Title: The gravitational Wilson loop and the non-Abelian Stokes' theorem

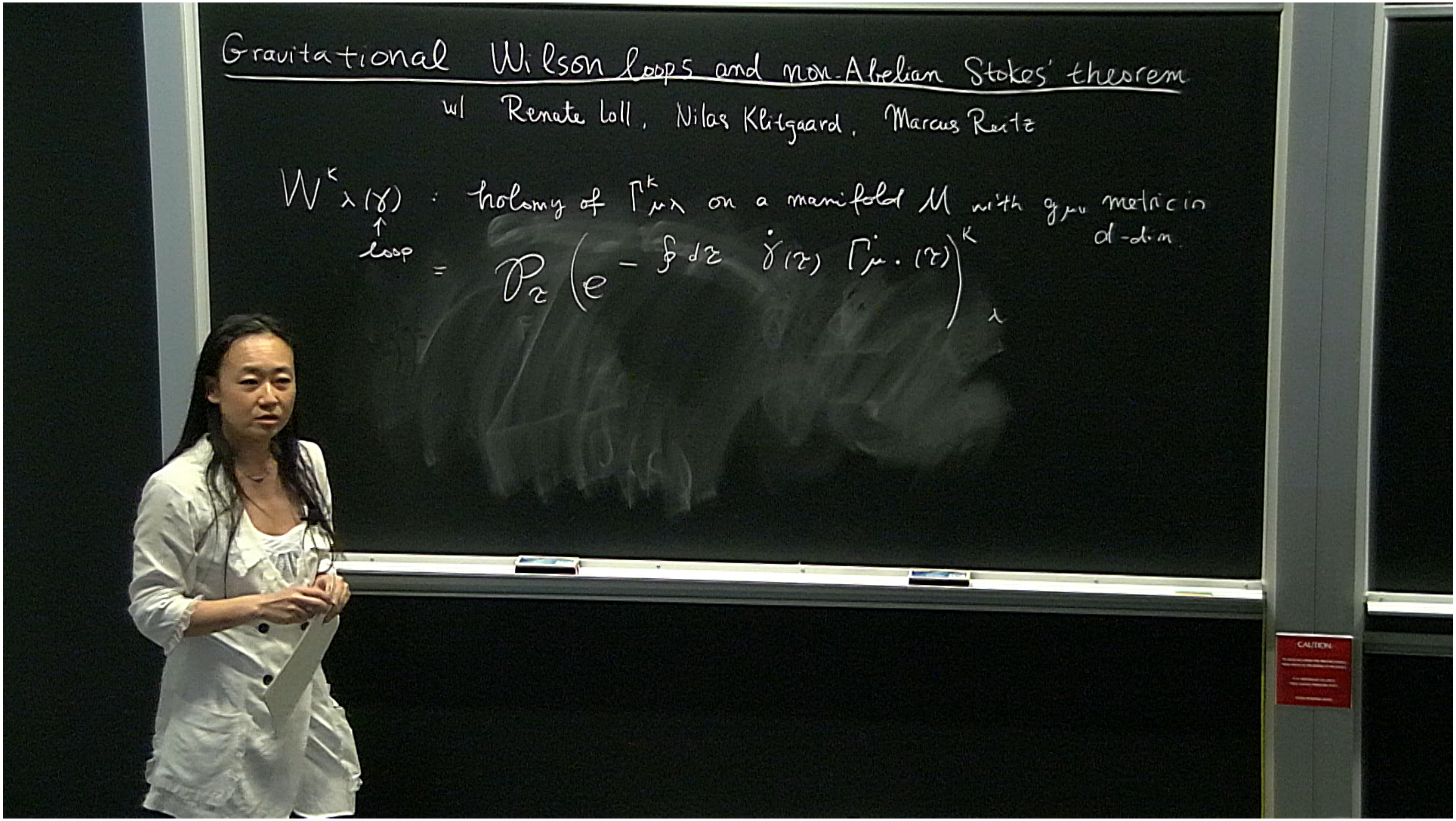
Speakers: Reiko Toriumi

Collection: Emmy Noether Workshop: The Structure of Quantum Space Time

Date: November 18, 2019 - 11:40 AM

URL: http://pirsa.org/19110086

Abstract: Finding suitable diffeomorphism-invariant observables to probe gravity at the Planck scale is essential in quantum gravity. The Wilson loop of the 4-dimensional Christoffel connection is a potentially interesting ingredient for the construction of such an observable. We have investigated to what extent and what form of curvature information of the underlying spacetime may be extracted from Wilson loops through a Stokesâ€™ theorem-like relation. We present an expression for the conservation of geometric flux as the quantity related to the gravitational Wilson loop.
This expression is surface-independent and it holds for a certain class of manifolds with global symmetries.


Gravitational Wilson loops and non-Abelian Stokes' theorem
w/ Renate Loll, Nilas Klitgaard, Marcus Reitz
$W^{\kappa}{}_{\lambda}(\gamma)$: holomy of $\Gamma^{\kappa}_{\mu\lambda}$ on a manifold $\mathcal{M}$ with $g_{\mu\nu}$ metric in d-dim
loop
$= \mathcal{P}_{\tau}\left(e^{-\oint d\tau \; \dot{\gamma}(\tau) \; \Gamma_{\mu\cdot}(\tau)}\right)^{\kappa}{}_{\lambda}$

Gravitational Wilson loops and non-Abelian Stokes' theorem

w/ Renate Loll, Nilas Klitgaard, Marcus Reitz

$W^{\kappa}{}_{\lambda}(\gamma)$: holonomy of $\Gamma^{\kappa}_{\mu\lambda}$ on a manifold M with $g_{\mu\nu}$ metric in d-dim.

