Title: The Diamond Lemma for (multiplicative) preprojective algebras

Speakers:

Series: Mathematical Physics

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Abstract: Bergman's Diamond Lemma for ring theory gives an algorithm to produce a (non-canonical) basis for a ring presented by generators and relations. After demonstrating this algorithm in concrete, geometrically-minded examples, I'll turn to preprojective algebras and their multiplicative counterparts. Using the Diamond Lemma, I'll reprove a few classical results for preprojective algebras. Then I'll propose a conjectural basis for multiplicative preprojective algebras. Finally I'll explain why the set is a basis in the case of multiplicative preprojective algebras for quivers containing a cycle, the subject of joint work with Travis Schedler.

The Diamond Lemma for (mult.) preprojective algebras

jt w/ Travis Schedler

Ex: V vector space $\quad \mu : V \otimes V \longrightarrow V$

Define a deformation

$$\mathcal{U}_t(V, \mu) := \frac{T(V)[[t]]}{\left(x \otimes y - y \otimes x - t\mu(x,y) \right)_{x,y \in V}}$$

Q: When is $\underset{\substack{\cong \\ \mathrm{Sym}(V)[[t]]}}{\mathcal{U}_t(V, \mu)}$ flat?

e.g. $h_{U_1(V,\mathfrak{su})}(s) = h_{\mathrm{Sym}(V)}(s) = 1 + \dim(V)s + \cdots$

Diamond Lemma (Bergman 1978)

$$A = T(V)/(R) \rightsquigarrow T(V) \overset{\text{v.s.}}{\cong} A \oplus (R)$$

$$A \cong T(V)_{irr}$$

Idea: • Fix an order on V $\overset{\text{lex.}}{\rightsquigarrow}$ get order $\leq$ on $T(V)$

• R_i relation $\quad r_i : T(V) \longrightarrow T(V)$

$$lt(R_i) \longmapsto lt(R_i) - R_i$$

• Monomials without subwords of form $lt(R_i)$ are reduced (irred.)

Define $\mathrm{red}: T(V) \longrightarrow T(V)_{irr} \qquad p \longmapsto r_1 \circ \cdots \circ r_n(p)$

finite composition

irred

① $\exists$ of such a sequence

$\hookrightarrow$ (eg $k[x]/(x^2-x)$ $\quad x \to x^2$)

