Title: How complement maps can cure divergences

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Abstract: Complements offer a separating device which proves useful for renormalisation purposes. A set and its set complement are disjoint, a vector space and its orthogonal complement have trivial intersection. Inspired by J. Pommersheim and S. Garoufalidis, we define a class of complement maps which give rise to a class of binary relations that generalise the disjointness of sets and the orthogonality of vector spaces. We discuss how these reflect locality in quantum field theory and how they can be used for renormalisation purposes.
This talk is based on joint work with Pierre Clavier, Li Guo and Bin Zhang.

Can complements cure divergences?

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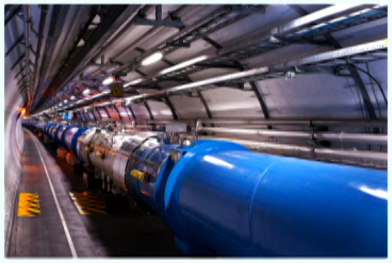
On leave from the University Clermont-Auvergne
joint work with Pierre Clavier, Li Guo and Bin Zhang

Perimeter Institute, Waterloo

November 18th 2019

How can taking a complement

be of any use for a **particle accelerator**?



They serve to separate subdivergences by means of

- either a coproduct and the induced algebraic Birkhoff-factorisation (à la Connes and Kreimer) using the associated convolution product;
- or a locality relation and the induced multivariable (à la Speer) minimal subtraction scheme,

both of which provide a device to extract finite parts from divergent quantities arising in quantum field theory

Prototypes of complement maps

are

- the set complement $\mathfrak{P}(X) \longrightarrow \mathfrak{P}(X)$ which sends A to $X \setminus A$;
- the orthogonal complement $\mathfrak{s}(V) \longrightarrow \mathfrak{s}(V)$ on an euclidean space V which sends W to $W^\perp$.

These complement maps stablise the set $\mathfrak{P}(X)$ (resp. $\mathfrak{s}(V)$) of subsets (resp. subspaces) of the set X (resp. the vector space V).

Complement maps should also include

- the "conical complement" of a face of a convex cone;
- the "tree complement" of a subtree of a rooted tree;
- the "graph complement" of a subgraph of Feynman graph.

