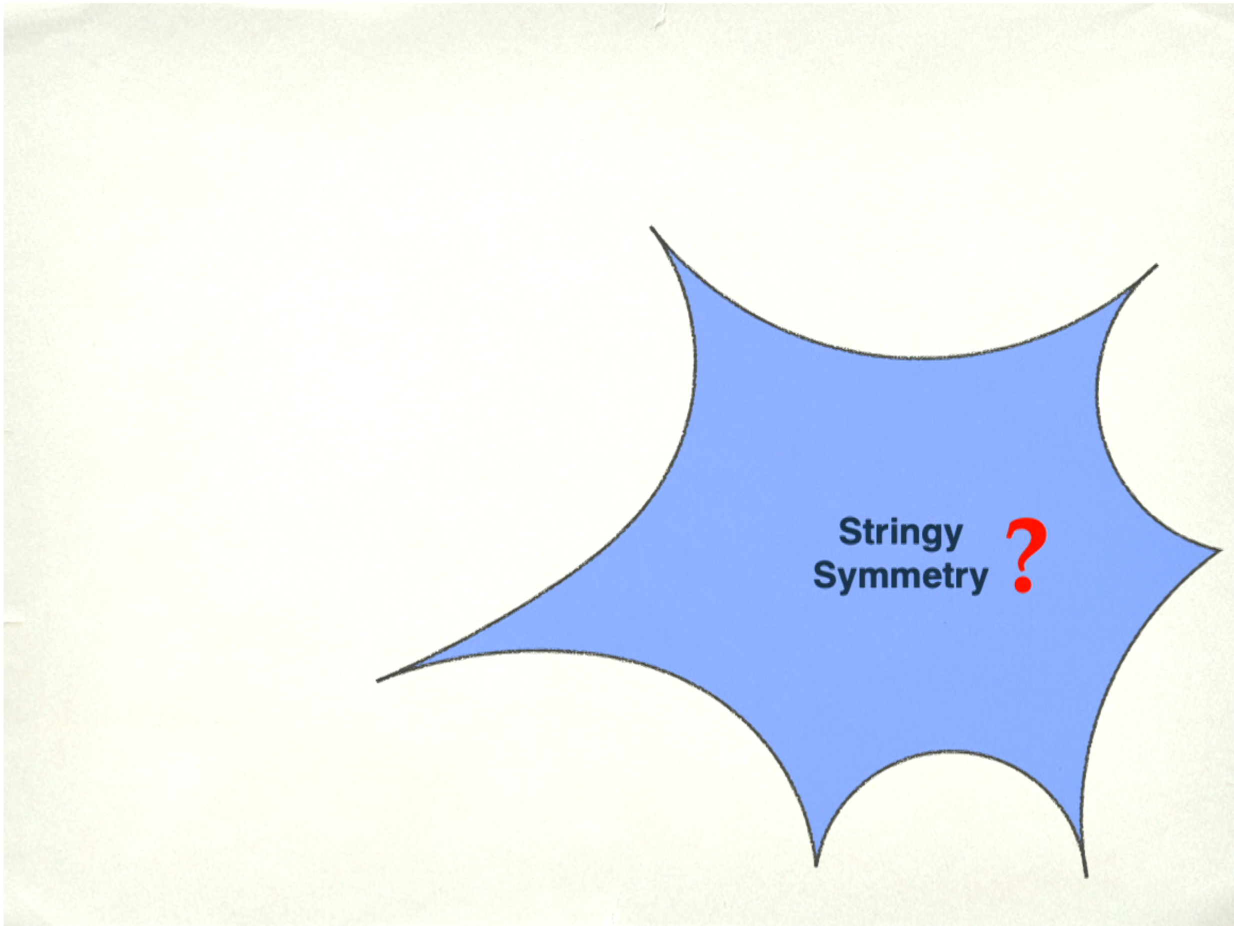
Bhatia '97

$$\text{growth of stringy symmetry} \sim \exp(c\, n) > \exp\left(c\, n^{\frac{d-1}{d}}\right)$$

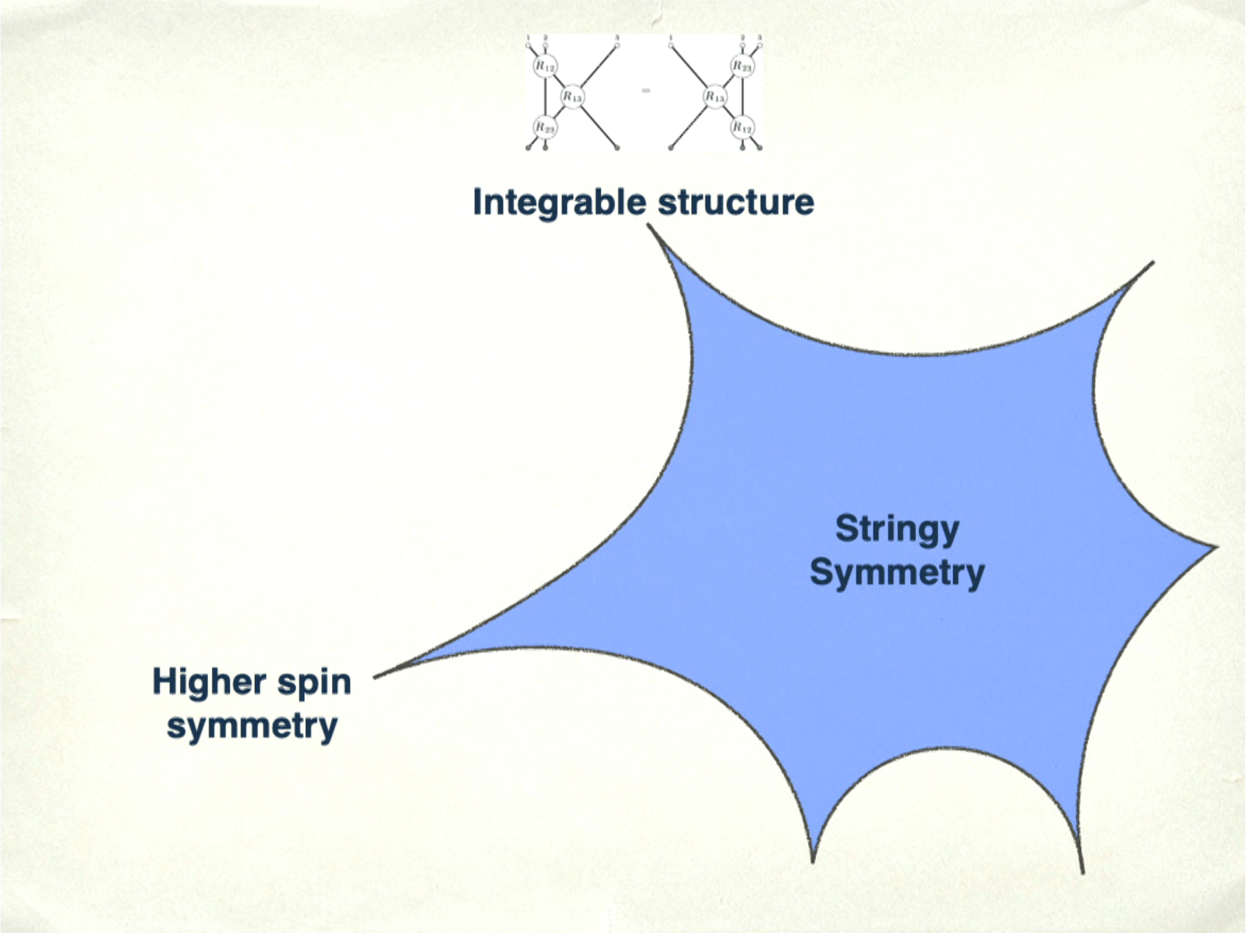
Embed higher-spin theory in string theory

	Bulk		Boundary
Higher spin	Vasiliev $\mathfrak{shs}_2[\lambda]$	⟷	Wolf coset
	$\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$		$\frac{\mathfrak{su}(N+2)_k \oplus \mathfrak{so}(4N+4)_1}{\mathfrak{su}(N)_{k+2} \otimes \mathfrak{u}(1)_k} \otimes \mathfrak{u}(1)$
	↓ $\lambda \to 0$		↓ $k \to \infty$
Free higher spin	Vasiliev $\mathfrak{shs}_2[0]$	⟷	$(T^4)^{\otimes(N+1)}/U(N)$
	$\mathrm{AdS}_3 \times S^3 \times T^4$		
	↓		↓
String	tensionless string theory	⟷	$(T^4)^{\otimes(N+1)}/S_{N+1}$
	$\mathrm{AdS}_3 \times S^3 \times T^4$		

