Title: Causal Set Action/Entropy

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Collection: Everpresent Lambda: Theory Meets Observations

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Scalar Field Theory on a Causal Set: Direct and Indirect Lessons for Everpresent Λ

Everpresent Lambda: Theory meets Observations
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Outline

- Free (gaussian) scalar field theory on a causal set.
- The Sorkin-Johnston vacuum state.
- Entanglement entropy from spacetime two-point correlation function.
- D'Alembertian operator on a causal set.
- Causal set action.

Quantum Field Theory

A quantum field theory is typically fully determined by the set of its n-point functions.

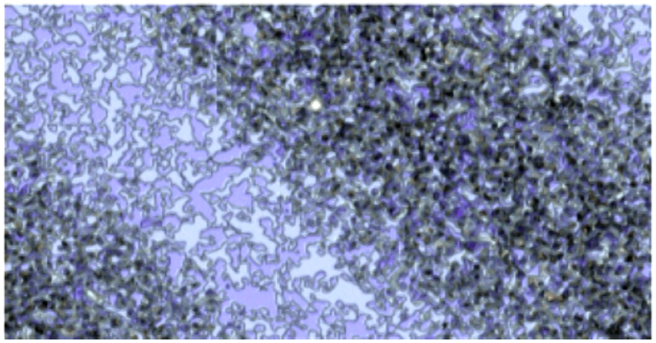
$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle, ..., \langle 0|\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle \quad (1)$$

If we consider a gaussian (for example scalar) field theory, then we only need to know

$$W = \langle 0|\phi(x_1)\phi(x_2)|0\rangle \quad (2)$$

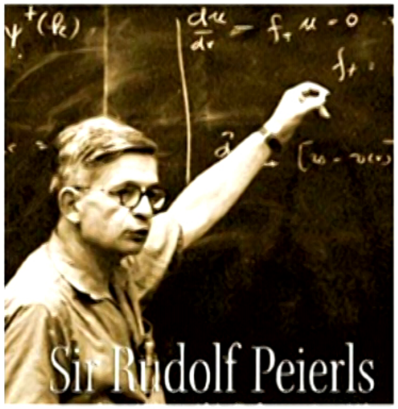
to fully determine the theory. All the higher n-point functions can be deduced from the 2-point function in this case.

So let's find W in a causal set.



Finding W in the Causal Set

The covariant commutation relations are given by the Peierls bracket



$$[\hat{\phi}(x), \hat{\phi}(x')] = i\Delta(x, x'), \qquad (3)$$

where the Pauli-Jordan function is

$$i\Delta(x, x') \equiv i(G_R(x, x') - G_A(x, x')), \qquad (4)$$

with $G_{R,A}(x, x')$ being
the retarded and advanced Green functions.

$$\mathrm{Ker}(\hat{\Box} - m^2) = \overline{\mathrm{Im}(\hat{\Delta})}. \qquad (5)$$

Thus the eigenvectors in the image of $i\hat{\Delta}$ span the full solution space of the KG operator.

The Sorkin-Johnston Vacuum [1]

$i\Delta$ is a self-adjoint operator on a bounded region of spacetime.

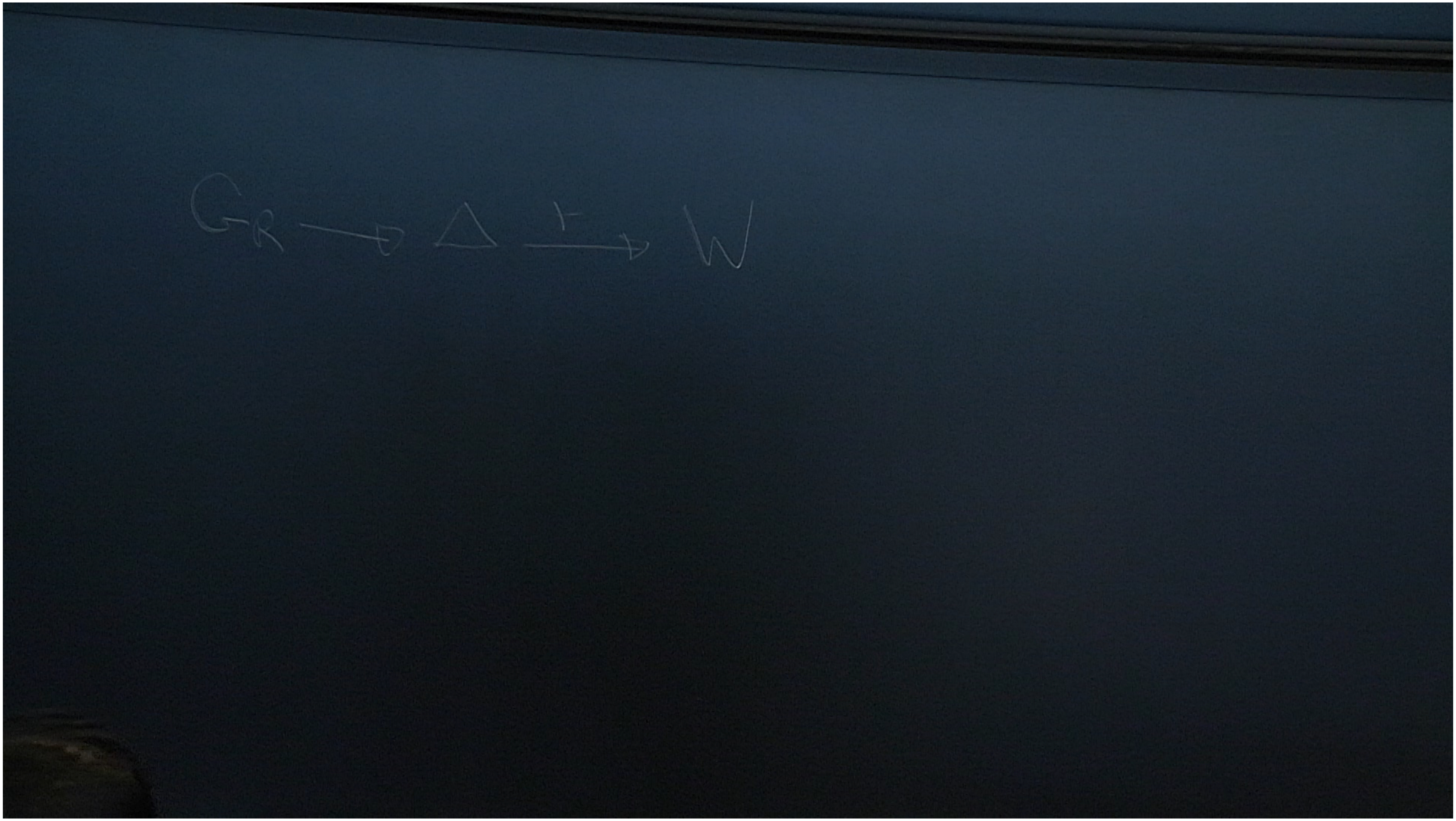
Write $i\Delta(x, x')$ in terms of its positive (u_k) and negative (v_k) eigenfunctions:

