Title: We Donâ€™t Live on Spatial Hypersurfaces, so Why Should Quantum Fields?

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Abstract: I will discuss a number of ongoing efforts to understand quantum field properties in a manifestly spacetime framework. Entanglement entropy and causal set theory are among the topics that I will especially touch on.

We Don't Live on Spatial Hypersurfaces, so Why Should Quantum Fields?

Emmy Noether Workshop: The Structure of Quantum Space Time
Perimeter Institute

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Around 1952, Rudolf Peierls found himself "looking for a new expression of some of the basic rules of quantum mechanics, namely the formulation of commutation laws of relativistic field theory"[1].

There already existed consistent formulations of relativistic quantum field theory, developed decades earlier by Heisenberg and Pauli, as well as more recent interaction picture frameworks developed by Schwinger, Tomonaga and others.

Despite this, Peierls was worried about the Lorentz invariance of these formulations. He wanted to find an alternative approach in which the Lorentz invariance of the formulation is manifestly evident.

Actual Size

[1] S. Lee, Sir Rudolf Peierls: . No. v. 2, World Scientific Publishing Company, Incorporated, 2009.

Peierls succeeded in deriving the covariant formulation of a relativistic QFT that he had in mind. His formulation[2], determines a general rule for the commutator at any pair of *spacetime* points. This formulation uses Green functions of the theory and makes no reference to a Hamiltonian. More precisely

$$i\Delta(x, x') \equiv i(G_R(x, x') - G_A(x, x')). \tag{1}$$

Peierls also held much correspondence about his work with other prominent physicists of his time. When his paper came out, many were interested in it but also expressed skepticism. Nonetheless, the Peierls bracket has stood the test of time and is an important framework for quantum field theory that shows that its time-dependence can be captured in a Lorentz invariant manner. It has also enabled important advances in approaches such as algebraic quantum field theory and causal set theory.

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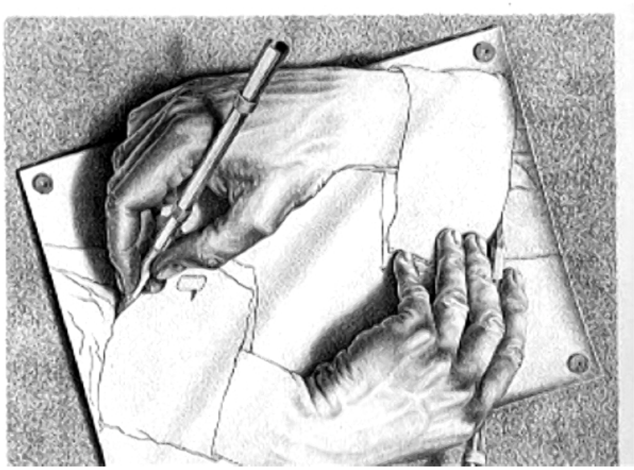
[2] R. E. Peierls, The Commutation laws of relativistic field theory, Proc. Roy. Soc. Lond., vol. A214,pp. 143157, 1952.

Outline

- Causal set theory.
- The Sorkin-Johnston vacuum state.
- Entanglement entropy from spacetime two-point correlation function.
- Free (gaussian) scalar field theory on a causal set.



Causal Set Theory: Spacetime is Fundamentally Discrete [3]

A Causal Set (or causet) is a locally finite partially ordered set. It is a set $\mathcal{C}$ along with an ordering relation $\preceq$ that satisfy:

- Reflexivity: for all $X \in \mathcal{C}$, $X \preceq X$.
- Antisymmetry: for all $X, Y \in \mathcal{C}$, $X \preceq Y \preceq X$ implies $X = Y$.
- Transitivity: for all $X, Y, Z \in \mathcal{C}$, $X \preceq Y \preceq Z$ implies $X \preceq Z$.
- And, local finiteness: for all $X, Y \in \mathcal{C}$, $|I(X, Y)| < \infty$, where $|\cdot|$ denotes cardinality and $I(X, Y)$ is the causal interval defined by $I(X, Y) := \{Z \in \mathcal{C} | X \preceq Z \preceq Y\}$.

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[3] L Bombelli, J H Lee, D Meyer, and R D Sorkin, 1987, Space-Time as a Causal Set, Phys. Rev. Lett. 59, 521.

Causal Set Theory

Sprinkling: generates a causal set from a given Lorentzian manifold $\mathcal{M}$, by placing points at random in $\mathcal{M}$ via a Poisson process with "density" ρ, such that $P(N) = \frac{(\rho V)^N}{N!} e^{-\rho V}$.