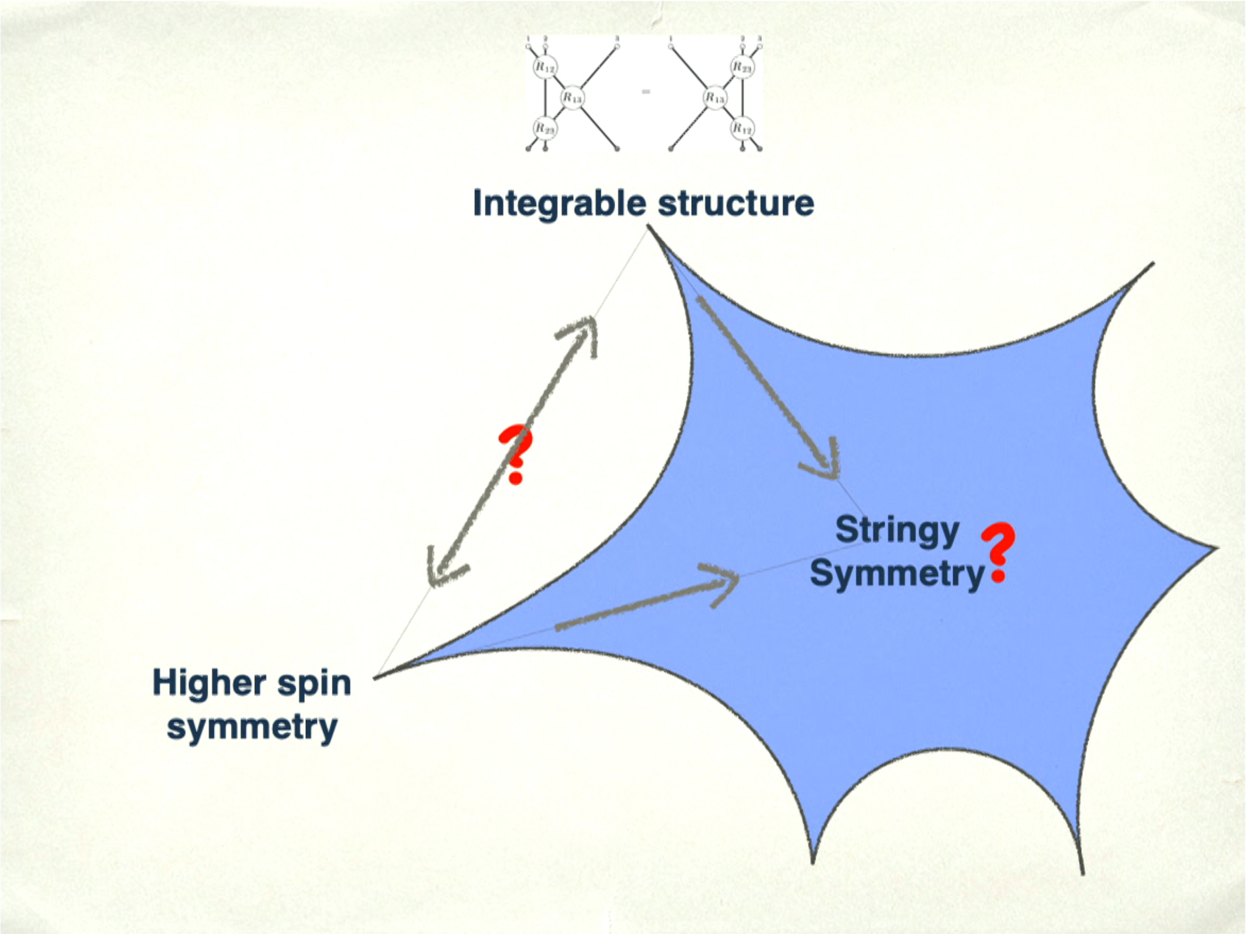
How to characterize stringy symmetry mathematically?



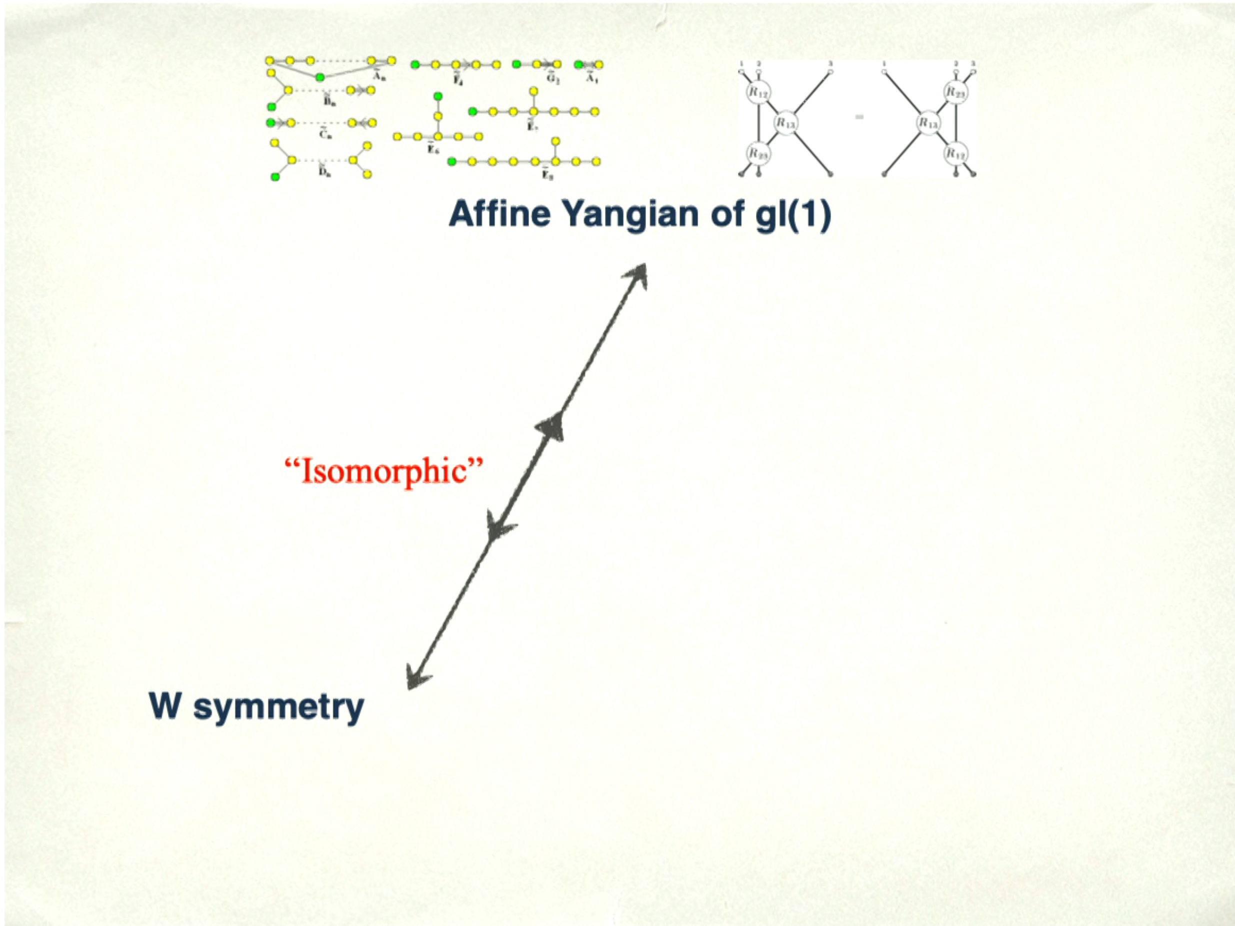
Different manifestation of stringy symmetry



Different manifestation of stringy symmetry



A concrete relation between HS and integrability



Regrouping the modes

$$W^{(s)}(z) = \sum_{n\in\mathbb{Z}} \frac{W_n^{(s)}}{z^{n+s}} \qquad s = 1, 2, 3, \ldots$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
spin-5	$\ldots$	X_{-3}	X_{-2}	$X_{-1}\sim e_4$	$X_0\sim \psi_5$	$X_1\sim f_4$	X_2	X_3	X_4
spin-4	$\ldots$	U_{-3}	U_{-2}	$U_{-1}\sim e_3$	$U_0\sim \psi_4$	$U_1\sim f_3$	U_2	U_3	U_4
spin-3	$\ldots$	W_{-3}	W_{-2}	$W_{-1}\sim e_2$	$W_0\sim \psi_3$	$W_1\sim f_2$	W_2	W_3	W_4
spin-2	$\ldots$	L_{-3}	L_{-2}	$L_{-1}\sim e_1$	$L_0\sim \psi_2$	$L_1\sim f_1$	L_2	L_3	L_4
spin-1	$\ldots$	J_{-3}	J_{-2}	$J_{-1}\sim e_0$	$J_0\sim \psi_1$	$J_1\sim f_0$	J_2	J_3	J_4

affine Yangian generators

$$e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \qquad \psi(z) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}$$