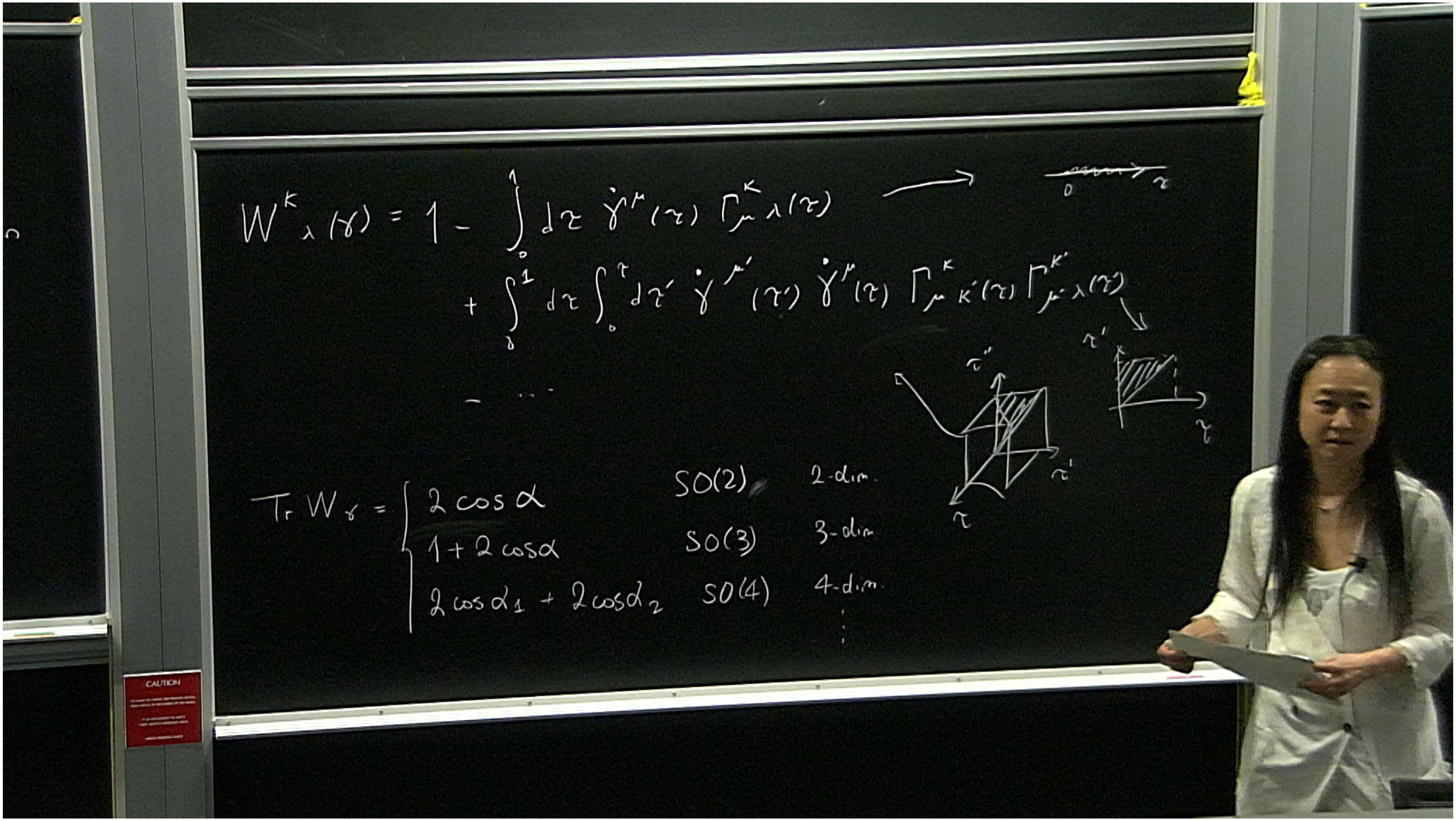
loop

$$= \mathcal{P}_{\tau}\left(e^{-\oint d\tau\, \dot{\gamma}(\tau)\, \Gamma_{\mu\cdot}(\tau)}\right)^{\kappa}{}_{\lambda}$$

