Title: Quantum Sine-Gordon model in perturbative AQFT

Speakers: Dorothea Bahns

Collection: Emmy Noether Workshop: The Structure of Quantum Space Time

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Abstract: We construct the Haag Kaster net of von Neumann algebras for the Sine-Gordon model. This is joint work with Klaus Fredenhagen and Kasia Rejzner.

# The Quantum Sine Gordon Model in pAQFT

Dorothea Bahns
Mathematical Institute, Universität Göttingen

joint with Kasia Rejzner (Commun. Math. Phys. 2017)
and with Klaus Fredenhagen and Kasia Rejzner (1712.02844, CMP 2019/20)
ongoing work with Kasia Rejzner and Nicola Pinamonti

$21^{\text{st}}$ November 2019
Emmy Noether Workshop: The Structure of Quantum Space Time
Perimeter Institute, 18 – 22 November, 2019

# Commercial break 1

## Emmy-Noether-Days 2019
## Göttingen

### 100 Jahre Habilitation

**Dienstag, 10. Dezember 2019 19:00 Uhr**
**Maximum**

**Gastspiel portraittheater**

**Mathematische Spaziergänge**
**mit Emmy Noether**

**Dienstag, 10. Dezember 2019 15:00 Uhr**
**Extra-Vorstellung für Schülerinnen, Schüler und Studierende**

**Donnerstag, 12. Dezember 2019 15:00 Uhr**
**Sitzungszimmer**

**Fest-Kolloquium**

**Dr. Mechthild Koreuber** - Freie Universität Berlin
**Prof. Dr. Mina Teicher** - Bar-Ilan University, Israel

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www.uni-math.gwdg.de/bahns/emmynoetherdays.html

## Commercial break 2 / Motivation

Need good tools to do perturbation theory in quantum field theory....

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Need good tools to do perturbation theory in quantum field theory....

.... but without Hilbert spaces (at least not starting from them).

.... for various reasons:

- Do QFT in general backgrounds (globally hyperbolic) [see below]
- Apply methods of perturbation theory to geometrically interesting situations,
  e.g. Quantization of gravity [Brunetti-Fredenhagen-Rejzner 16, ...],
  Quantization of strings and membranes [B-Rejzner-Zahn 14], ....
- This talk: Treat the (massless) sine Gordon model perturbatively, despite the fact that the free massless scalar field on 2-D Minkowski space is not a Wightman field.

since 2000: Brunetti+Dütsch+Fredenhagen, Rejzner+Fredenhagen, Rejzner+Hawkins, ... also Hollands+Wald, Fewster+Verch, Pinamonti ....