(Wedge subalgebra: removing e_0 and f_0)

Affine Yangian of $\mathfrak{gl}_1$

Def: Associative algebra with generators e_j, f_j and ψ_j, $j = 0, 1, \dots$

- Generators

$$\psi(z) = 1 + (h_1 h_2 h_3) \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \qquad e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}$$

- Parameters (h_1, h_2, h_3) with $h_1 + h_2 + h_3 = 0$
- One $\mathcal{S}_3$ invariant function $\varphi_3(z) = \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$
- Defining relations

$$[e(z)\,, f(w)] = -\frac{1}{h_1 h_2 h_3} \frac{\psi(z) - \psi(w)}{z - w}$$

$$\psi(z)\, e(w) \sim \varphi_3(z-w)\, e(w)\, \psi(z) \qquad \psi(z)\, f(w) \sim \varphi_3(w-z)\, f(w)\, \psi(z)$$

$$e(z)\, e(w) \sim \varphi_3(z-w)\, e(w)\, e(z) \qquad f(z)\, f(w) \sim \varphi_3(w-z)\, f(w)\, f(z)$$

- Initial conditions

$$[\psi_{0,1}, e_j] = 0 \quad [\psi_2, e_j] = 2e_j \quad [\psi_{0,1}, f_j] = 0 \quad [\psi_2, f_j] = -2f_j$$

- Serre relation

$$\mathrm{Sym}_{(j_1,j_2,j_3)}\, [e_{j_1}, [e_{j_2}, e_{j_3+1}]] = 0 \qquad \mathrm{Sym}_{(j_1,j_2,j_3)}\, [f_{j_1}, [f_{j_2}, f_{j_3+1}]] = 0$$

Schiffmann Vasserot '12 Maulik Okounkov '12 Tsymbaliuk '14

Affine Yangian of $\mathfrak{gl}_1$

Def: Associative algebra with generators e_j, f_j and ψ_j, $j = 0, 1, \ldots$

- Generators

$$\psi(z) = 1 + (h_1 h_2 h_3) \sum_{j=0}^{\infty} \frac{\psi_j}{z^{j+1}} \qquad e(z) = \sum_{j=0}^{\infty} \frac{e_j}{z^{j+1}} \qquad f(z) = \sum_{j=0}^{\infty} \frac{f_j}{z^{j+1}}$$

- Parameters (h_1, h_2, h_3) with $h_1 + h_2 + h_3 = 0$
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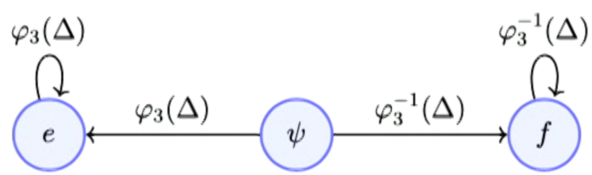
$$[e(z)\,, f(w)] = -\frac{1}{h_1 h_2 h_3} \frac{\psi(z) - \psi(w)}{z - w}$$

$$\psi(z)\, e(w) \sim \varphi_3(z-w)\, e(w)\, \psi(z) \qquad \psi(z)\, f(w) \sim \varphi_3(w-z)\, f(w)\, \psi(z)$$

$$e(z)\, e(w) \sim \varphi_3(z-w)\, e(w)\, e(z) \qquad f(z)\, f(w) \sim \varphi_3(w-z)\, f(w)\, f(z)$$



Schiffmann Vasserot '12 Maulik Okounkov '12 Tsymbaliuk '14

W algebra and affine Yangian

$$\mathcal{Y}[\widehat{\mathfrak{gl}_1}] \cong \mathrm{UEA}[\mathcal{W}_{1+\infty}[\lambda]]$$

Procházka '15

Gaberdiel Gopakumar Li Peng '17

for q-version $\mathcal{U}[\widehat{\widehat{\mathfrak{gl}_1}}] \cong \mathrm{UEA}[\text{q-}\mathcal{W}_{1+\infty}[\lambda]]$

Miki '07

Feigin Jimbo Miwa Mukhin '10-11

Advantages of affine Yangian over $\mathcal{W}_\infty$

1. number of generators
   - $\mathcal{W}_\infty$: ∞

$$J(z)\,,T(z)\,,W^{(3)}(z)\,,W^{(4)}(z)\,\ldots$$

   - affine Yangian of $\mathfrak{gl}_1$: only 3

$$\psi(z)\,,e(z)\,,f(z)$$

2. Defining relations
   - $\mathcal{W}_\infty$:

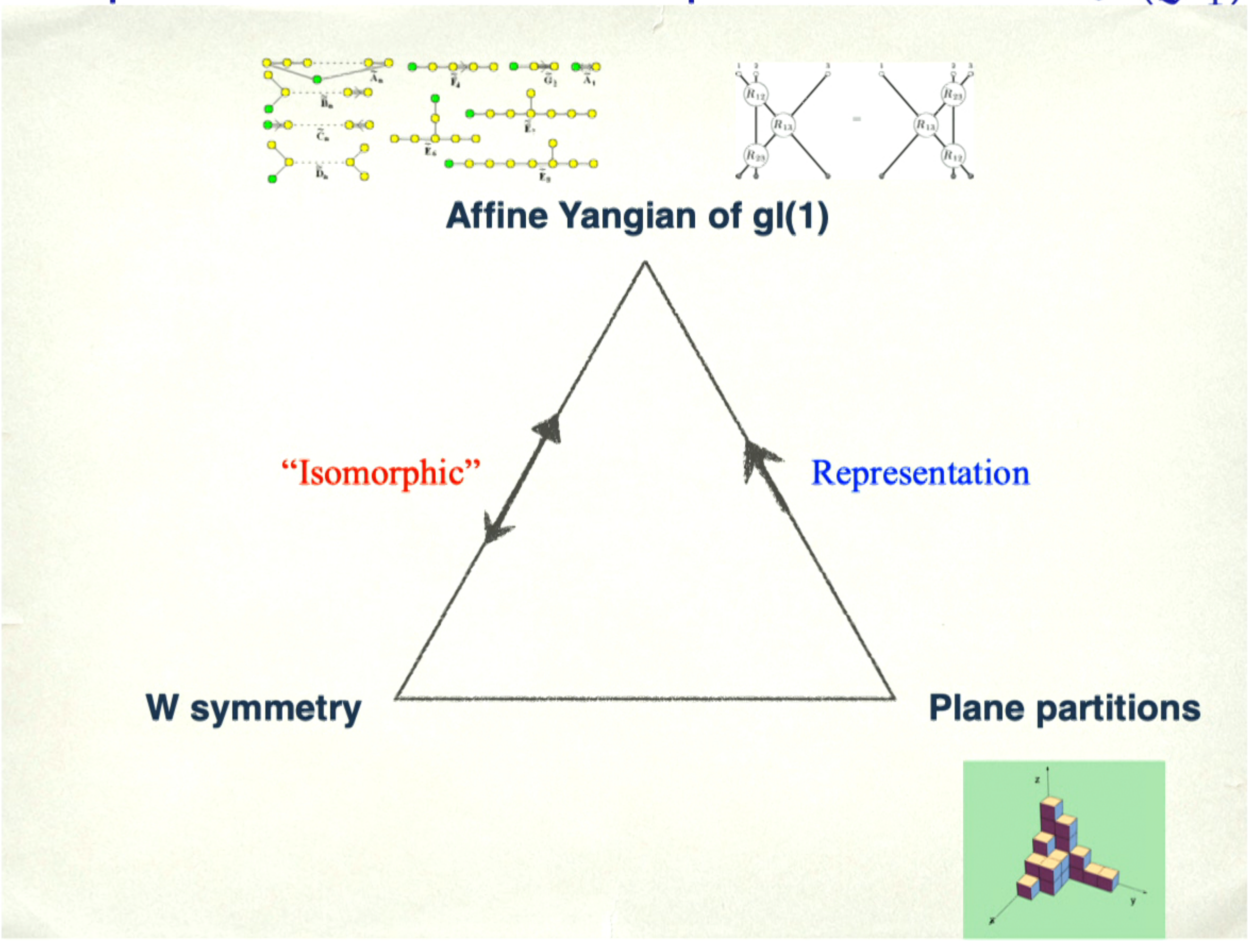
non-linear, fixed order by order by Jacobi-identities