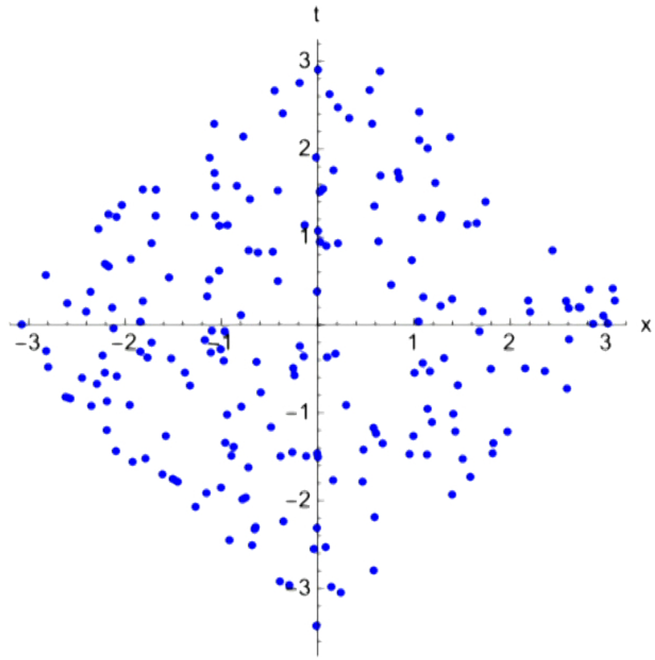


Figure: A causal set formed by sprinkling 500 elements into a finite interval in 1 + 1D Minkowski space, with $\rho = 1$.

Quantum Field Theory

A quantum field theory is typically fully determined by the set of its n-point functions.

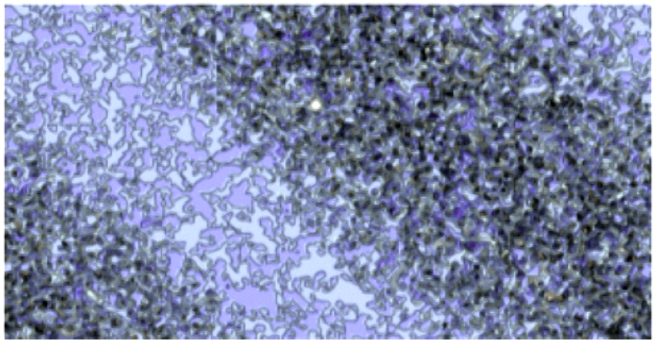
$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle, ..., \langle 0|\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle \qquad (2)$$

If we consider a free and gaussian (for example scalar) field theory, then we only need to know

$$W = \langle 0|\phi(x_1)\phi(x_2)|0\rangle \qquad (3)$$

to fully determine the theory. All the higher n-point functions can be deduced from the 2-point function in this case.

So let's find W in a causal set.



Finding W in the Causal Set

The covariant commutation relations are given by the Peierls bracket

$$[\hat{\phi}(x), \hat{\phi}(x')] = i\Delta(x, x'), \tag{4}$$

where the Pauli-Jordan function is

$$i\Delta(x, x') \equiv i(G_R(x, x') - G_A(x, x')), \tag{5}$$

with $G_{R,A}(x, x')$ being the retarded and advanced Green functions.

$$\mathrm{Ker}(\hat{\Box} - m^2) = \overline{\mathrm{Im}(\hat{\Delta})}. \tag{6}$$

Thus the eigenvectors in the image of $i\hat{\Delta}$ span the full solution space of the KG operator.

The Sorkin-Johnston Vacuum [4]

$i\Delta$ is a self-adjoint operator on a bounded region of spacetime.

Write $i\Delta(x, x')$ in terms of its positive (u_k) and negative (v_k) eigenfunctions:

$$i\Delta(x, x') = \sum_k \left[\lambda_k u_k(x) u_k^\dagger(x') - \lambda_k v_k(x) v_k^\dagger(x')\right]. \qquad (7)$$

Restrict to positive eigenspace to get the Wightman or two-point function in the SJ vacuum:

$$W_{SJ}(x, x') \equiv \text{Pos}(i\Delta) = \sum_k \lambda_k u_k(x) u_k^\dagger(x'). \qquad (8)$$

[4] R D Sorkin, J. Phys. Conf. Ser. 306 (2011) 012017.
S P Johnston (2010) arXiv:1010.5514.