② two sequences need to give the same irred expression

$r_1 \circ r_2 = r_2 \circ r_1$

$W = \cdots\cdots\ \ell t(R_1) \cdots\ \ell t(R_2) \cdots$

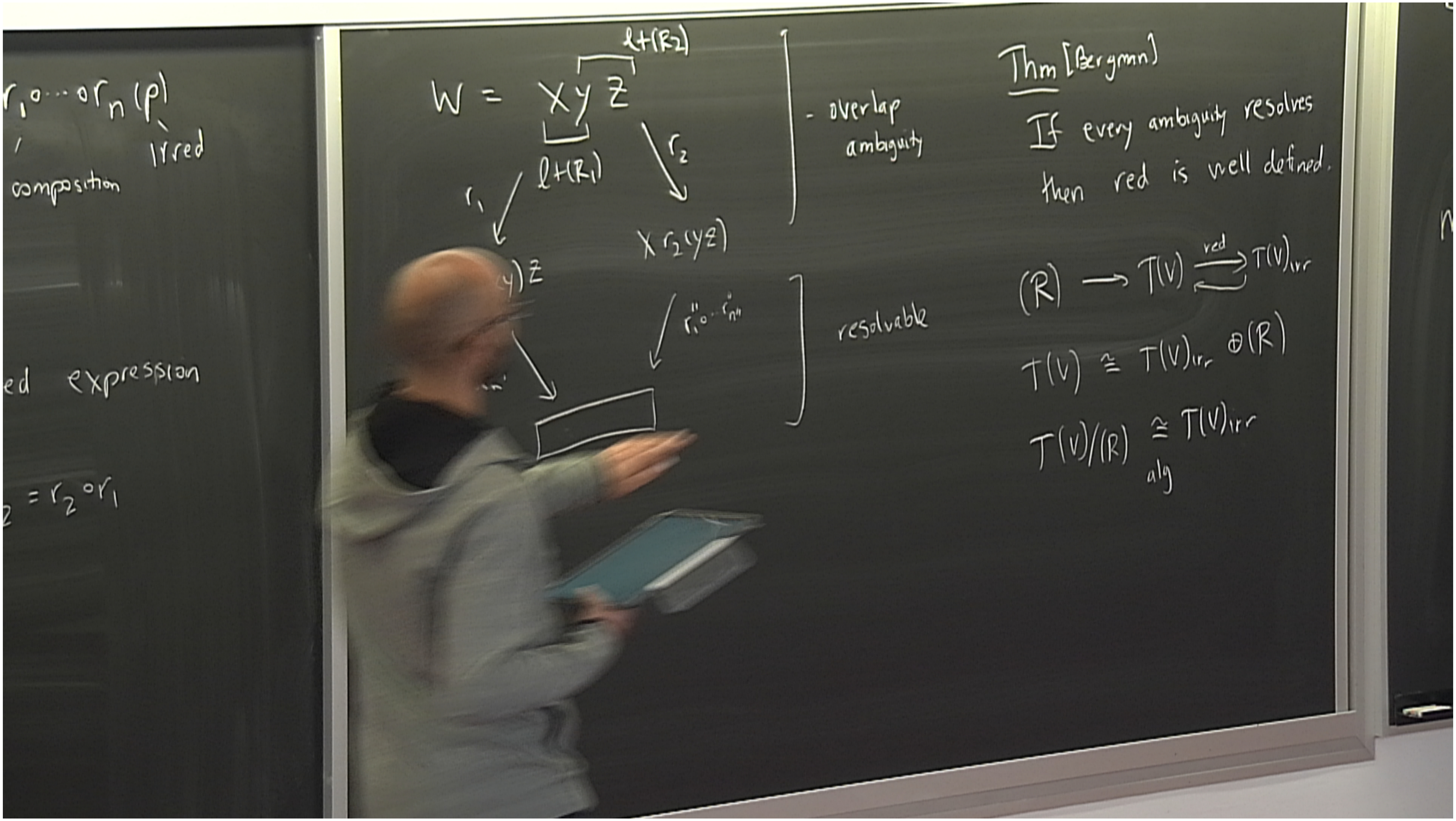
$A \oplus (R)$

$T(V)_{irr}$

$\leq$ on $T(V)$

$)-R_i$

$t(R_i)$ are reduced (irred.)



Thm [Bergman]
If every ambiguity resolves then red is well defined.
- overlap ambiguity
resolvable
irred
composition
expression

e.g. $h_{U_1(V,\mu)}(s) = h_{Sym(V)}(s) = 1 + \dim(V)\, s + \cdots$

Diamond Lemma (Bergman 1978)

Ex 1: $Sym(V) = T(V) / (x \otimes y - y \otimes x)$

Fix $\{x_1, .., x_d\}$ a basis V

$i<j$

$r_{ji}(x_j \otimes x_i) = x_i x_j$

$T(V)_{irr} = Span\{x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}\} \cong Sym(V)$

$z > y > x$

$$zyx \longmapsto xyz = (xy - yx)z - y(xz - zx) - (yz - zy)x$$

or

$$xyz = z(xy - yx) - (xz - zx)y - x(yz - zy)$$

$$[[x,y],z] + [y,[x,z]] + [[y,z],x] = 0$$

$$[y,[x,z]] = [[z,x],y]$$

$z > y > x$ Define $zyx \longmapsto xyz - (xy - yx)z - y(xz - zx) - (yz - zy)x$

or $xyz - z(xy - yx) - (xz - zx)y - x(yz - zy)$

$$[[x,y],z] + \underbrace{[y,[x,z]]}_{= [[z,x],y]} + [[y,z],x] = 0$$

Ex 2: $U_1(V,\mu) = T(V) / (x \otimes y - y \otimes x - \mu(x,y))$ $\quad x, y \in V$

For flatness, $r_{xx}(\mu(x,x)) = 0$ $\quad \mu$ alternating

$(yz - zy)x$

$- x(yz - zy)$

$= 0$

So $\mathfrak{g} = (V, M)$ is a Lie alg

$\Longleftrightarrow$

$\mathcal{U}(\mathfrak{g})$ has basis $\{x_1^{a_1} \cdots x_d^{a_d}\}$

$\Longleftrightarrow$

$\mathcal{U}_t(\mathfrak{g})$ is a flat def.