   - affine Yangian of $\mathfrak{gl}_1$:

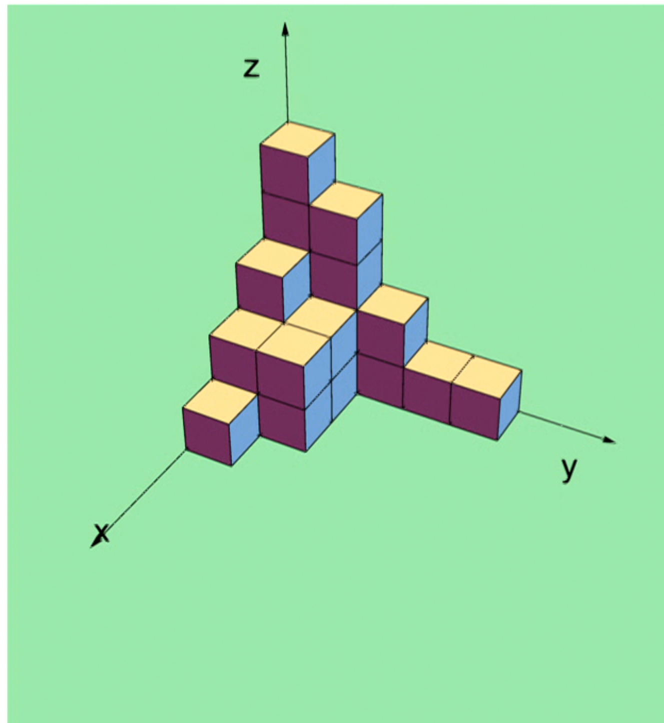
linear, given explicitly

3. $\mathcal{S}_3$ invariance
   - $\mathcal{W}_\infty$: Hidden
   - affine Yangian of $\mathfrak{gl}_1$: manifest
4. Plane partition representation

Plane partitions are faithful representations of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$

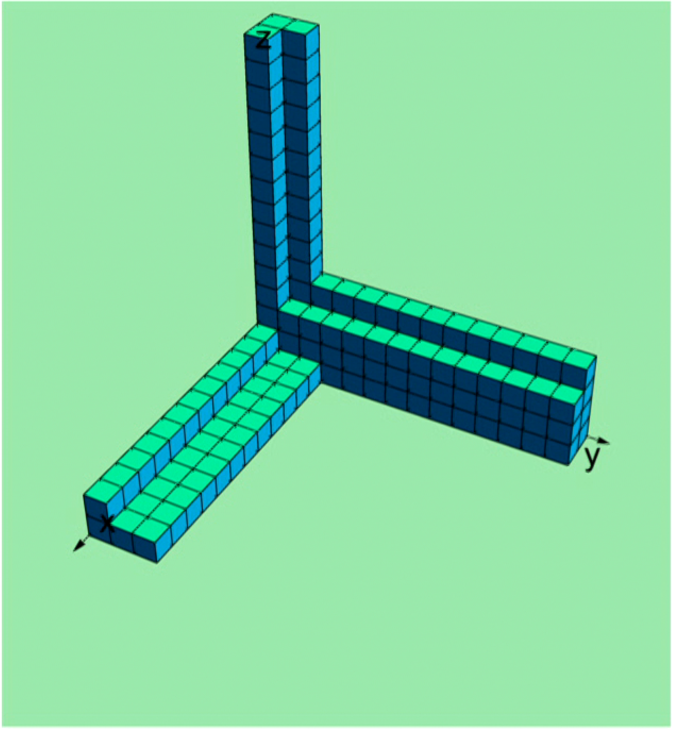


A descendant of vacuum module



Plane partition with non-trivial asymptotics

Ground state of $(\Lambda_x, \Lambda_y, \Lambda_z)$



Action of $\hat{\mathcal{Y}}(\mathfrak{gl}_1)$ on a plane partition

- $\psi(z)$ acts diagonally *Tsymbaliuk '14, Prochazka '15*

$$\psi(z)|\Lambda\rangle = \psi_\Lambda(z)|\Lambda\rangle$$

$$\psi_\Lambda(z) \equiv \left(1 + \frac{\psi_0 \sigma_3}{z}\right) \prod_{\square \in (\Lambda)} \varphi_3(z - h(\square))$$

$$h(\square) = h_1 x(\square) + h_2 y(\square) + h_3 z(\square)$$

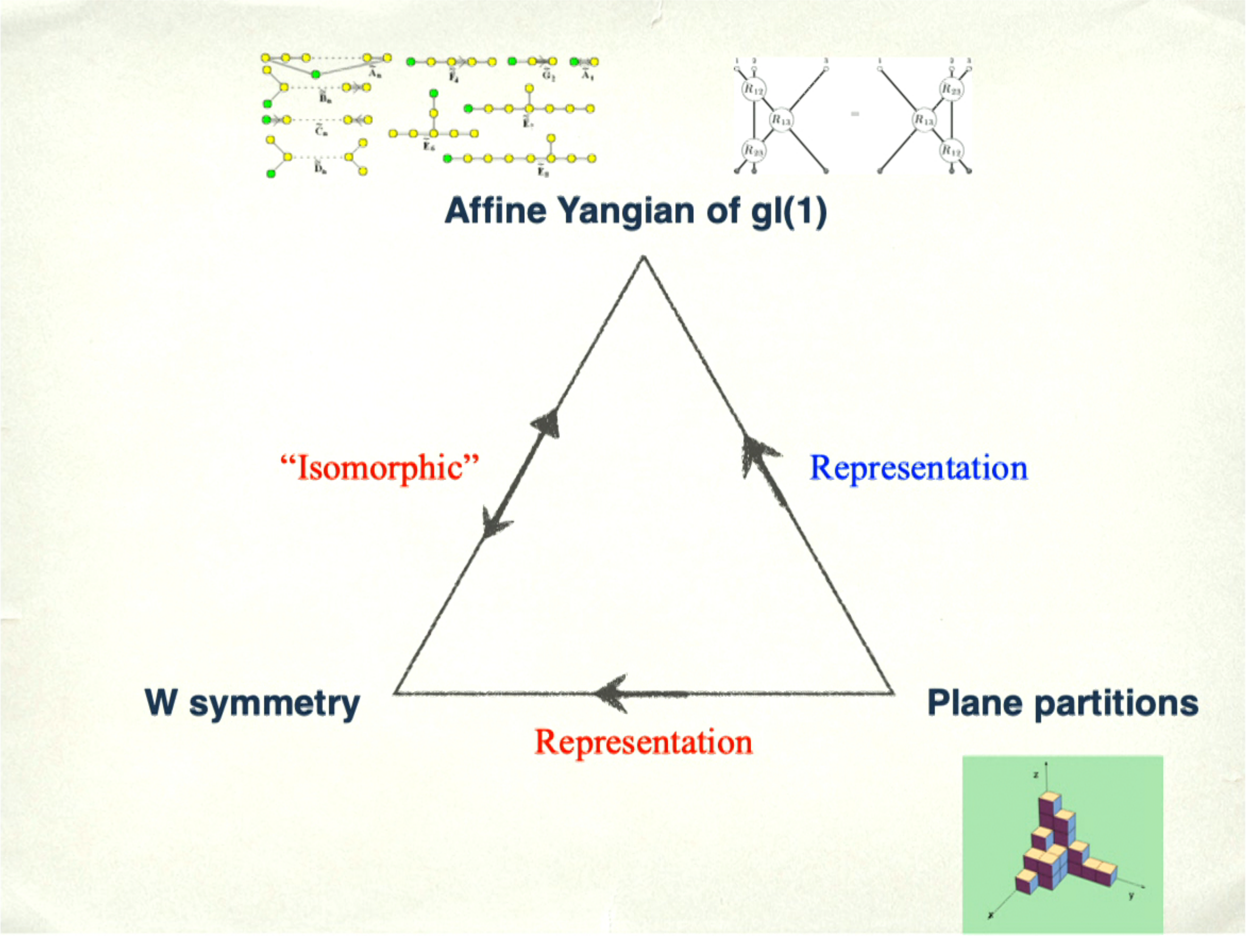
- $e(z)$ adds one box

$$e(z)|\Lambda\rangle = \sum_{\square \in \mathrm{Add}(\Lambda)} \frac{\left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)} \psi_\Lambda(w)\right]^{\frac{1}{2}}}{z - h(\square)} |\Lambda + \square\rangle$$