$$i\Delta(x, x') = \sum_k \left[\lambda_k u_k(x) u_k^\dagger(x') - \lambda_k v_k(x) v_k^\dagger(x')\right]. \qquad (6)$$

Restrict to positive eigenspace to get the Wightman or two-point function in the SJ vacuum:

$$W_{SJ}(x, x') \equiv \mathrm{Pos}(i\Delta) = \sum_k \lambda_k u_k(x) u_k^\dagger(x'). \qquad (7)$$

[1] R D Sorkin, J. Phys. Conf. Ser. 306 (2011) 012017.
S P Johnston (2010) arXiv:1010.5514.


G_R → Δ → W

Some Properties of the SJ Vacuum

- Uses the entire spacetime volume.
- An observer independent vacuum which is unique.
- In static spacetimes, the SJ state is the same one that is picked out by the timelike and hypersurface-orthogonal Killing vector.
- While not necessarily Hadamard itself, a family of Hadamard states can be constructed from it.
- Is a pure state for the spacetime definition of EE (while its restriction to a smaller subregion is not pure).

SJ State and Cosmology

- FRW: Results suggesting there are correlations on super-horizon scales. N Afshordi, S Aslanbeigi, R D Sorkin, JHEP 08 (2012) 137.
- de Sitter: Sometimes get $\alpha-$vacua. S Aslanbeigi, M Buck, JHEP 08 (2013) 039.
- de Sitter: New de Sitter invariant vacuum in 4d. S Surya, Nomaan X, and YKY, JHEP 1907 (2019) 009.

Can we learn something about the early universe from the SJ vacuum?

de Sitter invariant α-Vacua

α-vacua are a two-real-parameter family of dS invariant vacua. $\alpha = 0$ is special (Hadamard) and is called the Euclidean or Bunch-Davies vacuum[2].

The Wightman function for the Euclidean vacuum in d is given by

$$W_E(x,y) = \frac{\Gamma[h_+]\Gamma[h_-]}{(4\pi)^{d/2}\ell^2\Gamma[\frac{d}{2}]}\, {}_2F_1\left(h_+, h_-, \frac{d}{2}; \frac{1+Z(x,y)+i\epsilon\,\mathrm{sign}(x^0-y^0)}{2}\right)$$

where $Z(x,y) = \eta_{AB}X^A(x)X^B(y)$, $h_\pm = \frac{d-1}{2} \pm \nu$, $\nu = \ell\sqrt{\left(\frac{d-1}{2\ell}\right)^2 - m^2}$, and ${}_2F_1$ is a hypergeometric function.

It is usually said that there is no known de Sitter invariant Fock vacuum for the massless, minimally coupled theory.

[2] Also known by other names.

Results: 2d massless & massive, $ds^2 = \frac{1}{\cos^2 \tilde{T}} \left(-d\tilde{T}^2 + d\Omega^2_{d-1} \right)$