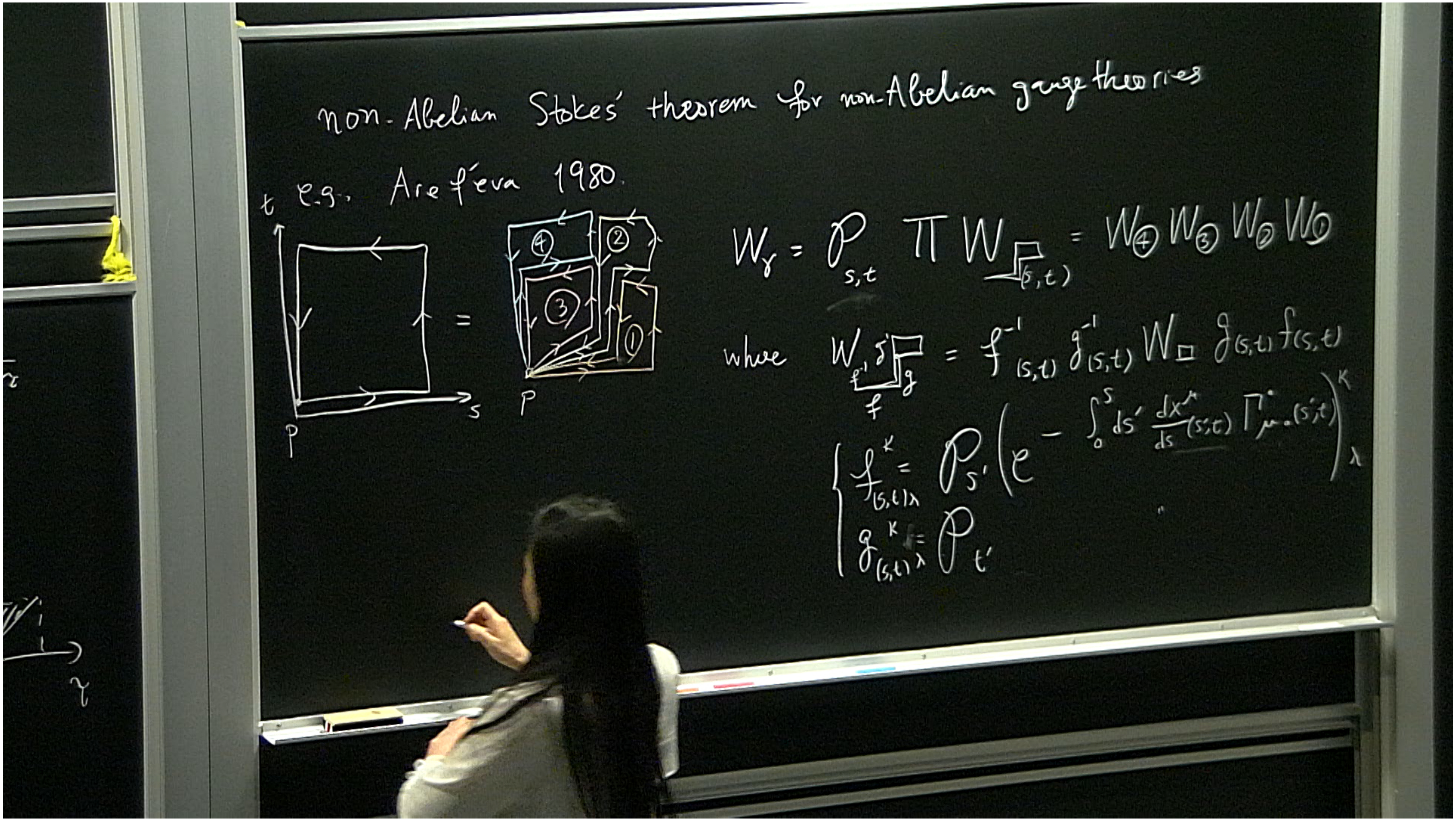
in Abelian-theories

$$W(c) = \cancel{\mathcal{P}}\, e^{-\oint_{c=\partial S} A_\mu dx^\mu} \doteq e^{-\int_S \vec{B}\cdot d\vec{A}}$$

Stokes' theorem



$W^{\kappa}{}_{\lambda}(\gamma) = 1 - \int_0^1 d\tau\, \dot\gamma^{\mu}(\tau)\, \Gamma^{\kappa}_{\mu\lambda}(\tau)$
$+ \int_0^1 d\tau \int_0^{\tau} d\tau'\, \dot\gamma^{\mu'}(\tau')\, \dot\gamma^{\mu}(\tau)\, \Gamma^{\kappa}_{\mu\kappa'}(\tau)\, \Gamma^{\kappa'}_{\mu'\lambda}(\tau')$
$- \dots$
0 τ
τ' τ
τ'' τ' τ
$\mathrm{Tr}\, W_\gamma = \begin{cases} 2\cos\alpha & SO(2) \quad \text{2-dim.} \\ 1 + 2\cos\alpha & SO(3) \quad \text{3-dim.} \\ 2\cos\alpha_1 + 2\cos\alpha_2 & SO(4) \quad \text{4-dim.} \end{cases}$
CAUTION