## What is AQFT?

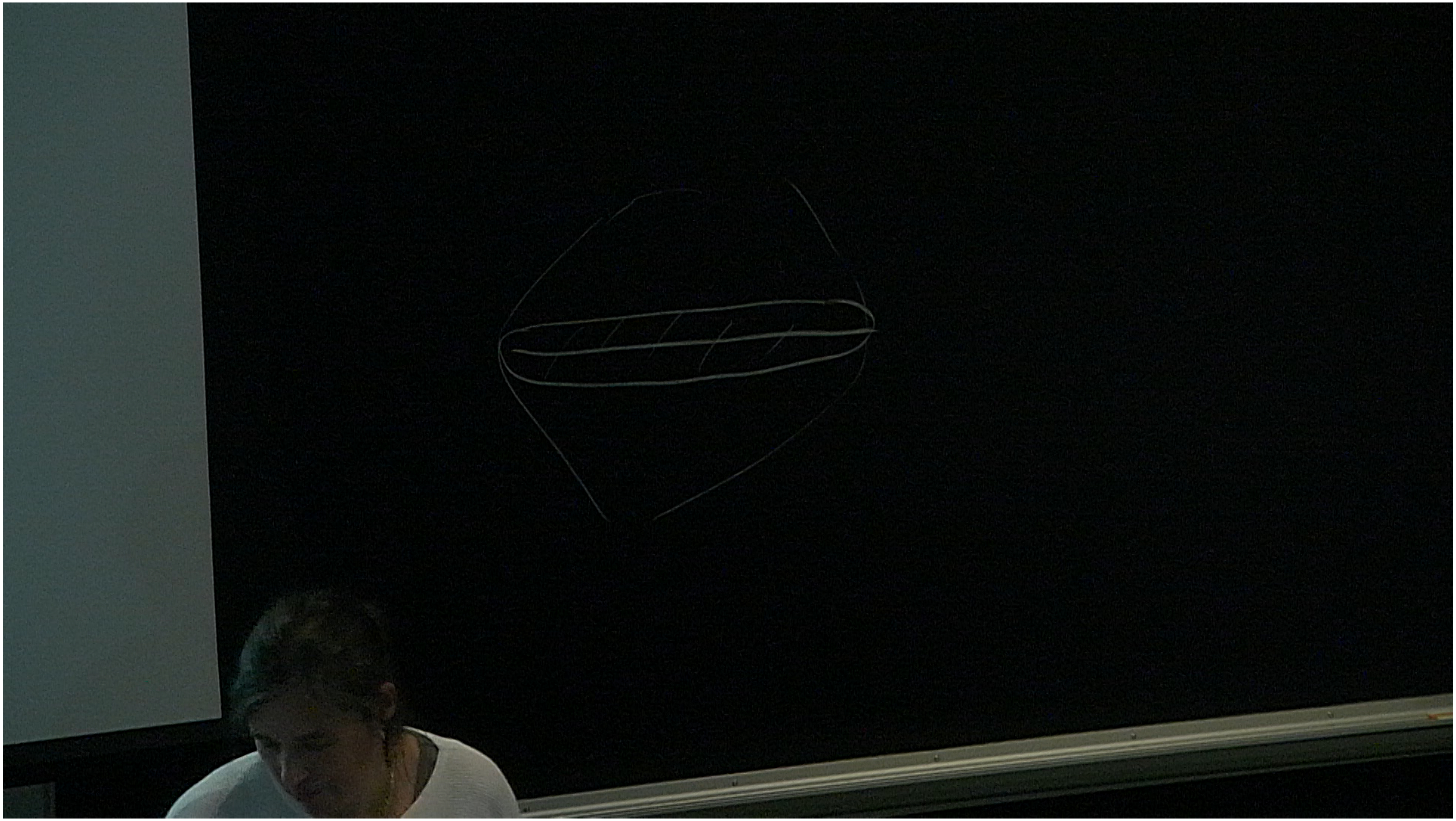
- Algebraic Quantum Field Theory [Haag, Araki, Kastler ...]

  Method to do "Quantum field theory without Hilbert spaces". One framework: Haag-Kastler-axioms.

- A model is defined by associating to each bounded (contractible) region $\mathcal{O}$ in Minkowski space $M$ (or more generally, a globally hyperbolic spacetime) the algebra of observables (of the model) $\mathfrak{A}(\mathcal{O})$.

- certain axioms are to be satisfied
  - Inclusion: $\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. Mathematically: net of algebras.
  - Causality: Algebras associated to spacelike separated regions commute with each other
  - Time slice axiom (initial value problem): if $\mathcal{U}$ is a nbhd of a Cauchy surface in $\mathcal{O}$, then $\mathfrak{A}(\mathcal{U})$ is isomorphic to $\mathfrak{A}(\mathcal{O})$
  - ... behaviour under symmetries, ...

- Problem: Good models?!



# What is pAQFT?

"real life QFT" (perturbation theory, Feynman graphs) meets AQFT

- Quantization = deformation quantization (formal power series) of a Poisson algebra of Functionals
- Interaction: S-matrix as a star exponential (formal power series, similar to Dyson's series). Main tool: time ordered products.
- Ingredients: Interaction Lagrangian, the retarted and advanced fundamental solutions of the underlying free theory, choice of a so-called Hadamard function
- Perturbative "construction" of Haag-Kastler net of observables in the sense of formal power series. Axioms proved to hold (in the sense of formal p.s.) [Fredenhagen-Rejzner 15]

Initiated by: Brunetti+Dütsch+Fredenhagen, Rejzner+Fredenhagen.