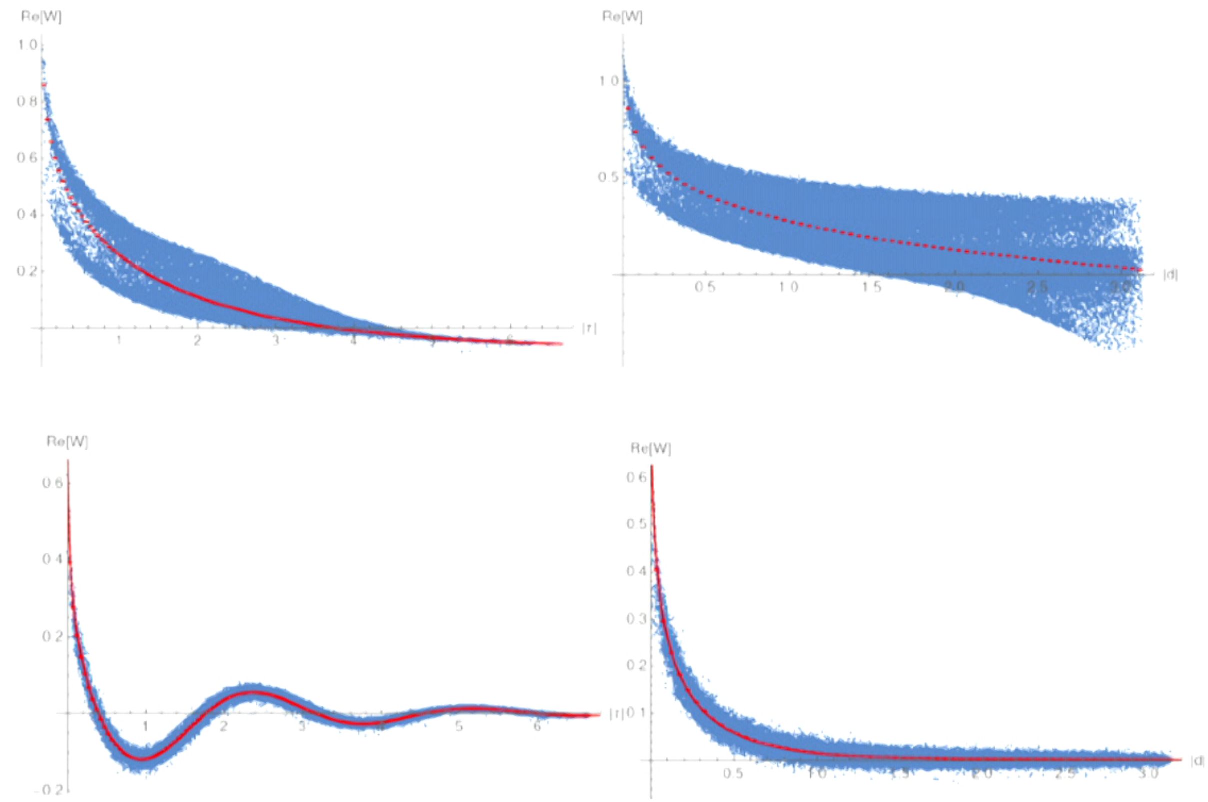


Figure: Upper: massless scatter plot with mean values in red. Lower: m=2.3 scatter plot with W_E in red. Left: causal. Right: spacelike. $T = \tilde{T}_{max} = 1.5$

Results: 4d massless & massive

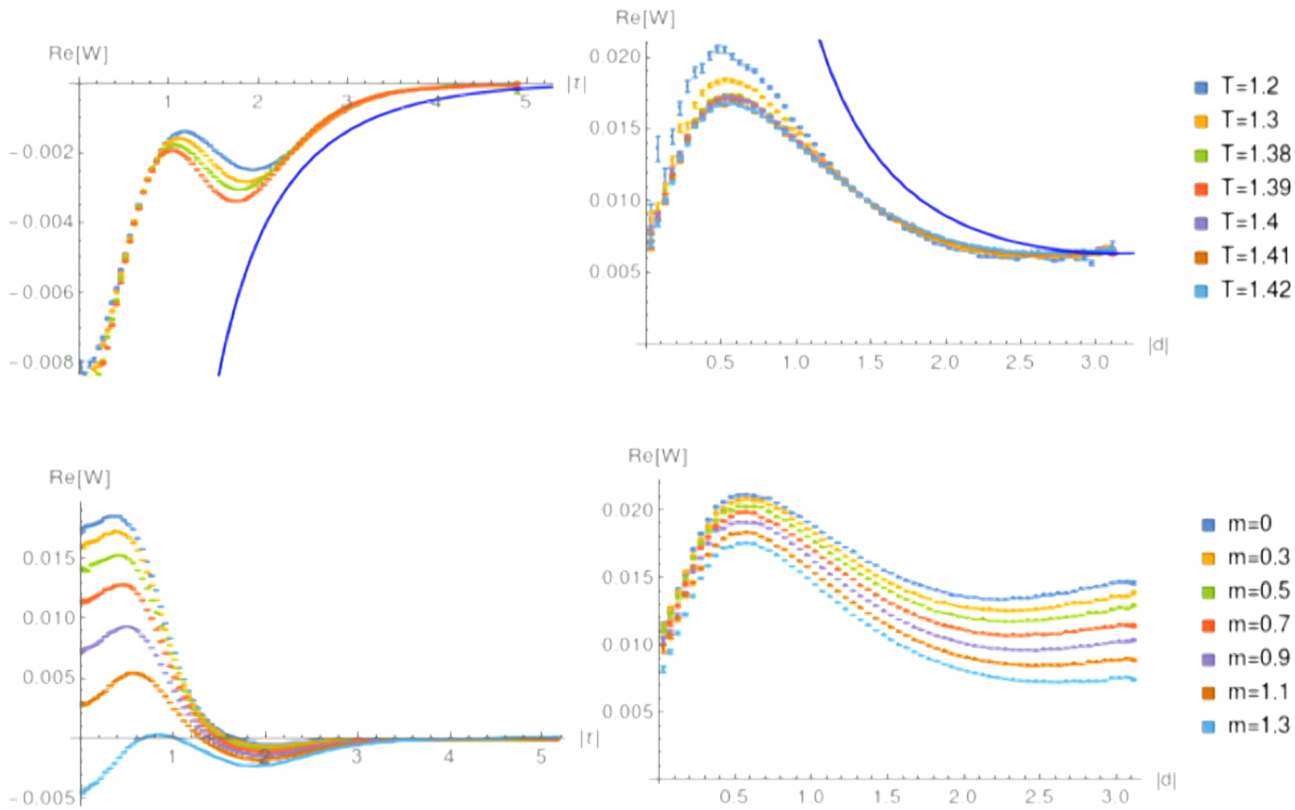


Figure: Upper: m=1.41 mean values with W_E in blue. Lower: T=1.42 mean values. Left: causal. Right: spacelike

Entanglement Entropy

In conventional treatments, entropy is defined as

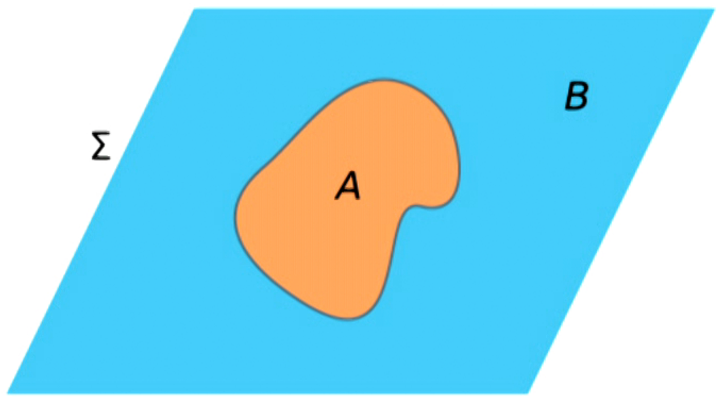
$$S = \mathrm{Tr}\rho \ln \rho^{-1} \tag{8}$$

where ρ is a density matrix on a spatial hypersurface Σ.
If Σ is divided into complementary subregions A and B, then the reduced density matrix for subregion A is

$$\rho_A = \mathrm{Tr}_B \rho \tag{9}$$

and its entanglement entropy with region B is

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A \ . \tag{10}$$



Entanglement Entropy in terms of W[3]

Express S directly in terms of the spacetime correlation function.