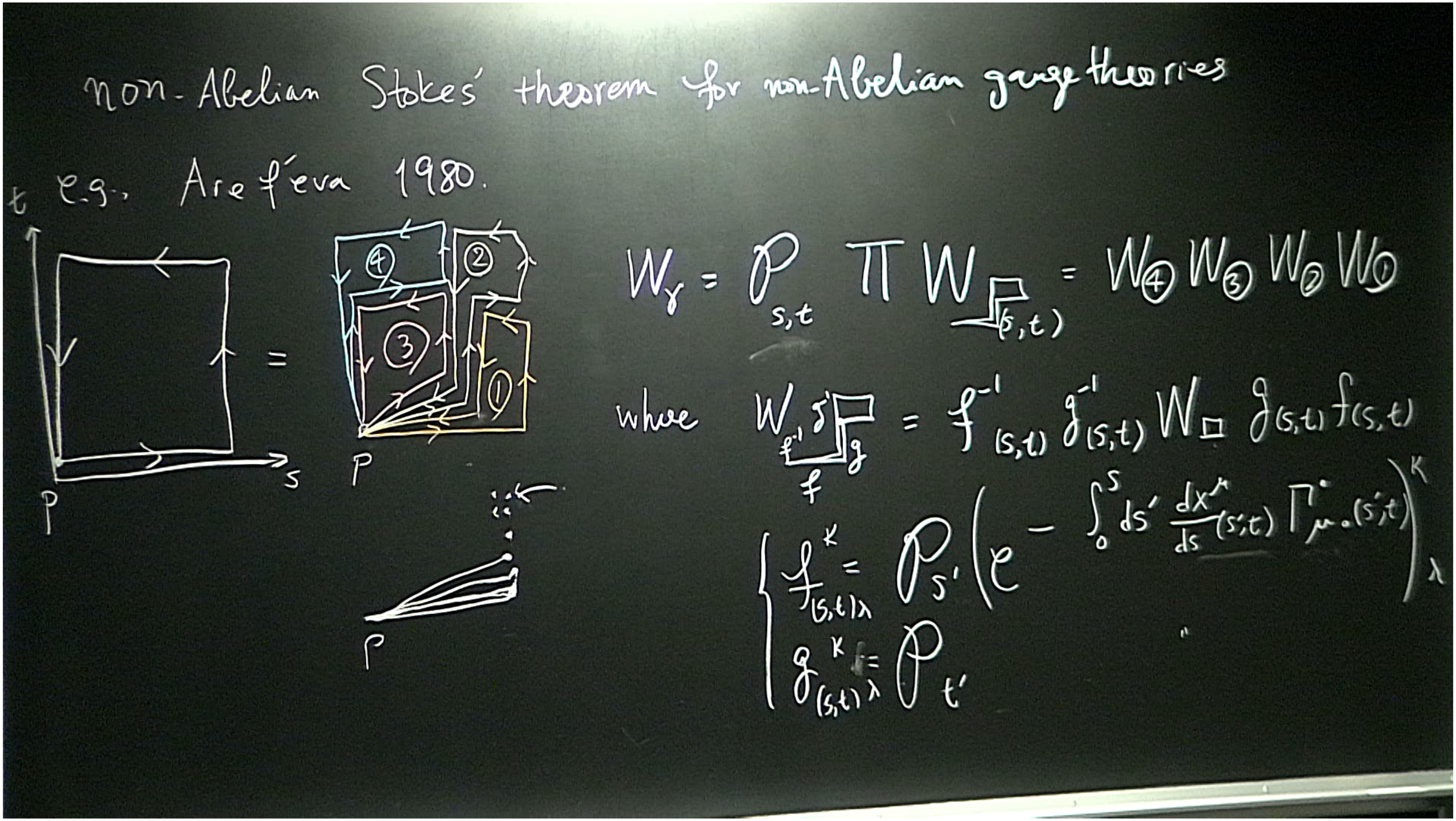
non-Abelian Stokes' theorem for non-Abelian gauge theories
e.g. Aref'eva 1980
$W_\gamma = \mathcal{P}_{s,t} \prod W_{\square(s,t)} = W_4 W_3 W_2 W_1$
where $W = f^{-1}_{(s,t)} g^{-1}_{(s,t)} W_\square g_{(s,t)} f_{(s,t)}$
$f^\kappa_{(s,t)\lambda} = \mathcal{P}_{s'} \left(e^{- \int_0^s ds' \frac{dx^\mu}{ds}(s',t) \Gamma^\kappa_{\mu\lambda}(s',t)} \right)^\kappa_\lambda$
$g^\kappa_{(s,t)\lambda} = \mathcal{P}_{t'}$

continuum limit,

$$(W_\gamma)^K{}_\lambda = \mathcal{P}_\tau\left(e^{-\int d\tau\, \dot{\gamma}(\tau)\, \Gamma_\mu{}^{\cdot}{}_{\cdot}(\tau)}\right)^K{}_\lambda$$

in the continuum limit

$$\longrightarrow \mathcal{P}_s\left(e^{-\iint f^{-1} g^{-1}\, \underline{R^{\cdot}{}_{\cdot\mu\nu}}\, \frac{dx^\mu}{ds}\frac{dx^\nu}{dt}\, g\, f\, ds\, dt}\right)^K{}_\lambda$$