## Setting: Quantization in pAQFT

- Free (=linear) theory $\rightarrow$ unique Pauli-Jordan commutator "function"

$$\Delta^{PJ} = E_{ret} - E_{adv} \in \mathcal{D}'(M)$$

- Functionals $\mathcal{F} = \{F : C^\infty(M) \rightarrow \mathbb{C} \mid$ $n$-th functional derivatives $F^{(n)}(\phi) \in \mathcal{D}'(M^n)$ $\}$

  Regular functionals $\mathcal{F}_{reg}$: $n$-th functional derivatives $F^{(n)}(\phi) \in C_c^\infty(M^n)$

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  Ex.: $F(\phi) = \int f(x)\phi(x)dx$, $f \in C_c^\infty(M)$, $F^{(1)}(\phi) = f$.

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- Peierls's bracket: For regular functionals $F, G$ set

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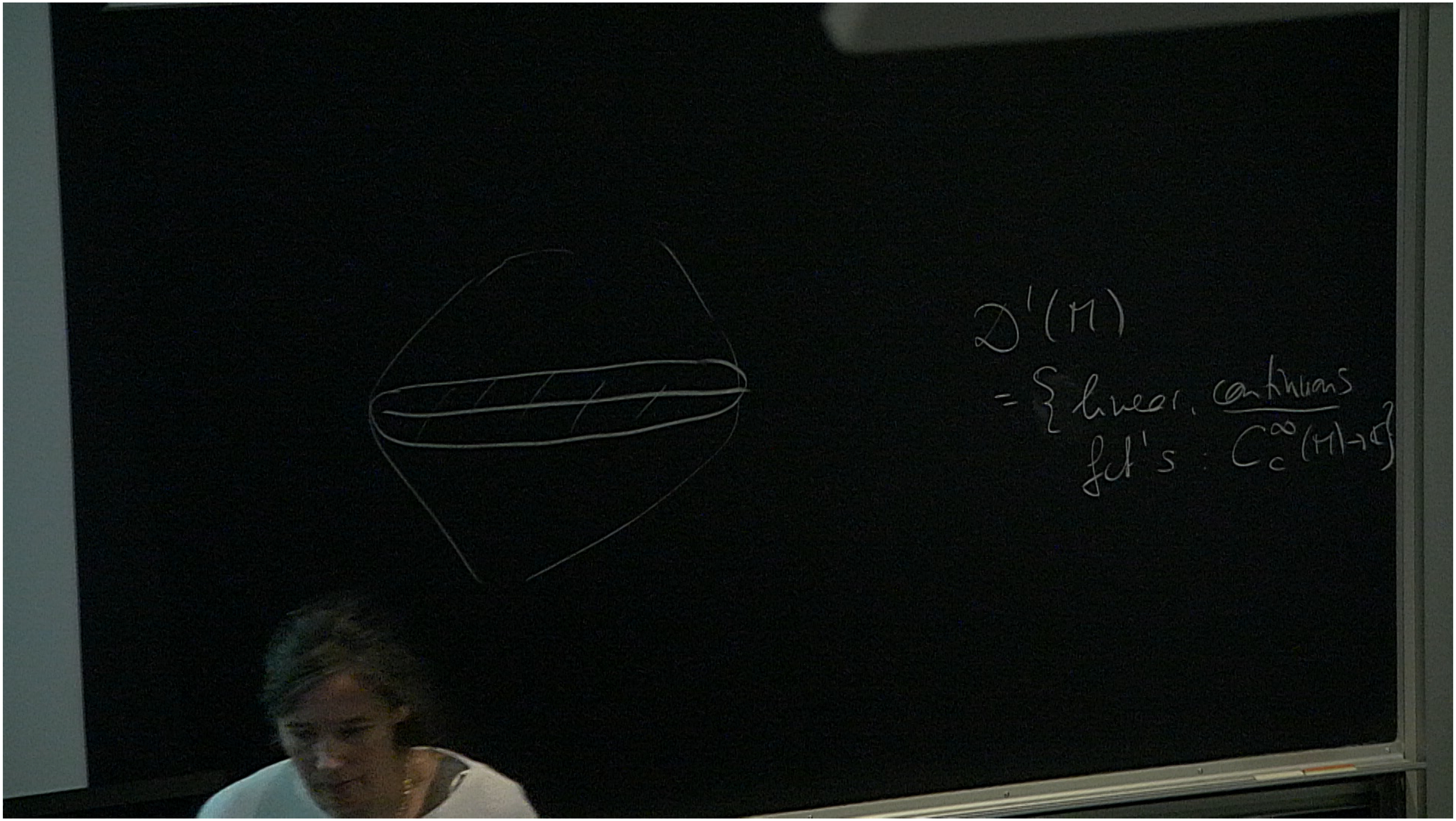
$$\{F, G\}(\phi) = \langle F^{(1)}(\phi), \Delta^{PJ} G^{(1)}(\phi)\rangle$$