The entropy can be expressed as a sum over the solutions λ of the generalized eigenvalue problem

$$W\,v = i\lambda\,\Delta\,v, \qquad (\Delta v \neq 0) \tag{11}$$

as

$$S = \sum_{\lambda} \lambda \ln|\lambda|\ . \tag{12}$$

W and $i\Delta$ are the Wightman and Pauli-Jordan matrices.

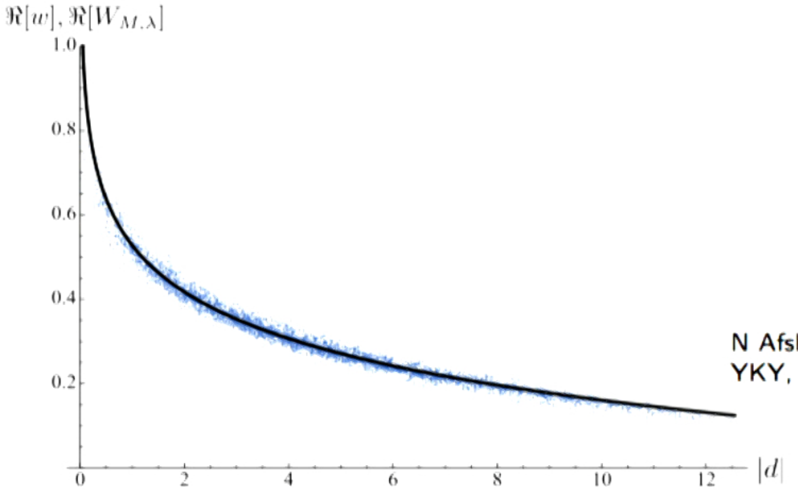
[3] R. D. Sorkin, arXiv:1205.2953

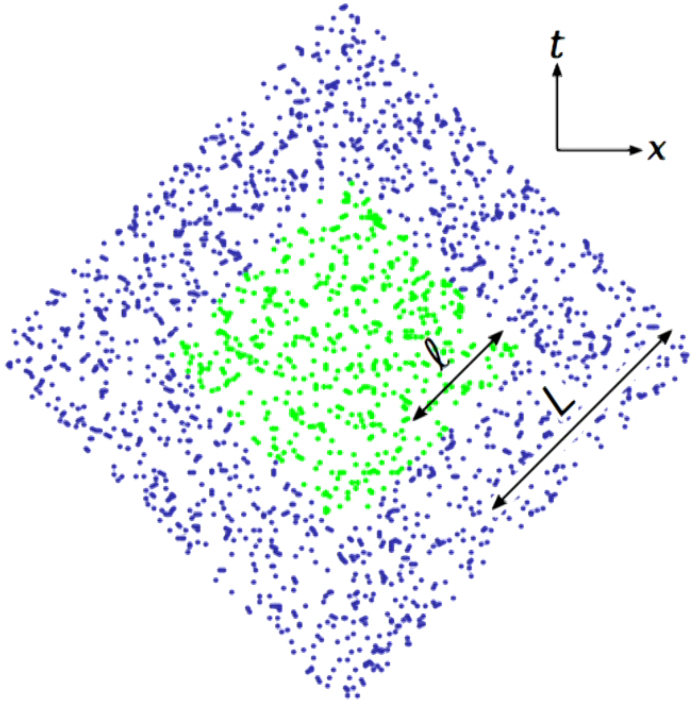
Nested Diamond Setup for Causal Set EE

$$\Delta(X, X') := G_R(X, X') - G_R(X', X)$$

For $m = 0$, we have that $G_R = \frac{1}{2}C$, where C is the causal matrix: $C_{xy} := \begin{cases} 1, & \text{if } x \preceq y. \\ 0, & \text{otherwise} \end{cases}$

For W, we choose W_{SJ}.





N Afshordi, M Buck, F Dowker, D Rideout, R D Sorkin, YKY, JHEP10(2012)08, arXiv:1207.7101.

$1+1$d Causal Set Results: Area Law[4]

The EE fits $S = b\ln(\sqrt{N_\ell}/4\pi) + c$ with $b = 0.346 \pm 0.028$ and $c = 1.883 \pm 0.035$. This is the usual result.

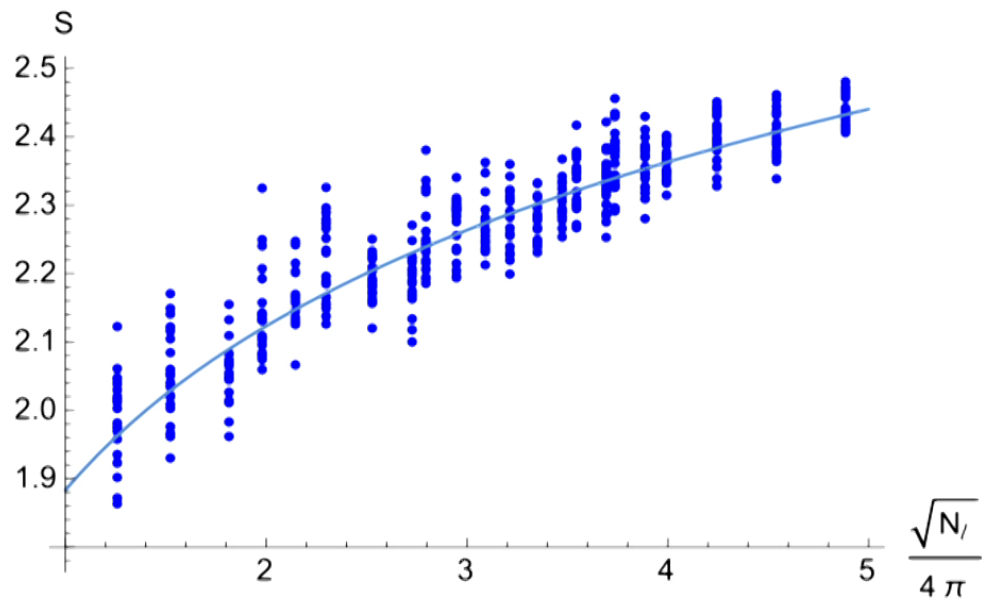


Figure: S vs. $\sqrt{N_\ell}/4\pi$. $\ell/L = 1/2$ in this example.