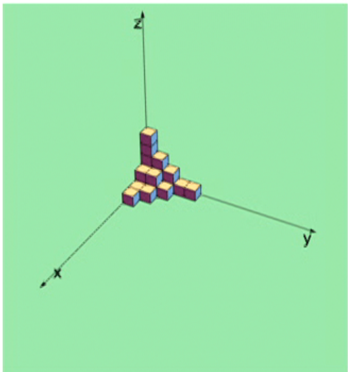
- $f(z)$ removes one box

$$f(z)|\Lambda\rangle = \sum_{\square \in \mathrm{Rem}(\Lambda)} \frac{\left[-\frac{1}{\sigma_3}\mathrm{Res}_{w=h(\square)} \psi_\Lambda(w)\right]^{\frac{1}{2}}}{z - h(\square)} |\Lambda - \square\rangle$$

Bosonic triangle *Datta Gaberdiel WL Peng '16, Gaberdiel Gopakumar WL Peng '17*

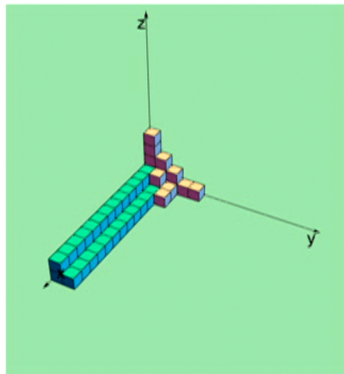


Plane partition as representations of W



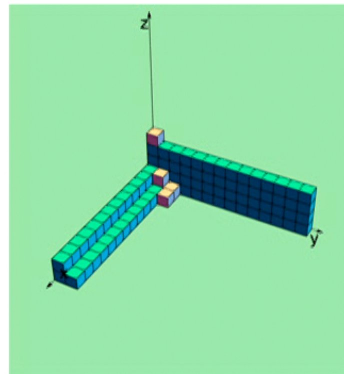
Trivial b.c.

vacuum



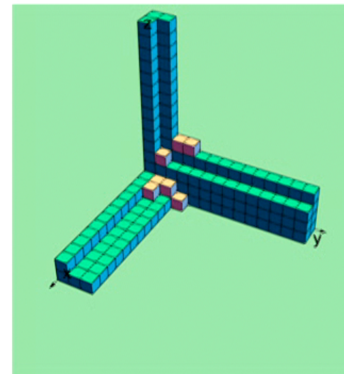
$(\Lambda_x; 0) = (\Lambda; 0)$

perturbative in Vasiliev



$(\Lambda_x; \Lambda_y) = (\Lambda_+; \Lambda_-)$

non-perturbative in Vasiliev



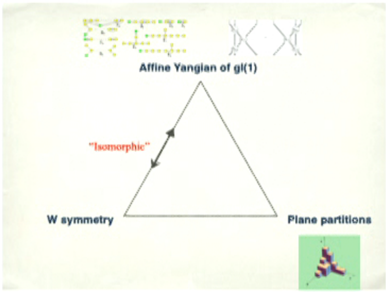
$(\Lambda_x; \Lambda_y; \Lambda_z)$

new representation

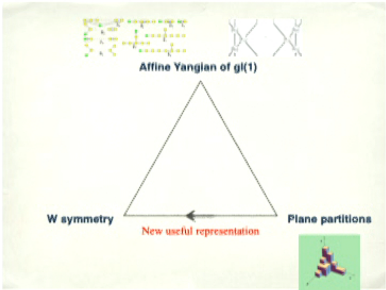
character of $\mathcal{W}_{1+\infty}$= generating function of plane partition

Feigin Jimbo Miwa Mukhin '10-11

Application



- Make S_3 symmetry in $\mathcal{W}$ CFT manifest



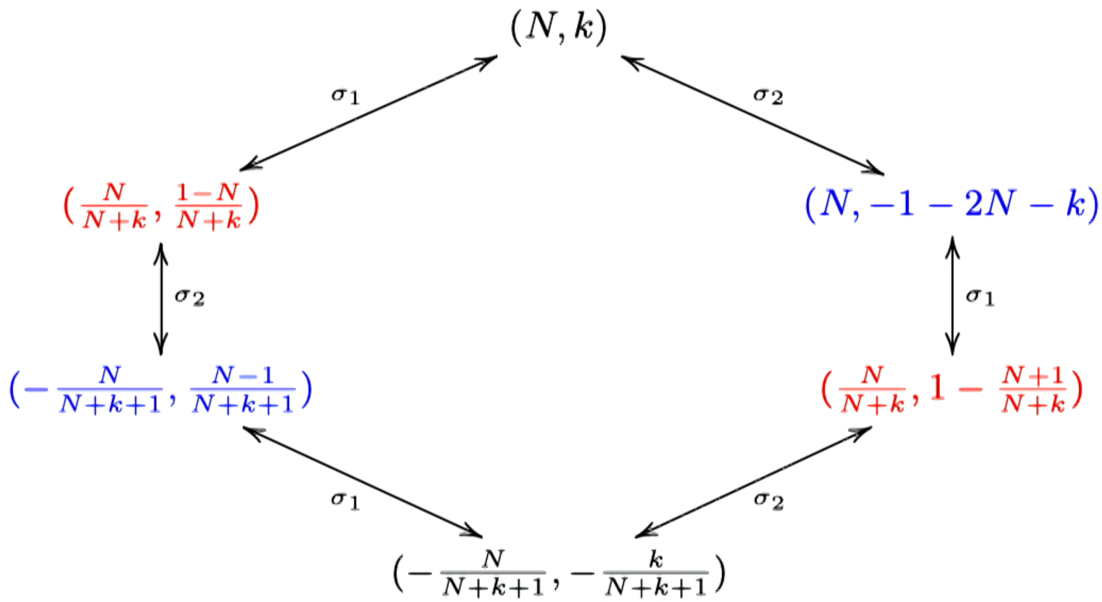
- Character computation more transparent

$\mathcal{S}_3$ action on $\mathcal{W}_{N,k}$ coset

$\mathcal{W}_{N,k}$ coset

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$$

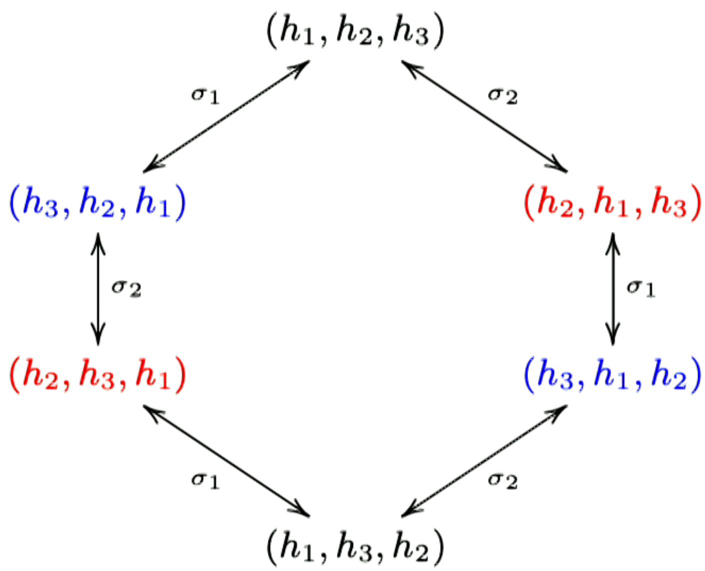
had hidden $\mathcal{S}_3$



$\mathcal{S}_3$ symmetry in $\mathcal{Y}[\mathfrak{gl}_1]$

- $\mathcal{S}_3$ symmetry in $\mathcal{W}_\infty$CFT is highly non-trivial (UV-IR)
  - hard to check/prove — *Gaberdiel Gopakumar '12, Linshaw '17*
- Manifest in $\mathcal{Y}[\widehat{\mathfrak{gl}_1}]$

Under $\mathcal{S}_3$ transformation on (N, k)