- Formal deformation quantization (star product, formal power series)

$$(F \star G)(\phi) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \langle F^{(n)}(\phi), (\tfrac{i}{2}\Delta^{PJ})^{\otimes n} G^{(n)}(\phi)\rangle$$


$\mathcal{D}'(M)$
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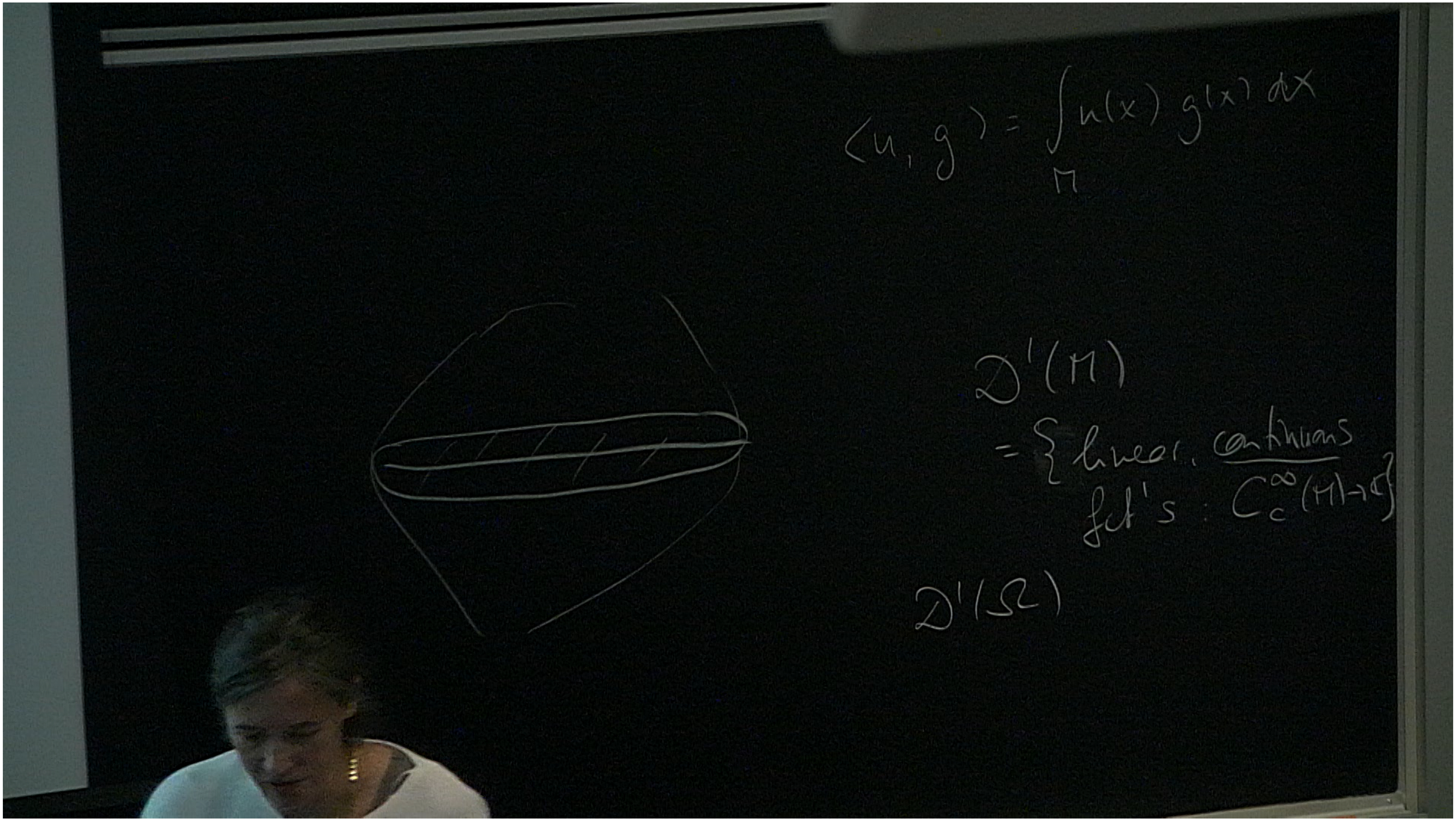
⟨u, g⟩ = ∫_M u(x) g(x) dx
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D'(Ω)