[4] R D Sorkin, YKY, CQG 35 074004 (2018).

EE of de Sitter Horizons in Causal Sets[5]

Currently investigating EE of horizons in dS.

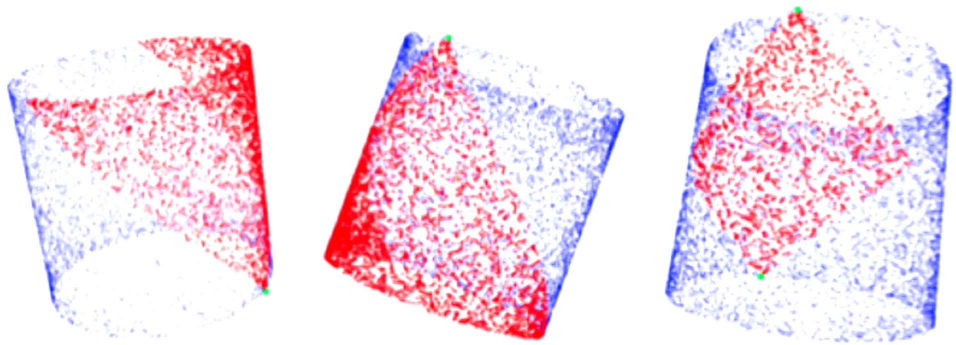


Figure: Horizons in 2d dS causal sets in conformal coordinates.

Preliminary results show that the spacetime entropy yields an area law in this case as well.

[5] S Surya, NX, YKY.

Entropy Formula has Broad Applications

The formulae

$$W\, v = i\lambda\, \Delta\, v, \qquad (\Delta v \neq 0) \tag{13}$$

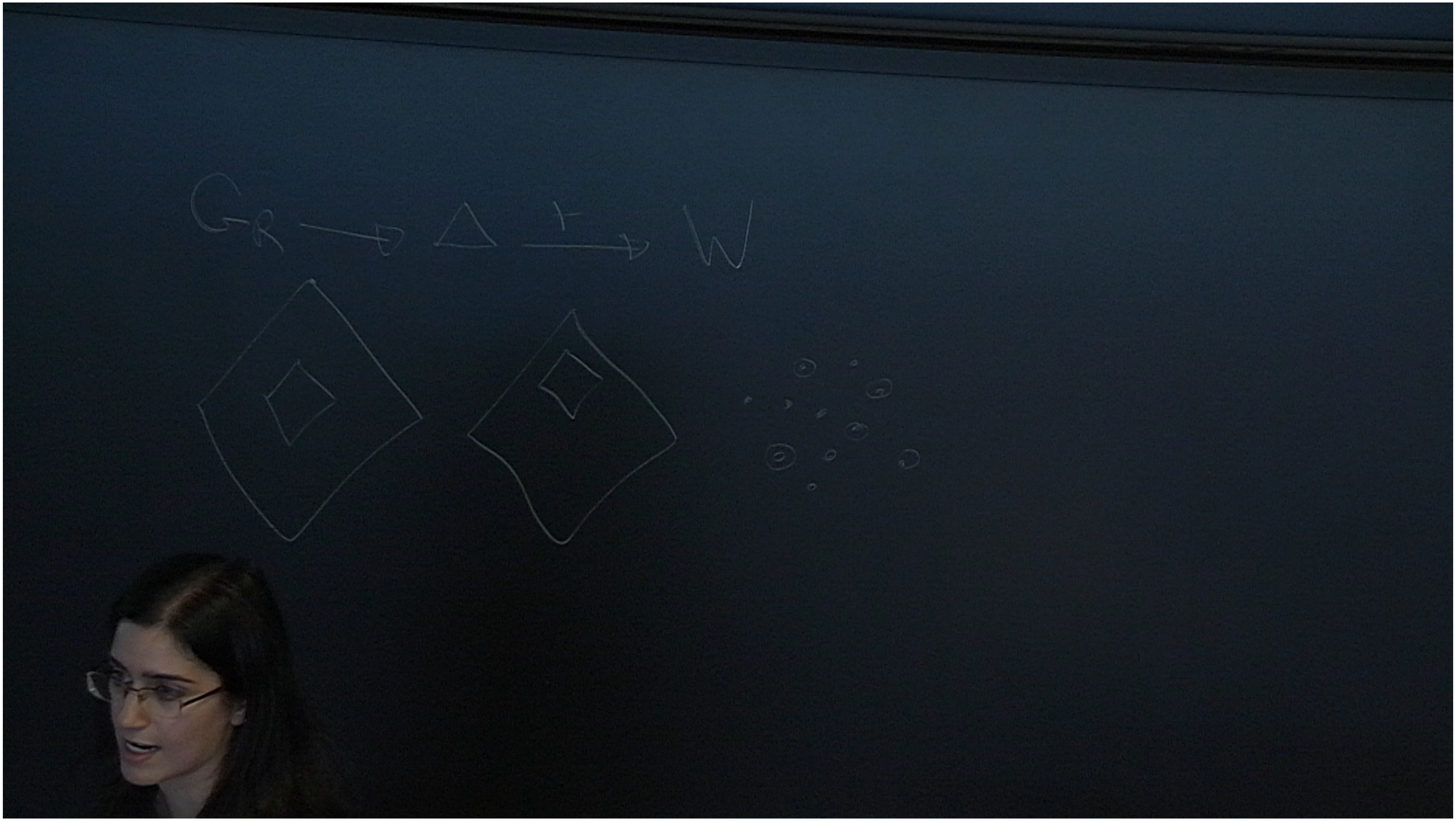
$$S = \sum_{\lambda} \lambda \ln|\lambda| \ . \tag{14}$$