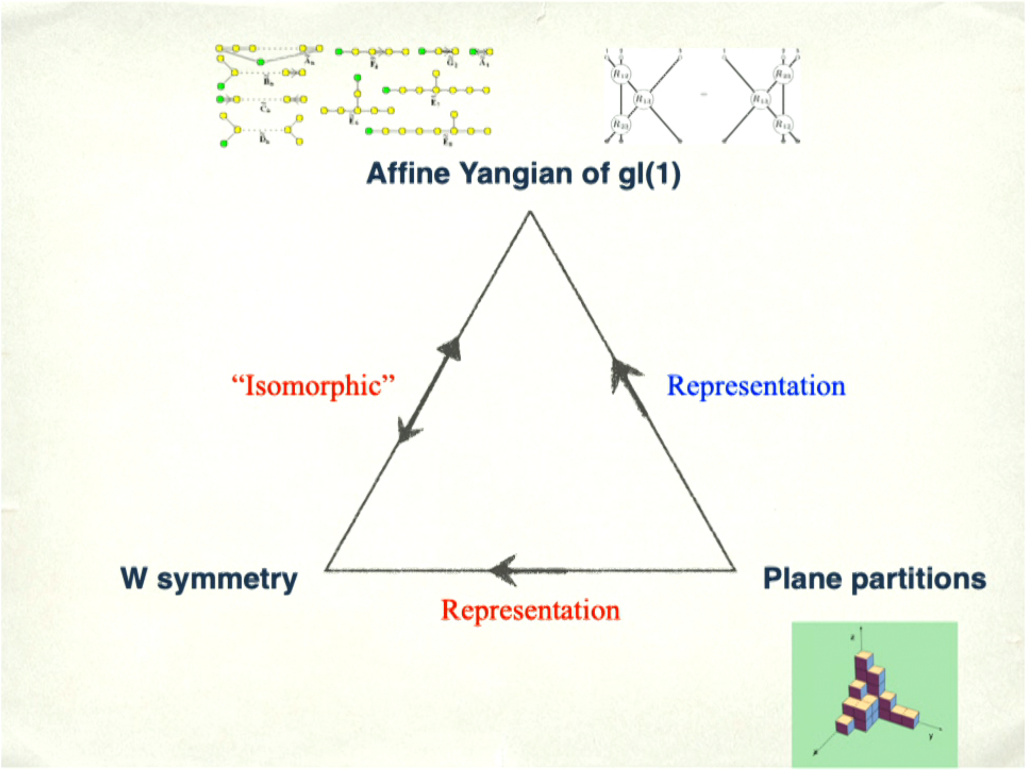


Procházka '15, *Gaberdiel Gopakumar Li Peng '17*

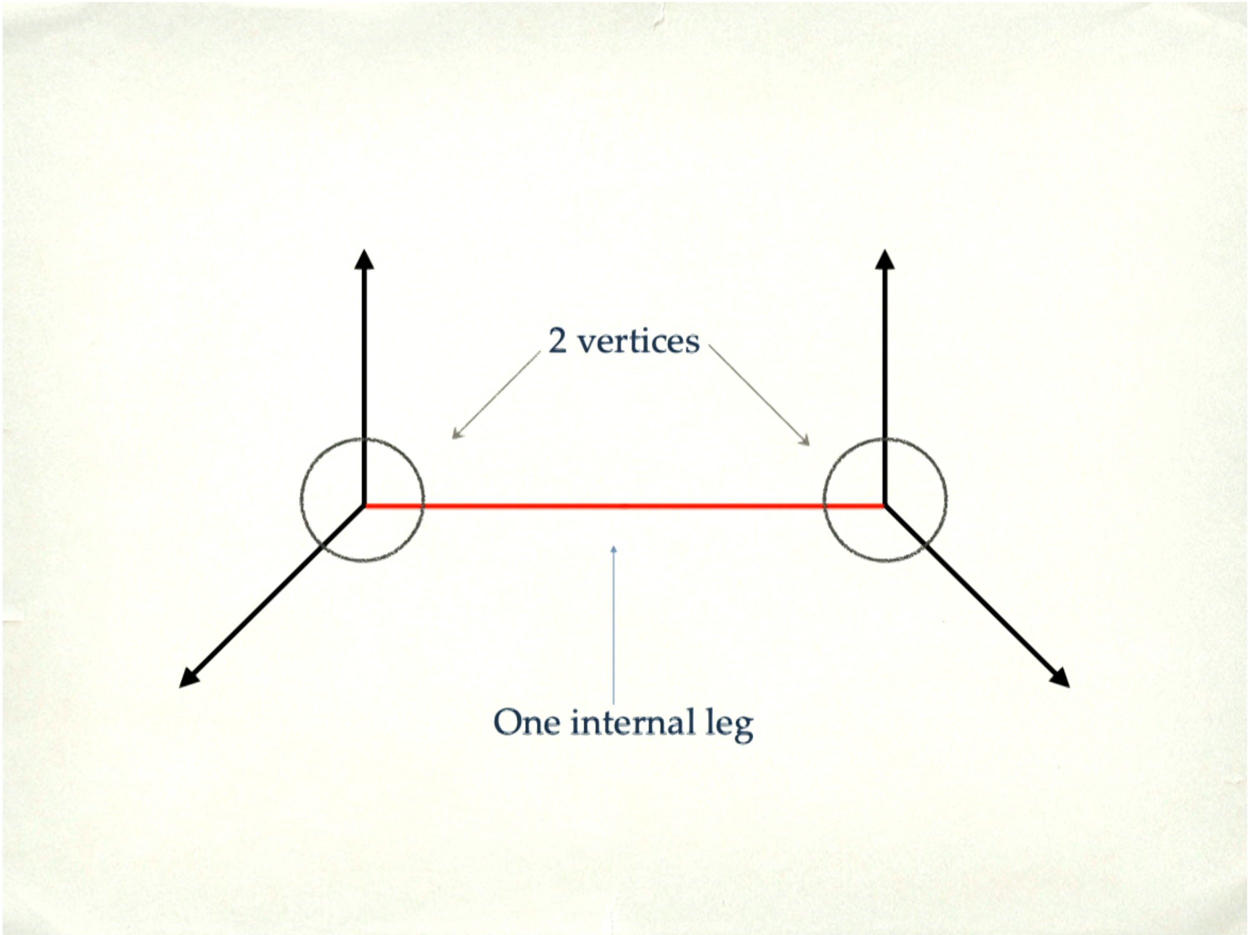
Advantage of affine Yangian

Easy to study

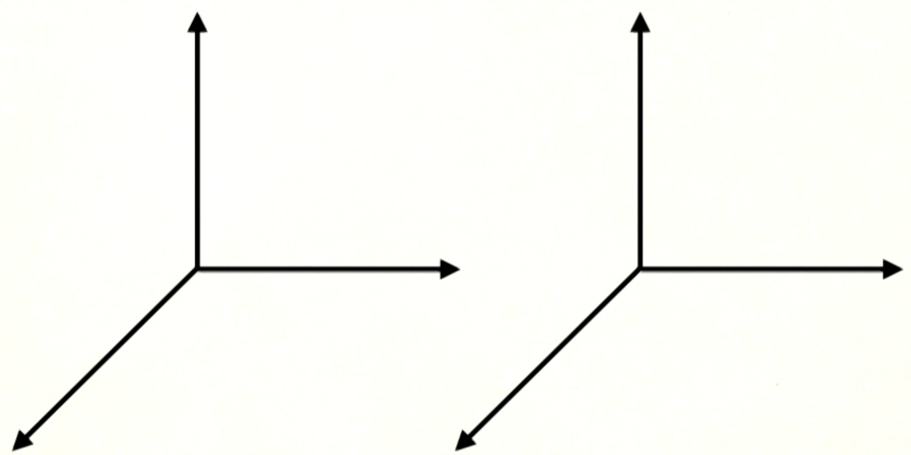
1. representation
2. truncation
3. gluing