$\Longleftrightarrow$

$gr(\mathcal{U}(\mathfrak{g})) \cong Sym(\mathfrak{g})$

Thm [Bergman]

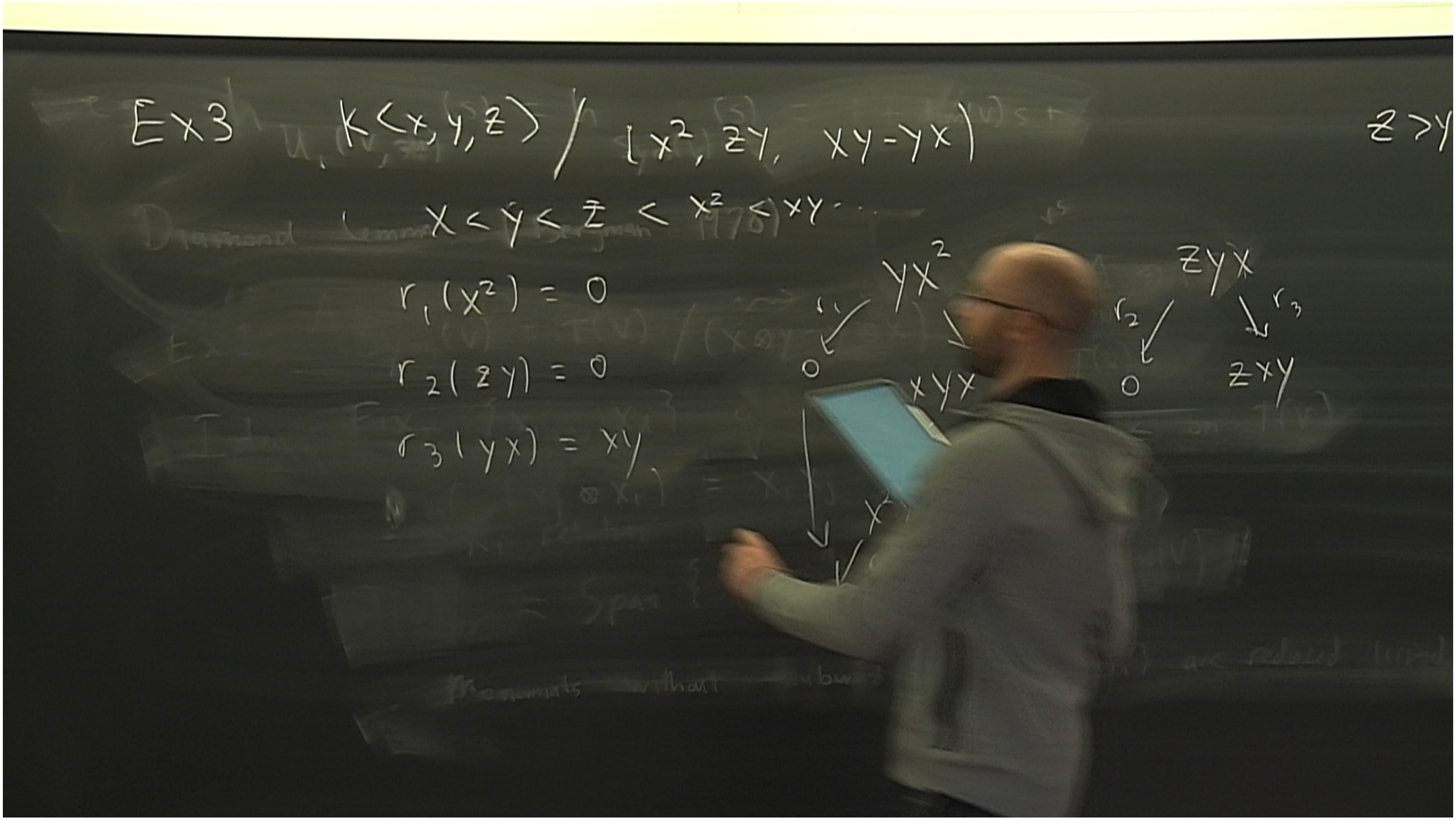
If every ambiguity resolves then red is well defined.