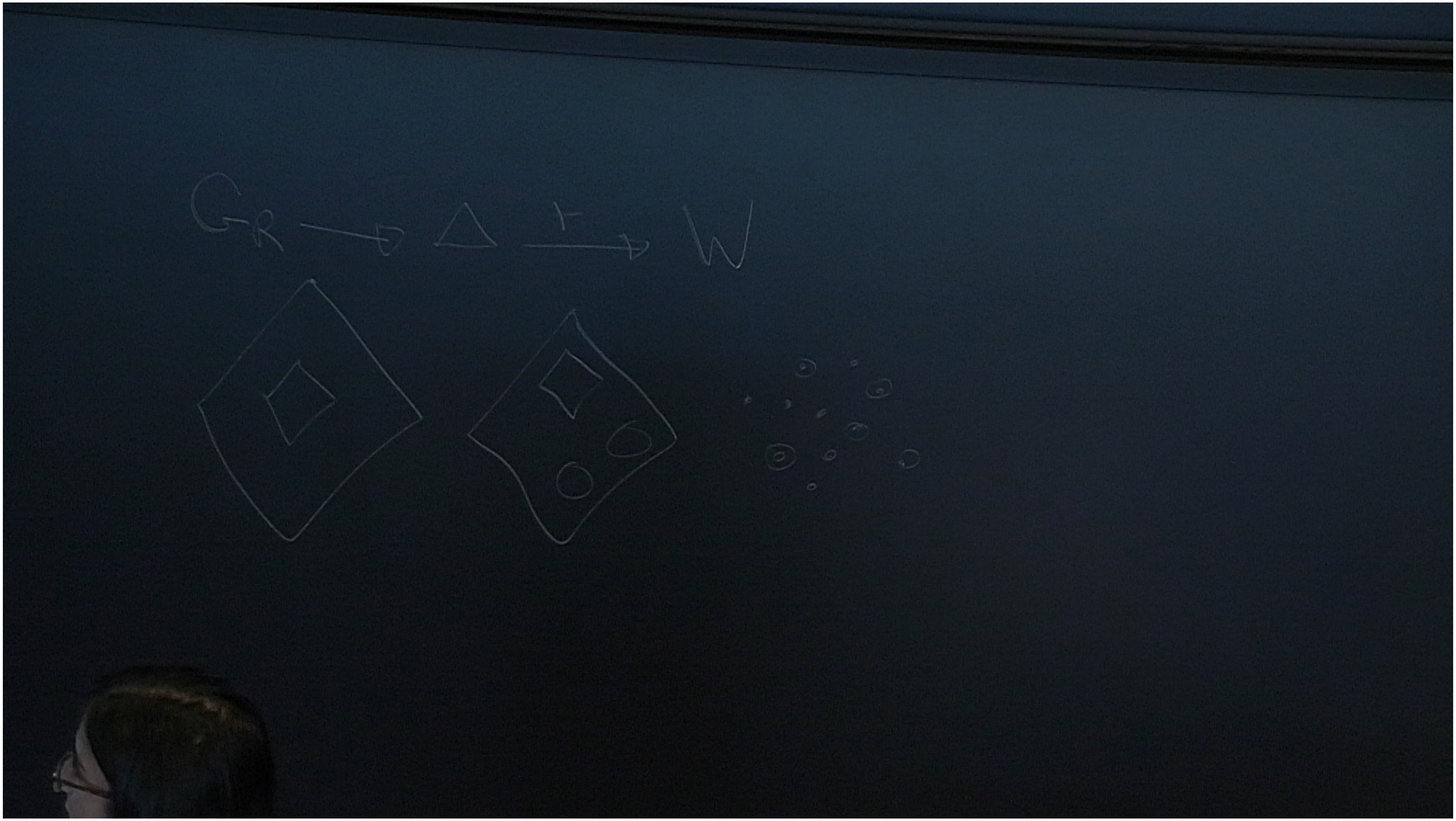
can be generally applied to any region(s) of spacetime within another.

For example we can consider entropy of coarse graining, or nested regions whose Cauchy surfaces are not subsets of each other.

They also turns out to approximately hold in some non-gaussian theories[6].

[6] Y Chen, R Kunjwal, H Moradi, YKY, and M Zilhão, in preparation.