non-Abelian Stokes' theorem for non-Abelian gauge theories
e.g., Aref'eva 1980.
t
s
P
=
④ ② ③ ①
$W_\gamma = \mathcal{P}_{s,t} \prod W_{(s,t)} = W_④ W_③ W_② W_①$
where $W_{f^{-1} g^{-1} \square g f} = f^{-1}_{(s,t)} g^{-1}_{(s,t)} W_\square g_{(s,t)} f_{(s,t)}$
$f^{\kappa}_{(s,t)\lambda} = \mathcal{P}_{s'} \left(e^{- \int_0^s ds' \frac{dx^\mu}{ds}(s',t) \Gamma_{\mu\cdot}^{\cdot}(s',t)} \right)^{\kappa}_{\lambda}$
$g^{\kappa}_{(s,t)\lambda} = \mathcal{P}_{t'}$

Stokes' theorem: $\displaystyle\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega$

$\partial\Sigma$: $n-1$ dim Σ: n-dim

e.g., in 3-dim, QED,

$$\oint_{l=\partial S} \vec{A}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{A}\cdot d\vec{S} \overset{\vec{B}=\vec{\nabla}\times\vec{A}}{=} \int_S \vec{B}\cdot d\vec{S} \equiv \Phi_B$$

magnetic flux

$$\oint_{S=\partial V} \vec{B}\cdot d\vec{S} = \int_V dV\, \vec{\nabla}\cdot\vec{B} = 0 \rightarrow \text{surface indep.}$$

$\because \vec{\nabla}\cdot\vec{B} = 0$

$$\int_{S_1} \vec{B}\cdot d\vec{S} = \int_{S_2} \vec{B}\cdot d\vec{S}$$

$\partial S_1 = \partial S_2 = l$

$$\text{"}A_\mu\text{"} = \text{"}\Gamma_\mu\text{"}.$$

$$\text{"}\vec{B}\text{"} = \underset{\uparrow}{K_s} \frac{\vec{\xi}}{|\xi|} \quad \leftarrow \text{"Killing vector"} \qquad K_s = \frac{R^\kappa{}_{\lambda\mu\nu}{}^{(M)}\, u_\kappa v^\lambda u^\mu v^\nu}{|u|^2|v|^2 - \langle u, v\rangle^2}$$

sectional curvature. spanned by u & v orthogonal to $\vec{\xi}$.

$$\text{"}\vec{\nabla}\cdot\vec{B}\text{"} = \vec{\nabla}\cdot K_s \frac{\vec{\xi}}{|\xi|} = 0$$

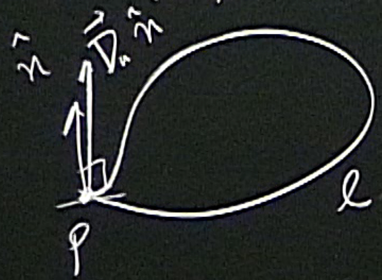
$$\text{so} \quad \text{"}\Phi\text{"} = \int_S \underbrace{K_s \frac{\vec{\xi}}{|\xi|}} \cdot d\vec{A} = \alpha_\gamma$$

in 3-dim.