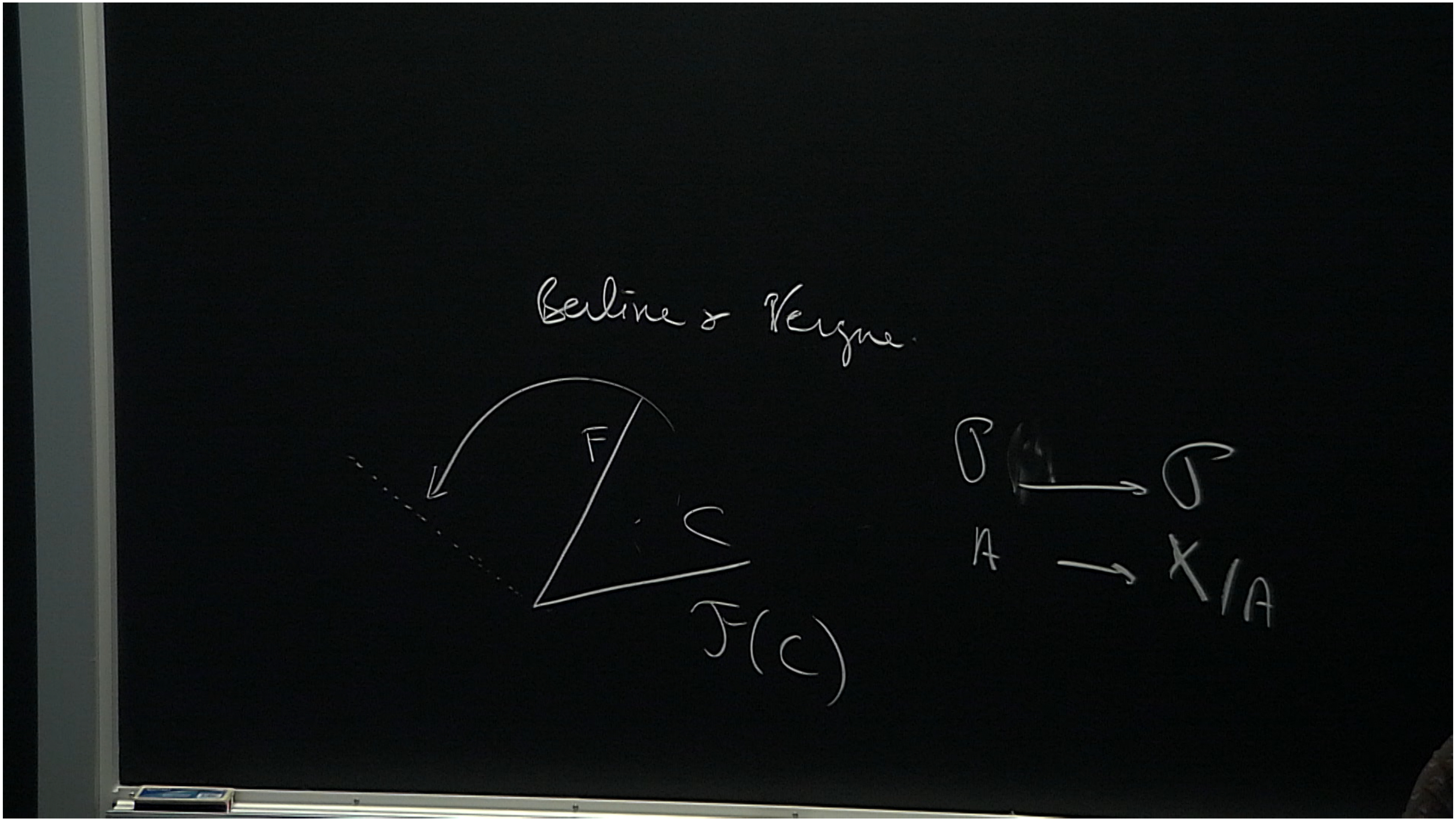
Common features of complement maps

The above complement maps Ψ are defined on posets $(\mathcal{P}, \leq)$ with a smallest element

1. On the power set $(\mathfrak{P}(X), \subset)$ of a set X whose smallest element is $\emptyset$;
2. On vector the set $(\mathfrak{s}(V), \preceq$ "to be a subspace of") of linear subspaces of a vector space V whose smallest element is $\{0\}$;
3. On the set $(\mathfrak{F}(C), \preceq$ "to be a face of") of faces of a convex poyhedral cone C, whose smallest element is $\{0\}$;
4. On the set $(\mathcal{T}(\mathfrak{t}), \preceq$ "to be a rooted subtree of") of subtrees of a tree $\mathfrak{t}$ whose smallest element is the root ($t' \preceq t$ if t' is the trunk that remains below an admissible cut of t);
5. On the set $(\mathfrak{G}(\Gamma), \preceq$ "to be a 1 PI subgraph of") of subgraphs of a graph Γ whose smallest element is the empty graph ($\Gamma' \preceq \Gamma$ is either empty or a nonempty (connected or disconnected) set of internal edges in Γ together with the vertices they encounter).

Complement maps mostly use an orthogonal or a set complement

1. On cones: the transverse cone $\Psi(F) = t(F, C) := \pi_{F^\perp}(C) \in \mathfrak{C}(\mathbb{R}^k)$ for a face F of $C \in \mathfrak{C}(\mathbb{R}^k)$;
2. On trees: $\Psi(\mathfrak{t}') = P_c(\mathfrak{t}') \in \mathcal{F}(\mathfrak{t})$ for a subtree $\mathfrak{t}'$ of $\mathfrak{t}$ obtained from the admissible cut c, here $P_c(\mathfrak{t}')$ is the crown above the cut, which might be a forest and not a tree;
3. On graphs: $\Psi(\Gamma') \in \mathfrak{G}(\Gamma)$ for a subgraph Γ' of Γ, is the contracted graph $\Gamma \backslash \Gamma'$ obtained by replacing all connected components of Γ' by their residues inside Γ.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Complement maps on posets (inspired by Garoufalidis and Pommersheim)

For E in a poset $(\mathcal{P}, \leq)$, we set $s(E) := \{A \in \mathcal{P} \,|\, A \leq E\}$.