$(R) \longrightarrow T(V) \underset{\longleftarrow}{\xrightarrow{red}} T(V)_{irr}$

$T(V) \cong T(V)_{irr} \oplus (R)$

$T(V)/(R) \underset{alg}{\cong} T(V)_{irr}$

Multi


Ex3 k<x, y, z> / (x², zy, xy − yx)
z > y
x < y < z < x² < xy ...
r₁(x²) = 0
r₂(zy) = 0
r₃(yx) = xy
yx²
xyx
0
zyx
zxy
0

Ex 4 $\Pi^{\lambda}(Q)$ is a deformation of $\Pi^{0}(Q)$

Q: When is it flat?

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "a^*"] \end{tikzcd}$$

Ex 4a: $Q = A_2$ $\quad R = aa^* - a^*a - \lambda_1 e_1 - \lambda_2 e_2$

$e_1 R = aa^* - \lambda_1 e_1 \qquad e_2 R = -a^*a - \lambda_2 e_2$

$r_1(aa^*) = \lambda_1 e_1 \qquad r_2(a^*a) = -\lambda_2 e_2$

$$\begin{tikzcd} & aa^*a \arrow[dl] \arrow[dr] & \\ \lambda_1 a & & -\lambda_2 a \end{tikzcd}$$

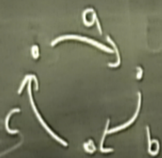
$\Rightarrow$ either $\lambda_1 + \lambda_2 = 0$

or $a = 0$

zx^2y

zxy

Ex 4b. $Q = \tilde{A}_2$



$$aa^* - c^*c = \lambda_1 e_1$$
$$bb^* - a^*a = \lambda_2 e_2$$
$$cc^* - b^*b = \lambda_3 e_3$$

no overlaps $\Rightarrow$ basis doesn't depend on λ_i

= (no clockwise after counterclockwise)

Thm [Bergman]

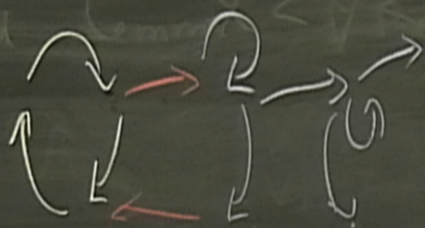
If every ambiguity resolves then red is well defined.

$$(R) \longrightarrow T(V) \underset{\longleftarrow}{\xrightarrow{\text{red}}} T(V)_{irr}$$

$$T(V) \cong T(V)_{irr} \oplus (R)$$

$$T(V)/(R) \underset{alg}{\cong} T(V)_{irr}$$

<u>Induction</u>, Q not Dynkin $\supset Q_E$



Filter $\Pi(Q)$ using red arrows

$\mathcal{F}_0 = \Pi(Q)$ $\quad$ $\mathcal{F}_1 =$ has at least 1 red arrow

$$gr(\Pi(Q)) \cong \Pi(Q_E) * \Pi(Q', (Q_E)_0)$$

~~Conj~~ Thm [K-Schedler]

If Q contains a cycle

non-Dynkin

then • $\Lambda^q(Q)$ is a 2-CY algebra

if $Q \overset{?}{\neq} \tilde{A}_n$ then CY is unique up to scaling

• $\Lambda^{q,dg}(Q)$ is formal, with homology $\Lambda^q(Q)$ in deg 0

• $\Lambda^q(Q)$ is prime

$\Lambda^1(\tilde{A}_n)$ is a NCCR over its center $\dfrac{K[x,y,z]}{(z^{n+1}+xy+xyz)}$

$\underline{\Lambda^q(\tilde{A}_2)}$ $\quad c \uparrow \overset{a}{\curvearrowright} \swarrow b$

$(1+aa^*) = q_1(1+c^*c)$

$(1+\alpha\alpha^*) = q^{(1+\alpha^*\alpha)}$

$\alpha = a+b+c$

$r_1(aa^*) = (aa^*+1)^{-1}$

- Inverse red

$(1+aa^*)a = a(1+a^*a)$

$a(1+a^*a)^{-1} = (1+aa^*)^{-1}a$

$(1+\alpha\alpha^*)^n\alpha^m \quad (1+\alpha\alpha^*)^n$

$= kQ_0\{(1+\alpha\alpha^*)^n\alpha^m,$