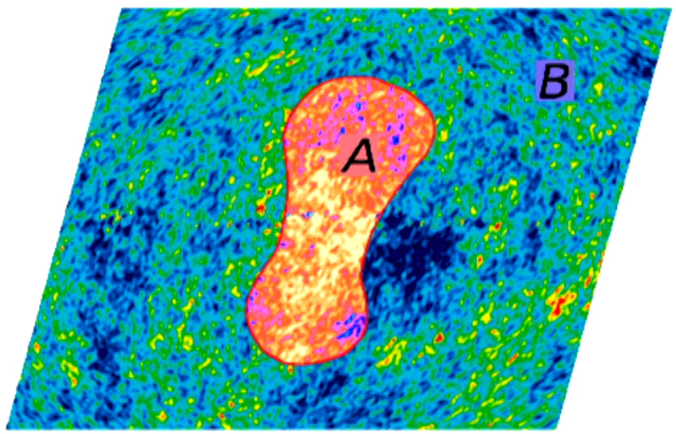

G_R
Δ
W


GR → Δ → W

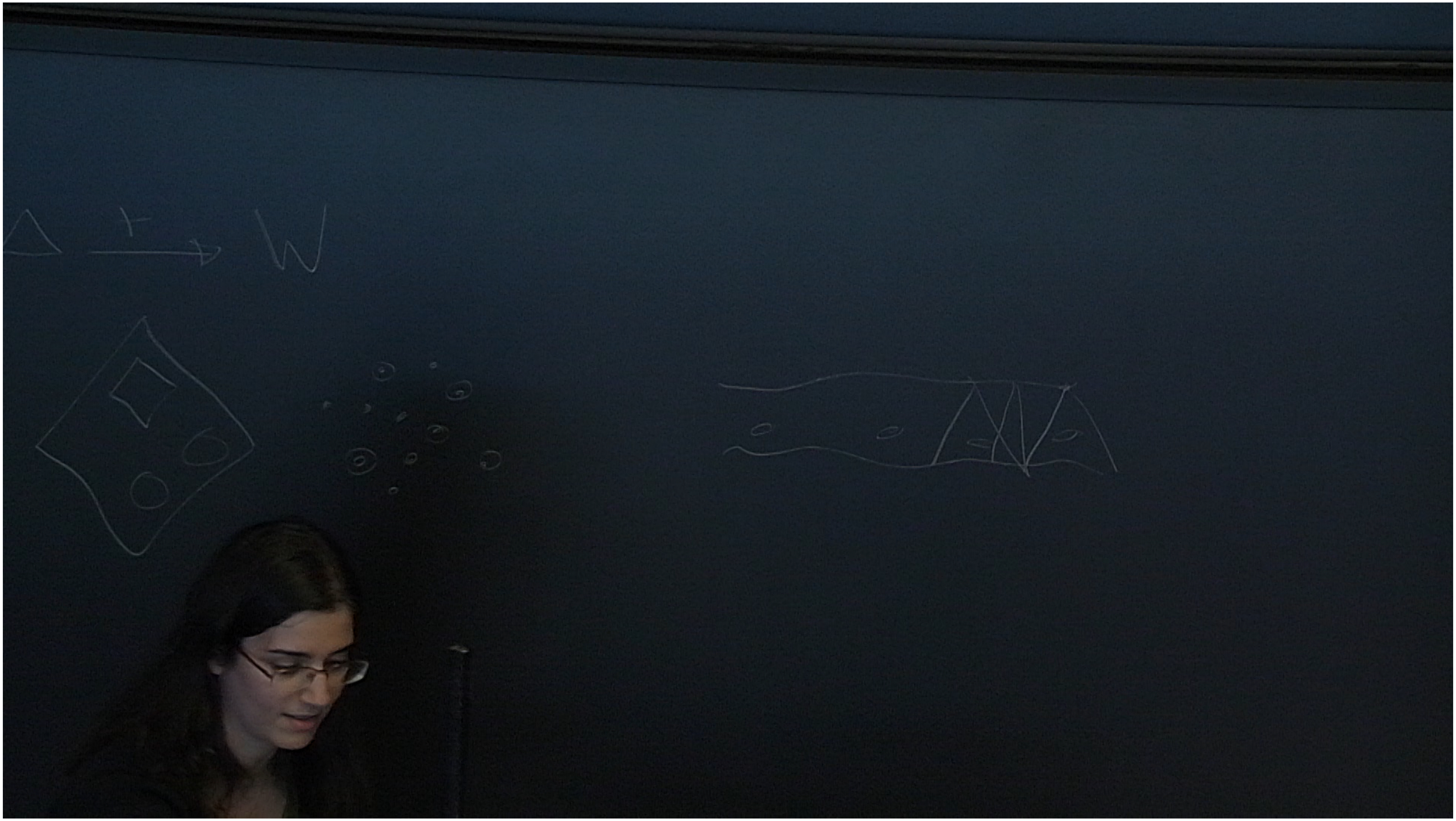
EE as a Probe of Inhomogeneities?

Can we use entanglement entropy to model inhomogeneities? This would also allow us to study it quantum mechanically.

Results on the SJ vacuum in FRW hinted at correlations on superhorizon scales[7]. These appeared without the help of inflation. Since EE measures information loss due to loss of correlation information (across inaccessible regions such as horizons), these structures may be quantifiable with EE.