## Quantization II: Algebra of free fields and Normal ordering

- Let $H$ be a bisolution (of the free theory), s.t. $W := \frac{i}{2}\Delta^{PJ} + H$ satisfies the Hadamard condition (condition on $W$'s singular support). Existence $H$ (even on glob. hyp. mfd): deformation argument.
- New star product on $\mathcal{F}_{\mu c}[[\hbar]]$,

$$(F \star_H G)(\phi) := \sum \frac{\hbar^n}{n!} \langle F^{(n)}(\phi), (\tfrac{i}{2}\Delta^{PJ} + H)^{\otimes n} G^{(n)}(\phi)\rangle$$

  microcausal functionals $\mathcal{F}_{\mu c}$, i.e. $F$ with $WF(F^{(n)}) \subset M^n \times \overline{V_+^n}$.
- On $\mathcal{F}_{reg} \subset \mathcal{F}_{\mu c}$, the products $\star_H$ and $\star$ are equivalent,

$$\alpha_H^{-1}(F) \star \alpha_H^{-1}(G) = \alpha_H^{-1}(F \star_H G) \qquad (1)$$

  linear, invertible map $\alpha_H$ on $\mathcal{F}_{reg}$ explicitly known (as a formal p.s.). Different choices of $H \to$ equivalent star products on $\mathcal{F}_{\mu c}[[\hbar]]$.

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  linear, invertible map $\alpha_H$ on $\mathcal{F}_{reg}$ explicitly known (as a formal p.s.). Different choices of $H \to$ equivalent star products on $\mathcal{F}_{\mu c}[[\hbar]]$.
- Certain completion/extension process of $\mathcal{F}_{reg}$ using (1) leads to algebra $(\mathfrak{A}[[\hbar]], \star)$.
- Ordinary setting: $\alpha_H^{-1} \leftrightarrow$ normal ordering. Write $:F:_H = \alpha_H^{-1}(F)$ for $F \in \mathcal{F}_{reg}$, and $:F:_H = \lim_n \alpha_H^{-1}(F_n) \in \mathfrak{A}[[\hbar]]$ for $F \in \mathcal{F}_{\mu c}$, $F_n \to F$.

## More formal power series: Interaction

$S$-matrix: Dyson's series (formal p.s., Feynman graphs), formalized in pAQFT on the level of functionals. No need for a representation (yet). Building block are time ordered products.

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**Def:** Time-ordering operator $\mathcal{T}$

$$(\mathcal{T}F)(\phi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(2n)}(\phi), (iE_D)^{\otimes n} \right\rangle \ , \quad F \in \mathcal{F}_{reg}$$

with the Dirac propagator (mean) $E_D = \frac{1}{2}(E_R + E_A)$.