Some Properties of the SJ Vacuum

- Is defined using the entire spacetime volume.
- An observer independent vacuum which is unique.
- In static spacetimes, the SJ state is the same one that is picked out by the timelike and hypersurface-orthogonal Killing vector.
- While not necessarily Hadamard itself, a family of Hadamard states can be constructed from it.
- Is a pure state for the spacetime definition of EE (while its restriction to a smaller subregion is not pure).

SJ State and Cosmology

- FRW: Results suggesting there are correlations on super-horizon scales. N Afshordi, S Aslanbeigi, R D Sorkin, JHEP 08 (2012) 137.
- de Sitter: Sometimes get $\alpha-$vacua. S Aslanbeigi, M Buck, JHEP 08 (2013) 039.
- de Sitter: New de Sitter invariant vacuum in 4d. S Surya, Nomaan X, and YKY, JHEP 1907 (2019) 009.

4d massless & massive SJ Vacuum in dS[5]

It is usually said that there is no known de Sitter invariant Fock vacuum for the massless, minimally coupled theory.

$$ds^2 = \frac{1}{\cos^2 \tilde{T}} \left(-d\tilde{T}^2 + d\Omega^2_{d-1} \right)$$

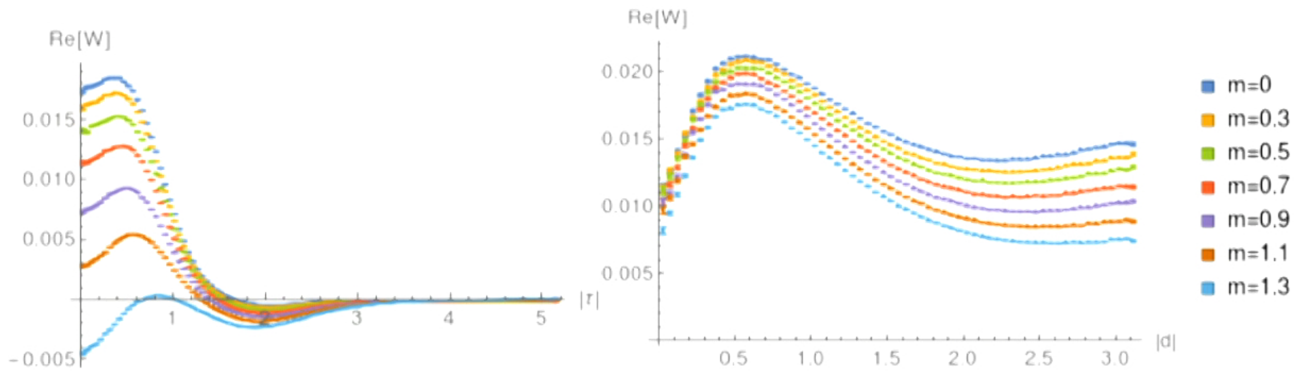


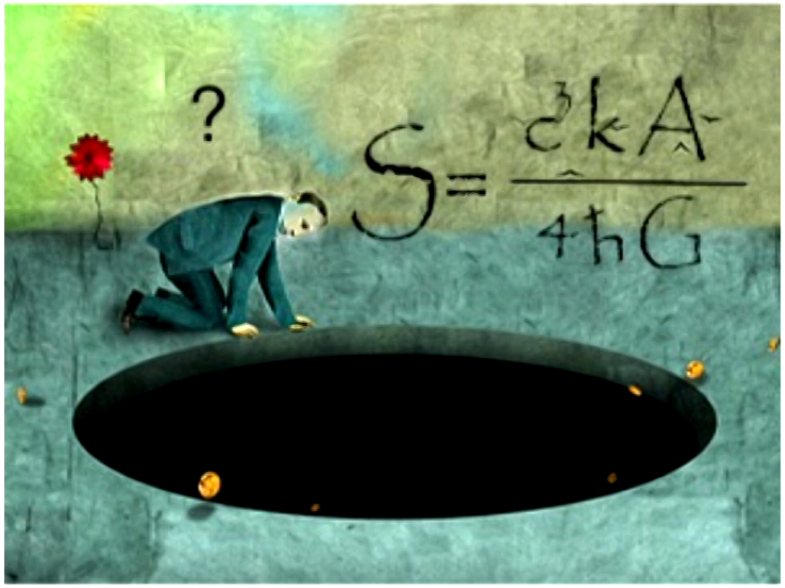
Figure: $\tilde{T}_{max} = 1.42$ mean values. Left: causal. Right: spacelike

[5] S Surya, Nomaan X, and YKY, JHEP 1907 (2019) 009.