A poset complement map

on $\mathcal{P}$ is a family of maps

$$\Psi_E : s(E) \to \mathcal{P}, A \mapsto \Psi_E(A) = E \,/\, A, \quad E \in \mathcal{P},$$

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2. $(C \,/\, A) \,/\, (B \,/\, A) = C \,/\, B$ for $A \leq B \leq C$;
3. $E \,/\, 1 = E$ for any $E \in \mathcal{P}$.

All the above examples are complement maps on posets.

A coproduct from a complement map

A complement map Ψ on a connected poset $(\mathcal{P}, \leq, 1)$ gives rise to a coproduct

$$\Delta(E) := \sum_{A \in s(E)} \Psi_E(A) \otimes A, \tag{1}$$

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The coproduct used to preserve multiplicativity

The data

- A graded algebra $\mathcal{P} = \oplus_{n=0}^{\infty} \mathcal{P}_n$ and a target algebra $\mathcal{M}$.
- A **coproduct** $\Delta_{\mathcal{P}}$ on $\mathcal{P}$ and a related **convolution product** $\phi_1 \star \phi_2 := m_{\mathcal{M}} \circ (\phi_1 \otimes \phi_2) \circ \Delta_{\mathcal{P}}$ of maps $\phi_i : (\mathcal{P}, m_{\mathcal{P}}) \longrightarrow (\mathcal{M}, m_{\mathcal{M}})$.

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The role of the coproduct: Birkhoff-Hopf factorisation Connes and Kreimer 98'

The coproduct is used to undo "fake" finite terms arising from hidden subdivergences: $\phi = \phi_-^{\star -1} \star \phi_+$.

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Forest formula [BPHZ] 57-68

The renormalised map $\phi^{\rm ren} := {\rm ev}_0 \circ \phi_+$ is multiplicative:

$$\phi^{\rm ren}(p_1\, p_2) = \phi^{\rm ren}(p_1)\, \phi^{\rm ren}(p_2).$$

Coproducts for renormalisation use

The coproducts on

1. cones: $\Delta(C) = \sum_{F \preceq C} t(F, C) \otimes F$;
2. trees: $\Delta(t) = \sum_{t' \preceq t} P_c(t') \otimes t'$;
3. graphs: $\Delta(\Gamma) = \sum_{\Gamma' \preceq \Gamma} \Gamma \backslash \Gamma' \otimes \Gamma'$.

can be used to implement

algebraic Birkhoff factorisation à la Connes and Kreimer

on maps $\Phi : \mathcal{P} \longrightarrow \mathcal{M}(\mathbb{C})$ with values in meromorphic germs at zero:

1. discrete sums on cones: $\Phi(C)(\vec{a}) = \sum_{\vec{n} \in C \cap \mathbb{Z}^k} \prod_{i=1}^{k} n_i^{\vec{a}_i}$ for a simplicial cone $C \subset R^k$, $\vec{a} \in \mathbb{Z}^k_{\geq 0}$;
2. zeta functions on trees: $\Phi(t)(\vec{a}) = \zeta_t(-\vec{a})$, $|\mathcal{V}(t)| = k$ and $\vec{a} \in \mathbb{Z}^k_{\geq 0}$;
3. Feynman graphs: $\Phi(\Gamma) =$ Feynman integral associated with Γ.

Locality relation

1. A locality relation (or **independence relation**) on a set X is a symmetric binary relation $\top \subseteq X \times X$. For $x_1, x_2 \in X$, we write $x_1 \top x_2$ if $(x_1, x_2) \in \top$. We use the notation $X \times_\top X$ for $\top$ and call $(X, \top)$ a locality set.
2. For a subset U of a locality set $(X, \top)$, let

$$U^\top := \{x \in X \,|\, (U, x) \subseteq X \times_\top X\}$$

is the **polar subset** of U.