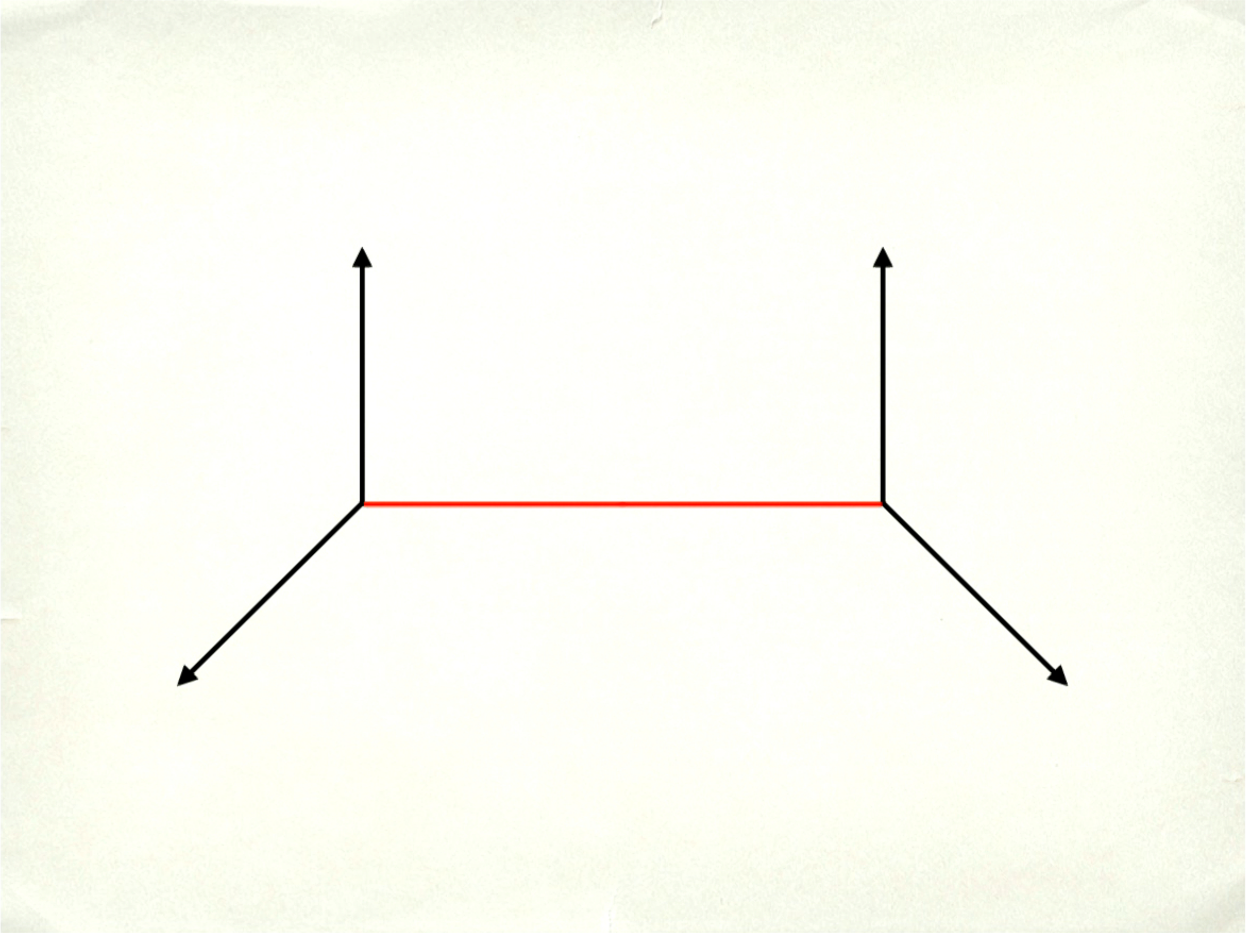
Simplest example: 2 vertices and 1 internal leg



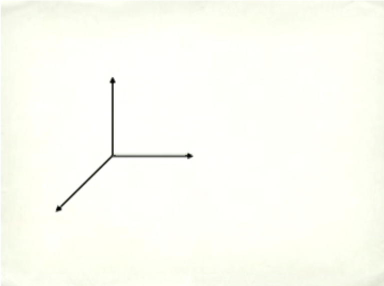
Two copies: left and right



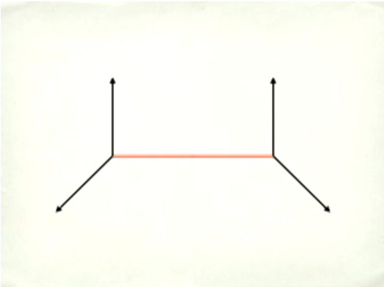
Gluing: two external legs fuse and become internal leg



Building blocks and gluing



1. Algebra: $\mathcal{W}_{1+\infty} \Rightarrow$ affine Yangian of $\mathfrak{gl}_1$
2. Representation: plane partitions



1. Algebra: internal leg $\Rightarrow$ additional operators
   conformal dimension
2. Irrep: bi-module $\Rightarrow$ non-trivial b.c. for vertices
   relative representation

Two parameter family of glued Yangian!

(Algebraic relations are fixed by demanding that the algebra acts on twin-plane-partition faithfully.)

Lessons

- plane partition is also very useful in the gluing process
  - visualize Fock space
  - Define algebra by faithful representation

Truncation and gluing

Gluing of these finite truncations of $\mathcal{W}_{1+\infty}$ should give chiral algebra of Y-junction webs *Rapcak Prochazka'17*

Finite truncation of $\mathcal{W}_{1+\infty}$ is easier to study as truncation of affine Yangian of $\mathfrak{gl}_1$ *Fukuda Matsuo Nakamura Zhu '15, Prochazka '15*

WL Longhi'19

1. Can use affine Yangian of $\mathfrak{gl}_1$ basis instead of $\mathcal{W}_{1+\infty}$ basis
2. Can first glue and then truncate

Higher spin gravity and holography

1. It is possible to define gravity coupled to higher spin gauge symmetry in AdS_d or dS_d

2. Simplest example: $\mathfrak{sl}(N) \oplus \mathfrak{sl}(N)$ Chern-Simons

3. higher spin holography is special

   simple, weak/weak, no need for supersymmetry

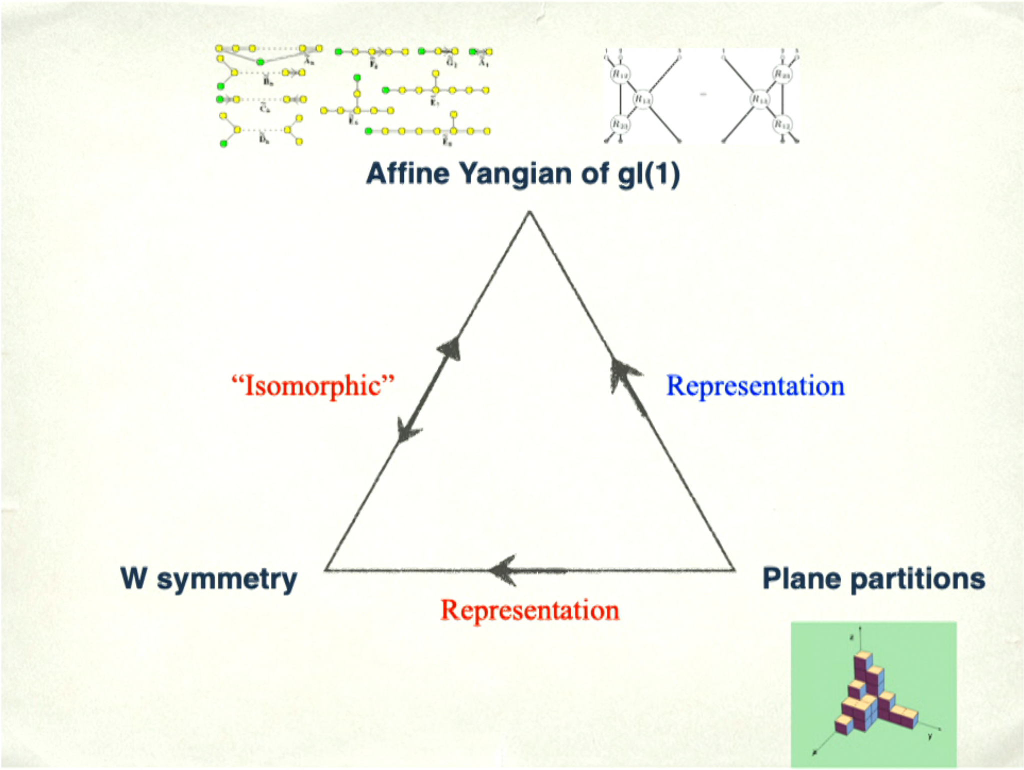
Higher spin symmetry v.s. stringy symmetry

1. Vasiliev's higher spin gravity can be embedded into string theory
2. Tensionless limit $\longrightarrow$ stringy symmetry
3. Vasiliev higher spin symmetry is subalgebra of stringy symmetry (from leading Regge trajectory)
4. In AdS_3: additional symmetry enhancement
5. Tensionless + AdS_3: strongly constrain the theory