Is Black Hole Entropy Entanglement Entropy?

- R D Sorkin, *On the Entropy of the Vacuum outside a Horizon*, (1983), arXiv:1402.3589.

EE is a promising candidate for the origin of black hole entropy, which is still an open question.



Entanglement Entropy

In conventional treatments, entropy is defined as

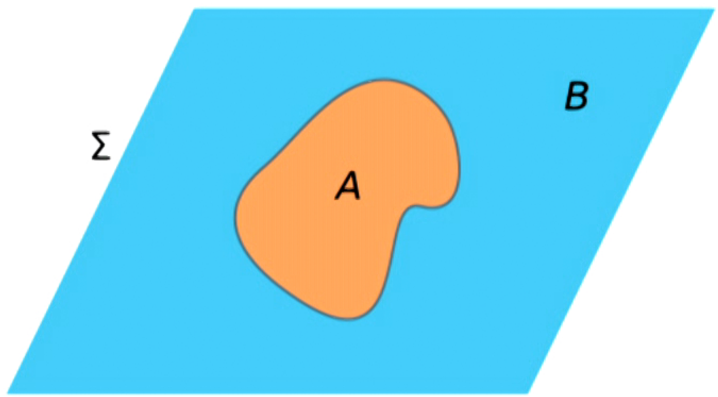
$$S = \mathrm{Tr}\rho \ln \rho^{-1} \tag{9}$$

where ρ is a density matrix on a spatial hypersurface Σ.
If Σ is divided into complementary subregions A and B, then the reduced density matrix for subregion A is

$$\rho_A = \mathrm{Tr}_B \rho \tag{10}$$

and its entanglement entropy with region B is

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A \ . \tag{11}$$



Some Reasons for Seeking a Spacetime Definition of EE

- Entanglement entropy requires a UV cutoff to render it finite. We need to use a covariant cutoff.
- Quantum fields are too singular to always admit meaningful restrictions to hypersurfaces.
- Necessary in the context of quantum gravity and its fluctuating causal structure.
- There is no meaningful notion of a state on a hypersurface in causal set theory.

Quantum Fields Live more Happily in Spacetime

Consider the normal-ordered ϕ^2 operator, and smear it with a test function with compact support on a time $t' =$ const slice,

$$:\phi^2(t', f) := \int d^4x f(\vec{x})\delta(x_0 - t') \int \frac{d^3p d^3k}{2(2\pi)^6\sqrt{E_{\vec{p}}E_{\vec{k}}}} \left(a_{\vec{p}}a_{\vec{k}}e^{i(p+k)\cdot x} + \dots\right)$$

where $E_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$. We then square the result and compute its expectation value in the Minkowski vacuum,

$$\langle 0| :\phi^2(t', f) :: \phi^2(t', f) : |0\rangle = \int \frac{d^3p\, d^3k |\tilde{f}(\vec{k} + \vec{p})|^2}{2(2\pi)^{12}E_{\vec{p}}E_{\vec{k}}} \qquad (12)$$

$$\propto \int d^3p' |\tilde{f}(\vec{p}\,')|^2 \int \frac{d^3k}{\sqrt{(\vec{p}\,' - \vec{k})^2 + m^2}\sqrt{\vec{k}^2 + m^2}}.$$

where $\tilde{f}$ is the Fourier inverse of f and $p' = p + k$. The k-integral diverges linearly in the large $|\vec{k}|$ limit.

Entanglement Entropy in terms of W[6]

Express S directly in terms of the spacetime correlation function.

The entropy can be expressed as a sum over the solutions λ of the generalized eigenvalue problem

$$W\,v = i\lambda\,\Delta\,v, \qquad (\Delta v \neq 0) \tag{13}$$

as

$$S = \sum_{\lambda} \lambda \ln|\lambda| \ . \tag{14}$$

W and $i\Delta$ are the Wightman and Pauli-Jordan matrices.