On a total geodesic surface, normal vector

$$\mathrm{II}(u,v) = -\langle D_u \hat{n}, v \rangle = 0$$

tangent vectors

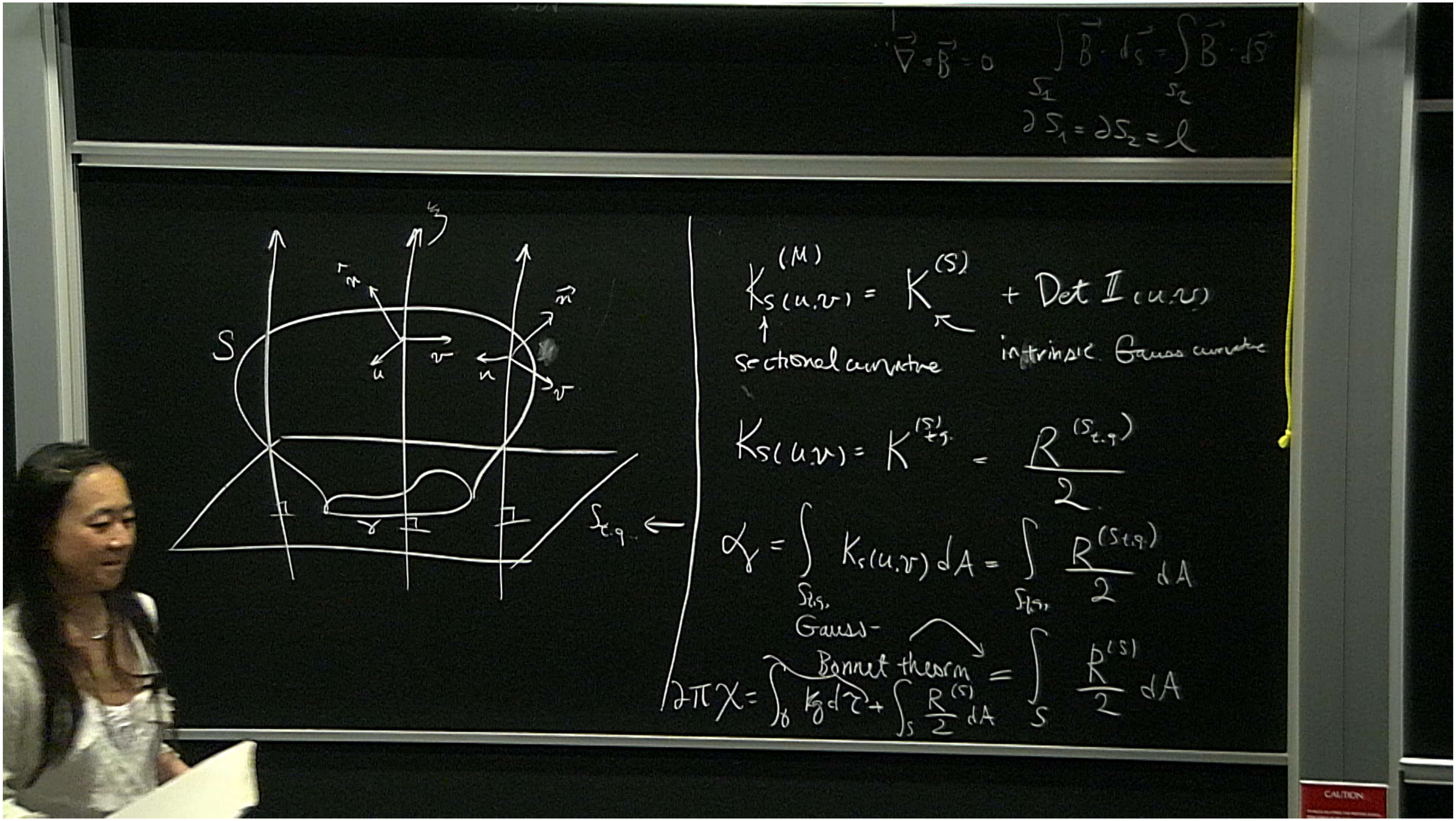


$\rightarrow W_\gamma \in SO(2)$

$\rightarrow [W_{\tilde{\gamma}(p)}, W_{\tilde{\tilde{\gamma}}(p)}] = 0$

no more surface ordering.

$$SO(d) \rightarrow SO(2) \oplus SO(d-2)$$


∇·B = 0   ∫_{S1} B·dS = ∫_{S2} B·dS
∂S1 = ∂S2 = ℓ
S
K_S(u,v)^(M) = K^(S) + Det II(u,v)
sectional curvature
intrinsic Gauss curvature
K_S(u,v) = K^(S_t.g.) = R^(S_t.g.)/2
S_t.g.
α_γ = ∫_{S_t.g.} K_S(u,v) dA = ∫_{S_t.g.} R^(S_t.g.)/2 dA
Gauss-Bonnet theorm
2πχ = ∫_γ k_g dτ + ∫_S R^(S)/2 dA = ∫_S R^(S)/2 dA