**Def:** Time-ordered product (inverse taken in the sense of f.p.s.)

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**Def:** formal $S$-matrix (for an interaction $F \in \mathcal{F}_{reg}$)

$$\mathcal{S}(\lambda F) \doteq \mathcal{T}\left(e^{i\mathcal{T}^{-1}(\lambda F)/\hbar}\right) = \sum_{n=0}^{\infty} \left(\frac{i\lambda}{\hbar}\right)^n \frac{1}{n!} F^{\cdot_{\mathcal{T}} n}$$

with a second formal parameter $\lambda$ (coupling constant).

## The Feynman propagator and normal ordering

Feynman propagator: Interaction needs normal ordering.

Time ordered products (Dirac propagator $E_D$)
**combined with**
normal ordering the interaction (choice of $H$)

leads to formal power series containing tensor powers of the Feynman propagator

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Renormalization: For nonregular functionals (even with normal ordering), the time ordered products are only defined on $M^n \setminus \Delta$. Extension of distributions (microlocal analysis). Not necessary here.

# The Sine Gordon model

The model:

- Free massless scalar field on 2-D Minkowski space

$$\Box\phi = 0$$

- Interaction potential

$$\cos(a\phi)$$

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- Free massless scalar field on 2-D Minkowski space

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- Interaction potential

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The trouble with $\phi$...

Standard approach (annihilation/creation operators on Fock space) does not work. Wightman axioms cannot be satisfied (!) for the free field. Explanation in the algebraic picture: algebra of free fields does exist, but there is no vacuum state. There are other states (but we know only few...).

## pAQFT Ingredients and 1st Example

In our setting (2D Minokwski massless field):

$$E_A, E_R, \Delta^{PJ} \quad \text{linear combination of step functions}$$

Choose

$$H_\mu(x,y) = \tfrac{-1}{4\pi} \ln(\mu^2 |(x-y)^2|) \qquad \text{with a scale } \mu > 0$$

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Example: Vertex operators: For $g \in C_c(M)$, $a \in \mathbb{R}$, let

$$v_a(g) \in \mathcal{F}_{\mu c}(M) \,, \qquad v_a(g)[\phi] = \int e^{ia\phi(x)} g(x) dx$$

The vertex operator $V_a(g)[\phi]$ is the normal ordered version of $v_a(g)[\phi]$ with star product

$$V_a(g)[\phi] \star V_b(h)[\phi] = v_a(g)[\phi] \star_{H_1} v_b(h)[\phi] \,.$$

It converges for $b = \pm a$, $\hbar |a|^2 < 4\pi$.

## States

**Def:** Quasifree state on $\mathfrak{A}$ given by a bisolution $H$, s.t.

$$\langle f, Hf\rangle \geq 0 \quad \text{and} \quad \langle f, \Delta^{PJ} g\rangle^2 \leq 4\langle f, Hf\rangle\langle g, Hg\rangle$$

via (with normal ordering defined w.r.t. any bisolution $\widetilde{H}$)

$$\omega_{H,\phi}(:F:_{\widetilde{H}}) = \alpha_{H-\widetilde{H}}(F)[\phi]$$

**Rem.** Observe: $H - \widetilde{H}$ is smooth!

Massless field in 2D: $\langle f, H_\mu f\rangle \geq 0$ only for test functions with vanishing integral... Schubert's modification (2013):

$$H(x,y) = H_\mu(x,y) + \text{extra terms}$$

Extra terms depend on $H_\mu$, $\Delta^{PJ}$, the choice of a test function $\psi$ with $\int \psi = 1$ and a parameter $r > 0$. Satisfies quasi free state condition!