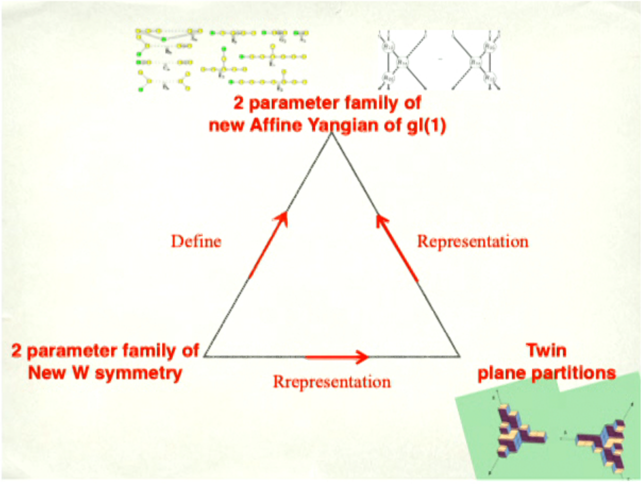
Advantage of affine Yangian

Easy to study

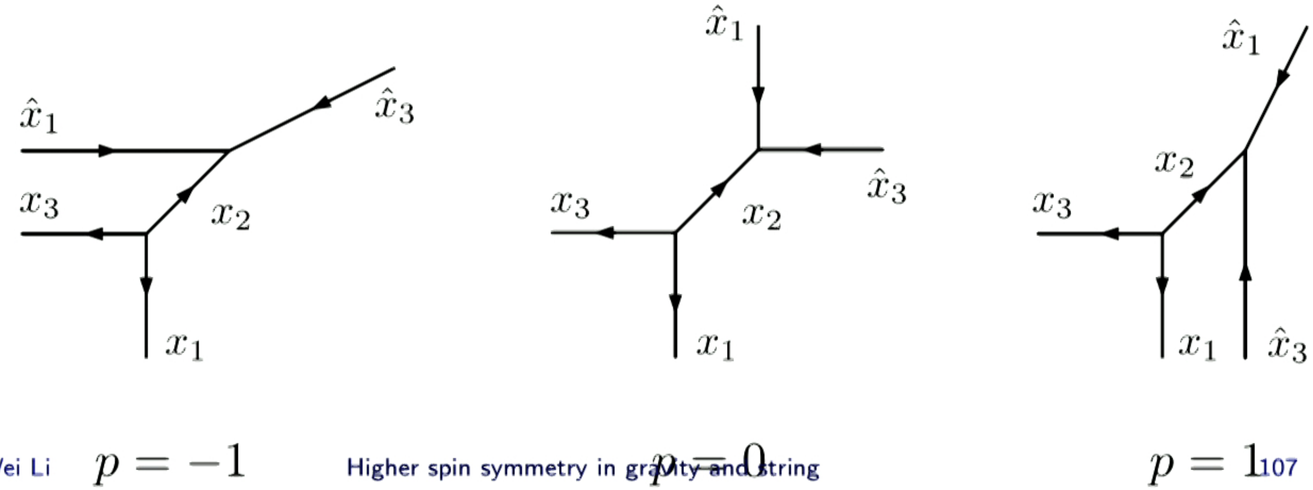
1. representation
2. truncation
3. gluing



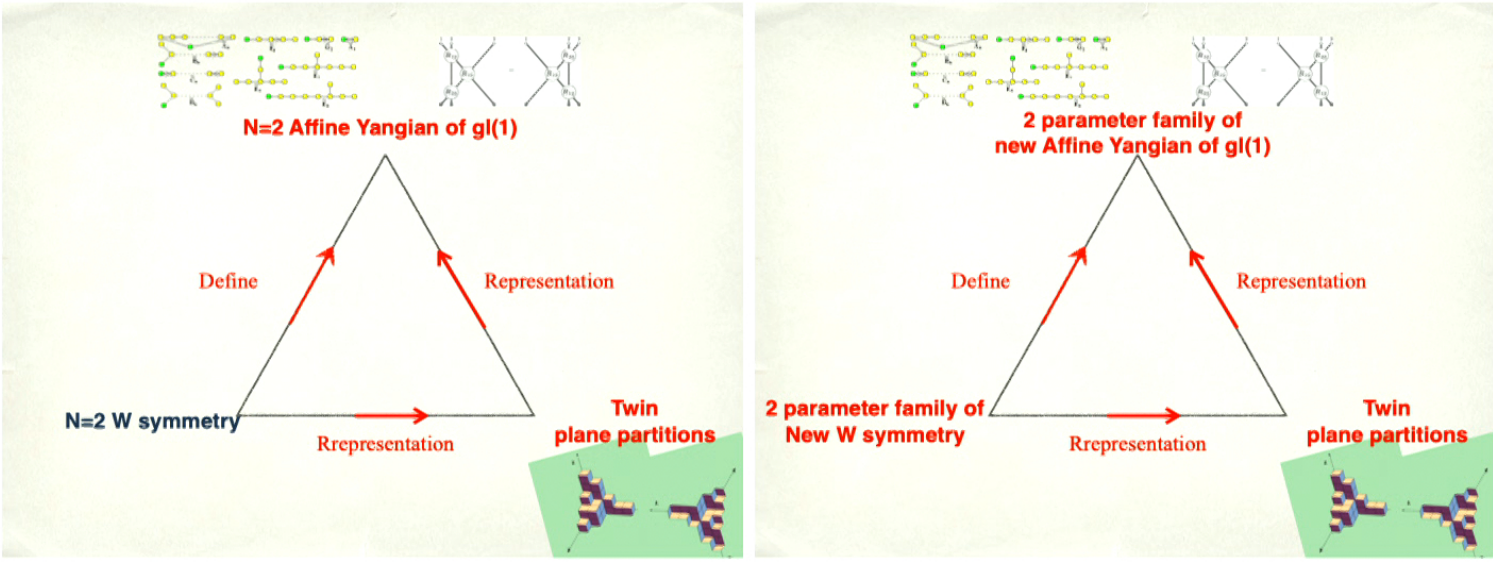
New W and Yangian algebras via gluing *WL Longhi '19; WL '19*



The 4 constraints from twin plane partition
match one-to-one
to the 4 constraints from (p, q) web.



New W and Yangian algebras via gluing plane partitions



Plane partition is also very useful in the gluing process

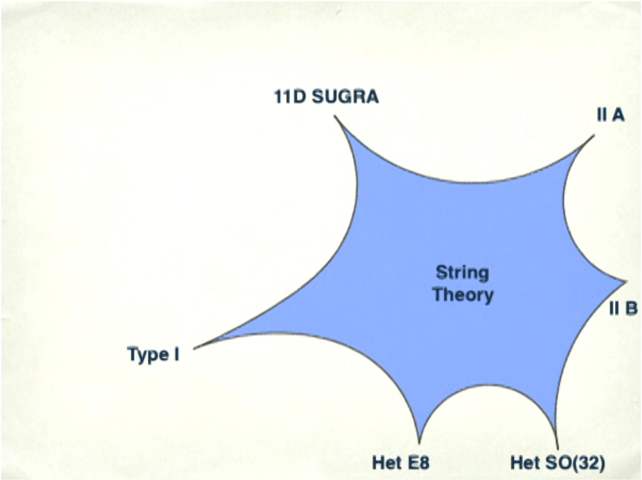
- visualize Fock space
- Define algebra by faithful representation

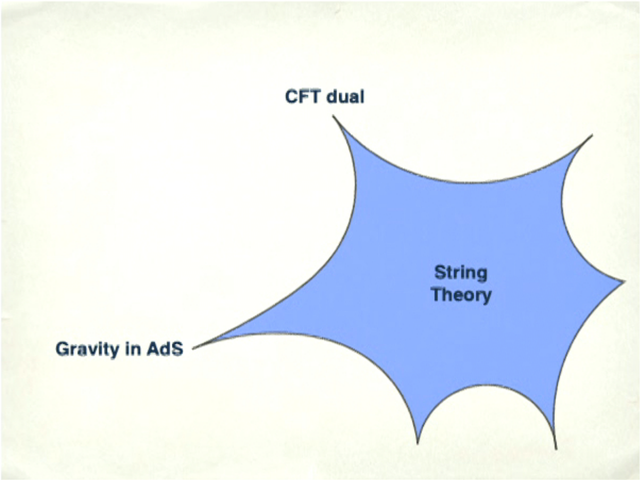
More open problems

1. Deeper relation between higher spin symmetry and integrable structure ?

2. Mathematical description of stringy symmetry?

3. Application of stringy symmetry? Can stringy symmetry explain dualities of string theory?





Thank you very much !