Prototypes of Locality relations

1. Disjointness $A \top B \Longleftrightarrow A \cap B = \emptyset$ on a power set $\mathcal{P}(X)$;

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \iff P(A \cap B) = P(A)\, P(B).$$

Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$: $m \top n \iff m \wedge n = 1$.

Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

$L_1 \top L_2 \iff L_1 \pitchfork L_2 \iff T_x L_1 + T_x L_2 = T_x M \quad \forall x \in L_1 \cap L_2$.

Analysis: Almost-separation of supports

Let $\epsilon \geq 0$ and $U \subset \mathbb{R}^n$ be an open subset. Two functions $\phi, \psi \in \mathcal{D}(U)$ are independent i.e., $\phi \top \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

Locality and complement map

A complement map Ψ on a poset $(\mathcal{P}, \leq, 1)$ with biggest element E gives rise to a locality relation

$$A \top_\Psi B \Longleftrightarrow (B \in \Psi_E(A) \vee A \in \Psi_E(B)).$$

Prototype examples

- Disjointness on $\mathcal{P}(X)$: $A \cap B = \emptyset \Longleftrightarrow (A \subset X/B \vee B \subset X/A)$;
- Orthogonality on $\mathfrak{s}(V)$: $A \perp B \Longleftrightarrow (A \subset B^\perp \vee B \subset A^\perp)$.

Warning: Not every locality relation arises from a complement map.

Other examples

1. cones: $C_1 \top C_2 \Longleftrightarrow C_1 \perp C_2$;

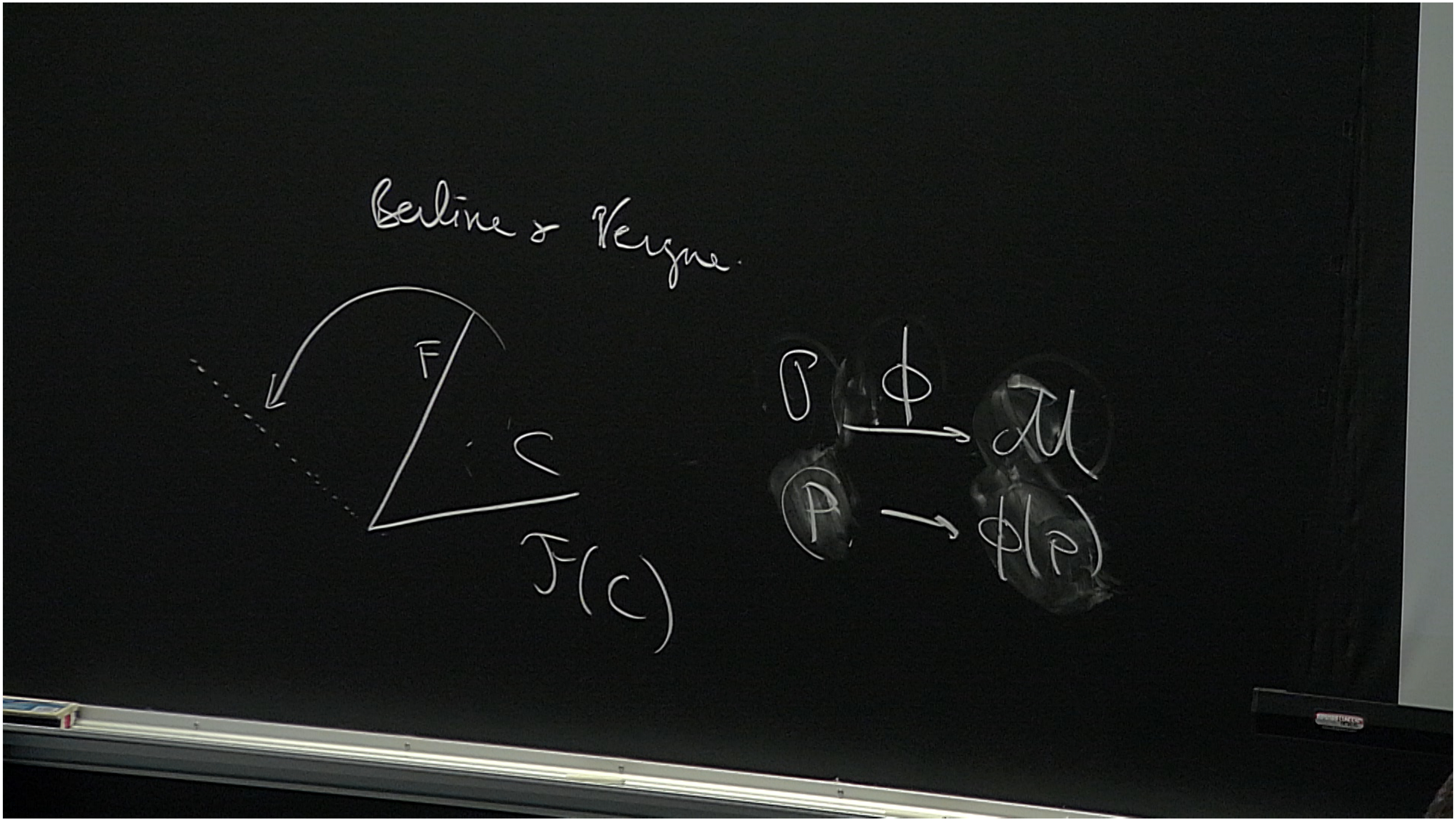
Algebraic locality versus locality in quantum field theory