## States II

Schubert's state can be understood in terms of a representation of the free scalar massless field $\Phi$ in 2D Minkowski used in string theory and formalized by Derezinski and Meissner 06:

- Carrier space: $\mathcal{H} = \mathcal{H}_0 \otimes L^2(\widehat{\mathbb{R}})$ with $\mathcal{H}_0$ ordinary massless Fock space.
- Representation: Choose test function $\psi$ with $\int \psi = 1$, and set

$$\pi_\psi \phi(g) = \Phi_c \left( g - \psi \int g \right) \otimes 1 + 1 \otimes \left( \int g \right) q - 1 \otimes \left( \int g \Delta \psi \right) p$$

  with $\Phi_c$ the free field on Fock space $\mathcal{H}_0$ (well defined because $g - \psi \int g$ has vanishing integral), $p$ and $q$ momentum and position operator (in momentum space).
- **Fact**: Schubert's $H$ equals

$$H(x,y) = \langle \underline{\Omega}, \pi_\psi \phi(x) \pi_\psi \phi(y) \underline{\Omega} \rangle$$

  where $\underline{\Omega} = \Omega_0 \otimes \Omega_r$, with $\Omega_0$ Fock vacuum, $\Omega_r$ harmonic oscillator ground state (freq dep on $r$).

# S matrix

**Theorem** [B-Fredenhagen-Rejzner19]
Let $\hbar a^2 < 4\pi$ (model's ultraviolet finite regime). Then the $S$-matrix of the Sine Gordon model, calculated within the framework of pAQFT converges strongly in the DM representation on a dense domain $D \subset \mathcal{H}$. The resulting operator is in fact unitary.

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Idea of proof: For $g, f \in C_c^\infty(\mathbb{R}^2)$

$$\|\pi_\psi(S_n(\lambda V(g)))\,\pi_\psi(e^{i\Phi(f)})\,\Omega_0 \otimes \xi\| \leq C^n (n!)^{\frac{1-p}{p}}$$

$1 < p < \frac{4\pi}{\hbar a^2}$. Estimate itself relies on [BR17] (finiteness of $S$-matrix in the vacuum rep), which in turn relies on older estimates in Euclidean approach [Fröhlich et al 70's]. Tied to 2 dimensions.

Improvement of our original estimate [BR17] for convergence in massless vacuum (where a condition on the test function $g$ (IR behaviour) was needed): consistent with local quasiequivalence of DM rep to massive vacuum.

## Haag Kastler net – Sine Gordon

Observables: the field, the interaction Lagrangian (Cosine), the interaction in the field equation (Sine). Hence, the test objects are $\underline{\mathcal{D}} = C_c^\infty(M, \mathbb{C}) \oplus C_c^\infty(M, \mathbb{R})$, and the Lagrangian

$$L(g, h) = v_a(g) + v_{-a}(\bar{g}) + \Phi(h)$$

with $v_a(g) = \int e^{ia\Phi(x)} g(x)$. Observe: $: L(0, h) :$ is the field and $: L(g, 0) :$ is the Sine Gordon theory's interaction term for $g$ real-valued.

**Thm** [BFR17] The formal power series $S(g, h)$ defined as usual in pAQFT (time ordered exponential of normal ordering $L(g, h)$) in the DM representation converges strongly on the dense domain $D \subset \mathcal{H}$ and gives rise to unitary operators on $\mathcal{H}$. The Bogoliubov factorization property is satisfied.

Completing each $\mathfrak{A}_G(\mathcal{O})$ with $G$ =constant w.r.t. the family of seminorms $\|A\|_{\Psi,\underline{g}} = |\langle \Psi, A(\underline{g})\Psi\rangle|$ (independent of $\underline{g} \in G_{\mathcal{O}}$), yields the net of local von Neumann algebras.

## Conclusion and Outlook

- Original motivation [B-Rejzner17]: pAQFT put to the test.
- [BFR19]: Construction of the net of local von Neumann algebras in the DM representation. No Wick rotation needed, no auxiliary mass.
- Comparison with traditional constr QFT: needs vacuum state!
  - 1st step: thermal states (ongoing jt work w/ Rejzner+Pinamonti) and true adiabatic limit (also getting rid of the cutoff function in the interaction)
  - Understanding of the representation theory (DM rep'n vs. massless with auxiliary mass vs. solitonic quantization vs. "true vacuum"?) interesting in its own right.
  - Long term goal: better understanding of "equivalence" of Osterwalder-Schrader and Wightman axioms