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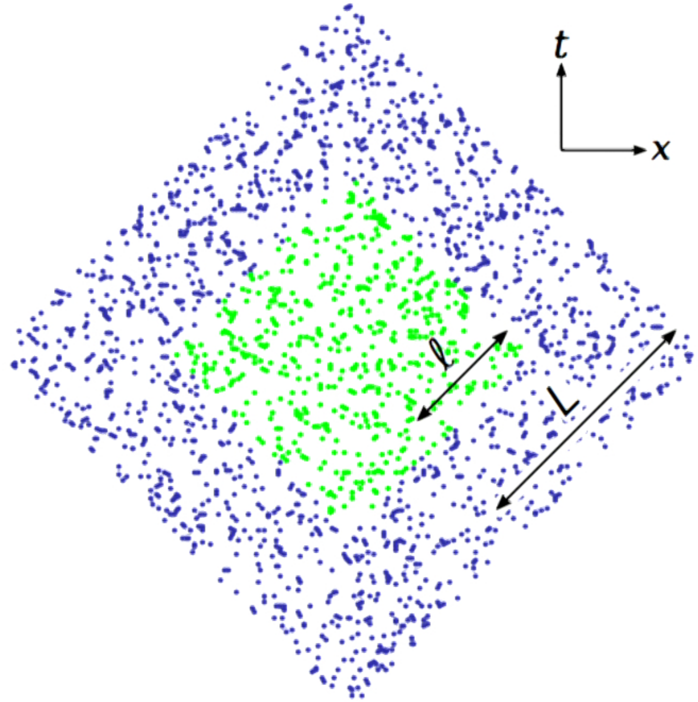
[6] R. D. Sorkin, arXiv:1205.2953

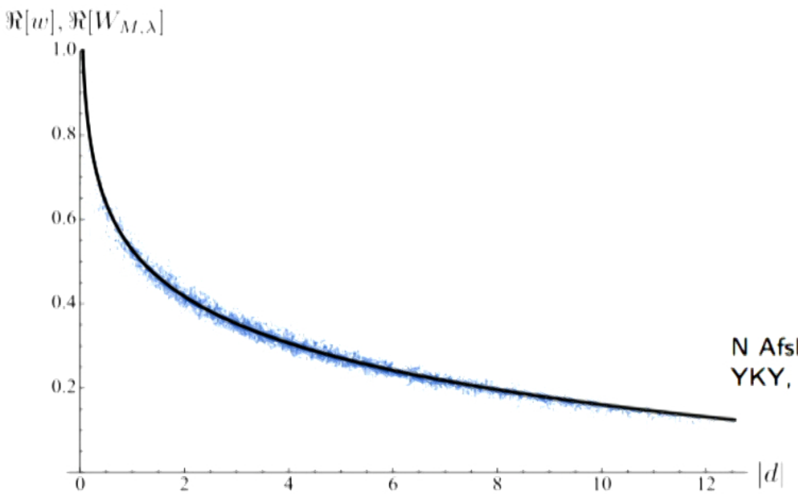
Nested Diamond Setup for Causal Set EE

$$\Delta(X, X') := G_R(X, X') - G_R(X', X)$$

For $m = 0$, we have that $G_R = \frac{1}{2}C$, where C is the causal matrix: $C_{xy} := \begin{cases} 1, & \text{if } x \preceq y. \\ 0, & \text{otherwise} \end{cases}$

For W, we choose W_{SJ}.





N Afshordi, M Buck, F Dowker, D Rideout, R D Sorkin, YKY, JHEP10(2012)08, arXiv:1207.7101.

$1 + 1$d Causal Set Results: Area Law[7]

The EE fits $S = b\ln(\sqrt{N_\ell}/4\pi) + c$ with $b = 0.346 \pm 0.028$ and $c = 1.883 \pm 0.035$. This is the usual result.