Independence of probabilistic events

One event has no effect on the probability of another event occurring.

Independence of events in QFT

An object is only directly influenced by its immediate surroundings. Two events situated in different locations do not influence each other.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Independence of measurements

Observable $\mathcal{O} \longrightarrow$ Measurement $\langle\mathcal{O}\rangle \in \mathbb{C}$

$$\underbrace{\mathcal{O}_1 \text{ and } \mathcal{O}_2}_{\text{independent}} \underbrace{\Longrightarrow}_{\text{locality}} \underbrace{\langle\mathcal{O}_1 \star \mathcal{O}_2\rangle = \langle\mathcal{O}_1\rangle \cdot \langle\mathcal{O}_2\rangle}_{\text{multiplicativity}}.$$

Multivariable renormalisation (inspired by Speer)

We swap

- the coproduct Δ on the source space $\mathcal{P}$ for a locality relation $\top_{\mathcal{M}}$ on the target space $\mathcal{M}$: $\Delta_{\mathcal{P}} \rightsquigarrow \top_{\mathcal{M}}$;
- univariate for multivariate meromorphic functions: $\mathcal{M}(\mathbb{C}) \rightsquigarrow \mathcal{M}(\mathbb{C}^{\infty})$;
- Birkhoff-Hopf factorisation for a (naive) multivariate projection $\phi_+ \rightsquigarrow \pi_+ \circ \phi$.