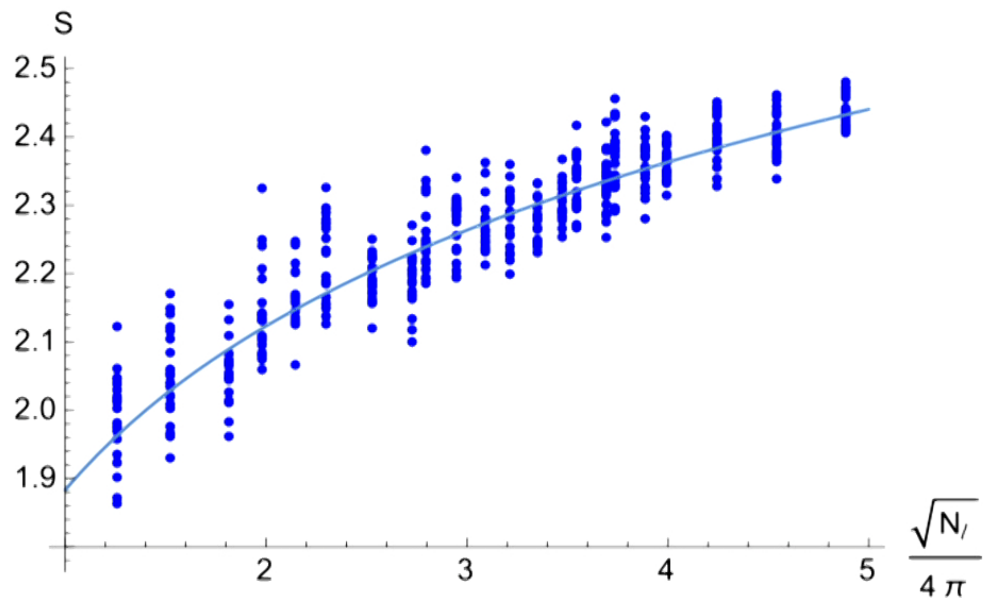


Figure: S vs. $\sqrt{N_\ell}/4\pi$. $\ell/L = 1/2$ in this example.

[7] R D Sorkin, YKY, CQG 35 074004 (2018).

EE of de Sitter Horizons in Causal Sets[8]

Currently investigating EE of horizons in dS.

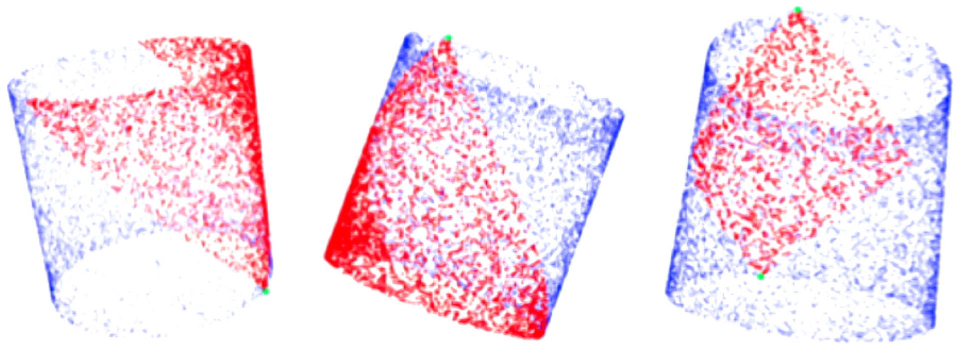


Figure: Horizons in 2d dS causal sets in conformal coordinates.

Preliminary results show that the spacetime entropy yields an area law in this case as well.

[8] S Surya, Nomaan X, YKY.

Towards Spacetime EE for Interacting Theories[9]

Considered quartic perturbations to the gaussian case

$$\begin{aligned}\rho_{qq'} &= \langle q|\rho|q'\rangle \\ &= Ne^{-A/2(q^2+q'^2)-C/2(q-q')^2-\left(\lambda_1\frac{q^4+q'^4}{2}+\lambda_2(q^3q'+qq'^3)+\lambda_3q^2q'^2\right)}\end{aligned}$$

Found that to first order in perturbation theory, the same formula

$$S=\sum_\lambda \lambda \ln|\lambda|$$

holds, where W and Δ in $Wv = i\lambda\Delta v$ now contain correction terms. Hints that S may more universally depend on W.

[9] Y Chen, R Kunjwal, H Moradi, YKY, and M Zilhão, in preparation.

Entropy Formula has Broad Applications

The formulae

$$W\, v = i\lambda\, \Delta\, v, \qquad (\Delta v \neq 0) \tag{15}$$

$$S = \sum_{\lambda} \lambda \ln |\lambda| \tag{16}$$

can be generally applied to any region(s) of spacetime within another.

For example we can consider entropy of coarse graining, or nested regions whose Cauchy surfaces are not subsets of each other.

We may also be able to learn something from the magnitude of the EE (in addition to its scaling behaviour with the cutoff).

Summary and Future Directions

- Maintaining explicit Lorentz-invariance is hard, but progress has been made.
- The approaches such as the ones I talked about can help us understand some aspects of QFTs in the spacetime domain that they truly live in.
- Causal set theory and spacetime entanglement entropy.
- Future directions: other fields, interactions, black hole EE, and ultimately a Lorentzian path integral formulation.