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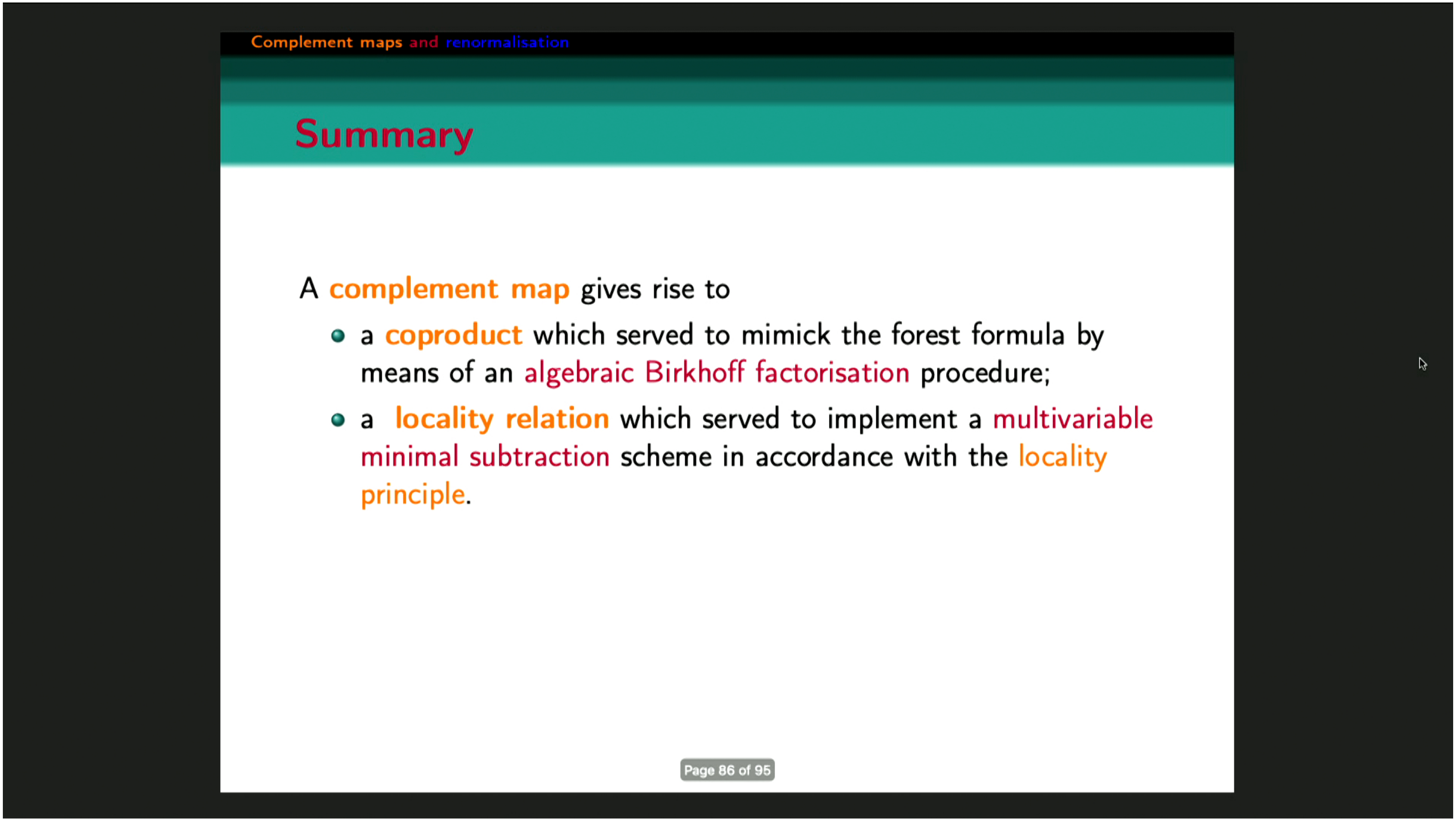
What for?

- It naturally encompasses the locality principle;
- Its universality: renormalisation π_+ takes place on the target space $\mathcal{M}(\mathbb{C}^\infty)$ common to various problems.

Summary

A complement map gives rise to

- a coproduct which served to mimick the forest formula by means of an algebraic Birkhoff factorisation procedure;


Complement maps and renormalisation
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a locality relation which served to implement a multivariable minimal subtraction scheme in accordance with the locality principle.
Page 86 of 95

Summary

A **complement map** gives rise to

- a **coproduct** which served to mimick the forest formula by means of an algebraic Birkhoff factorisation procedure;
- a **locality relation** which served to implement a multivariable minimal subtraction scheme in accordance with the locality principle.

Both serve to

- cure divergences and renormalise;

In progress

Locality relations versus complement maps

From a complement map we have built a locality relation.

Theorem: *There is a one to one correspondence between a class of locality relations and complement maps on finite dimensional vector spaces.*

Complement maps and Laurent expansions in multivariables

Appropriate complement maps yield a splitting of meromorphic germs.

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Complement maps and Laurent expansions in multivariables

Appropriate complement maps yield a splitting of meromorphic germs.

Theorem: *A class of complement maps ensure a splitting of the space* $\mathcal{M}(\mathbb{C}^k) = \mathcal{M}_+(\mathbb{C}^k) \oplus \mathcal{M}_-(\mathbb{C}^k)$ *and gives rise to a theory of Laurent expansions on* $\mathcal{M}(\mathbb{C}^k)$.

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THE END

THANK YOU !