[7] N Afshordi, S Aslanbeigi, R D Sorkin, JHEP 08 (2012) 137

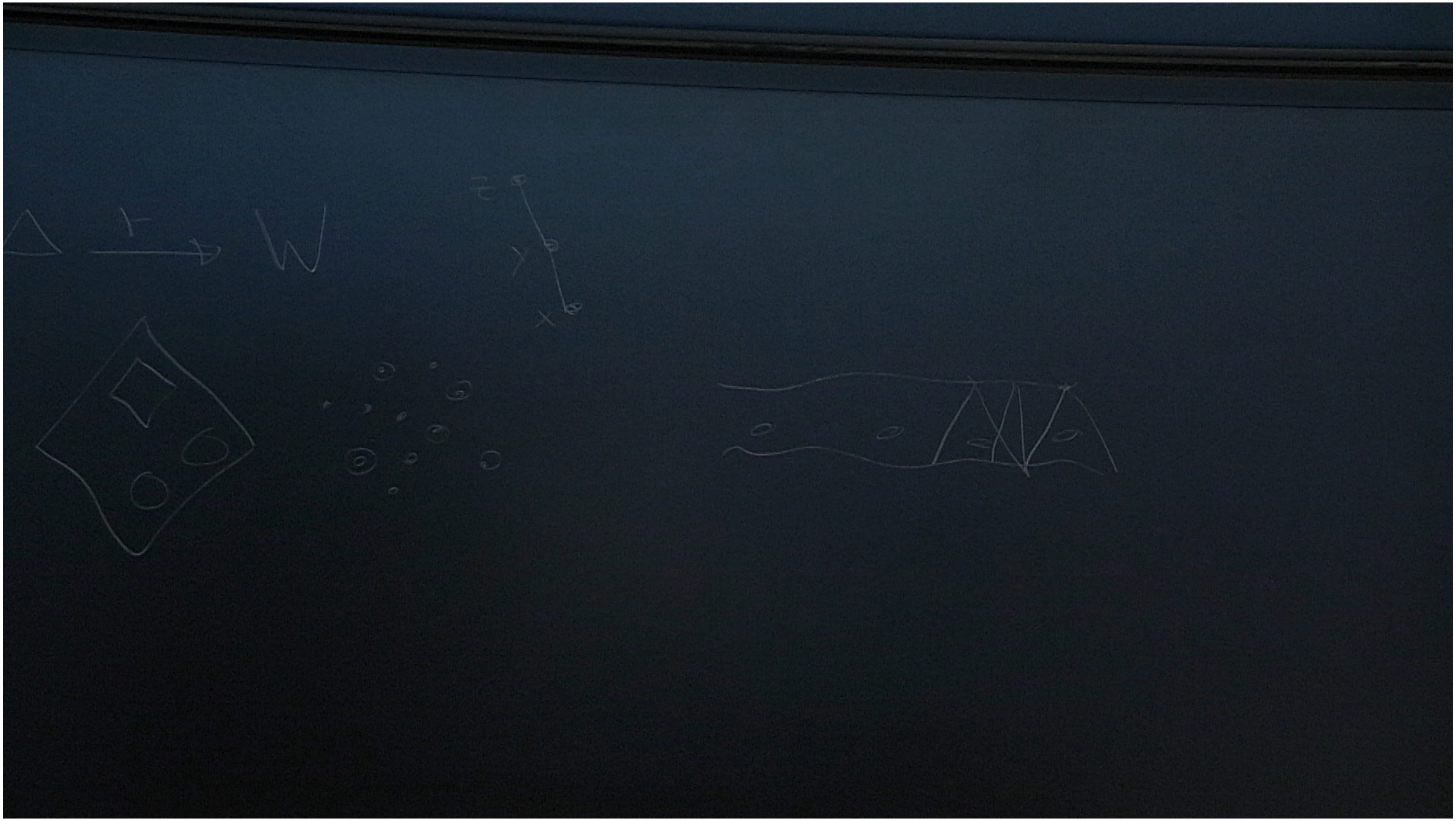


Causal Set d'Alembertian

In QFT one usually inverts the d'Alembertian to obtain Green functions. Since we have independently defined Green functions (as reviewed in the SJ discussion) we can reverse the process to obtain a d'Alembertian.

Since $G_{ret}^{(2d)} \sim C$ and $G_{ret}^{(4d)} \sim \sqrt{\rho}\, L$, we need to either set the diagonal to nonzero values or consider the symmetric part. This gives good agreement with $\Box$ but its frame-independence is not manifest. Also we do not know G_{ret} generally.

This led to a search for a more direct definition of $\Box$ that is more intrinsic to the causal set.


W
z
y
x

Causal Set $\Box \equiv B$ in $1+1$d

If we fix an element $x \in \mathcal{C}$, at which we would like to know the value of $\Box\phi$, this will be[8]:

$$B\phi(x) = \frac{4}{\ell^2}\left(-\frac{1}{2}\phi(x) + \left(\sum_{y\in L_1} -2\sum_{y\in L_2} + \sum_{y\in L_3}\right)\phi(y)\right),$$

where ℓ is the discreteness scale. More explicitly as a matrix, B is

$$\frac{\ell^2}{4}B_{xy} = \begin{cases} -1/2, & \text{for } x = y \\ 1, -2, 1, & \text{for } n(x,y) = 0, 1, 2, \text{ respectively, for } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

where $n(x, y)$ is the number of elements causally between y and x. Given B we can also go backwards and reconstruct[9] $\mathcal{C}$!

[8] R D Sorkin, *Does Locality Fail at Intermediate Length-Scales? In Approaches to QG*, ed by D Oriti, Cambridge University Press (2009) 2643.
[9] YKY, A Kempf, CQG 34 094001, 2017

Benincasa-Dowker Causal Set Action

Since $\lim_{\ell \to 0} \bar{B}\,\phi(x) = (\Box - \frac{1}{2}R(x))\,\phi(x)$, this led to a proposed action functional for causal sets.

More precisely, we get the action $S[\mathcal{C}]$ by acting with B on the constant $-\hbar\ell^2$ times another order one constant (related to units) and summing over the elements of $\mathcal{C}$. This gives[11]

$$\frac{1}{\hbar}S^{(2d)}[\mathcal{C}] = N - 2N_1 + 4N_2 - 2N_3 \qquad (16)$$

and

$$\frac{1}{\hbar}S^{(4d)}[\mathcal{C}] = N - N_1 + 9N_2 - 16N_3 + 8N_4 \qquad (17)$$

where N is the number of causal set elements and N_i is the number of $(i+1)$ element order intervals.

[11] D M T Benincasa, F Dowker, PRL 104 181301, (2010).

Non-local Causal Set d'Alembertians

Even though the average over sprinklings agrees well with the continuum expressions, any particular realization of it on a single sprinkling will not. There are fluctuations, and they grow with N.

Can damp out fluctuations by introducing an intermediate nonlocality scale $\ell \ll \ell_k \ll L_{IR}$

$$B\phi(x) \to B_k\phi(x) = \frac{4}{\sqrt{6}\ell_k^2}\left(-\phi(x) + \epsilon \sum_{y \prec x} f(n(x,y),\epsilon)\right)$$

where $\epsilon = (\ell/\ell_k)^4$, and we effectively smear out the expressions above over the new length scale ℓ_k.

Nonlocal Causal Set Action

Once again $\lim_{\ell \to 0} \bar{B}_k \, \phi(x) = (\Box - \frac{1}{2} R(x)) \, \phi(x)$, when ϕ varies slowly over scales ℓ_k and the radius of curvature $r \gg \ell_k$.

In this case the fluctuations are tamed, by the law of large numbers. In an interval in $\mathbb{M}^4$, the fluctuations were found[12] to die out as $1/\sqrt{N}$.

We now have a d'Alembertian operator (and action) defined in an intrinsic way with respect to a causal set and therefore explicitly Lorentz-invariant. It yields the expected answer not only on average but with high probability in each realization.

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[12] D M T Benincasa, F Dowker, PRL 104 181301, (2010).

Could there be a Λ term in the fluctuations of B?

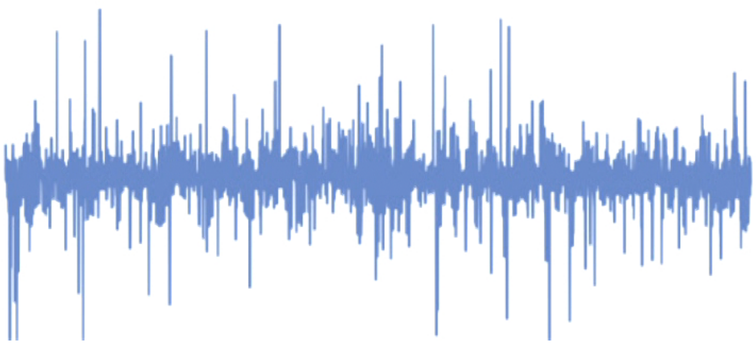
The cosmological constant enters the gravitational action as

$$S_G = \int \left[\frac{1}{16\pi G}(R - 2\Lambda) \right] \sqrt{-g}\, d^4x. \qquad (18)$$

After we remove the R term from the causal set action, could the remaining residue be the Λ term?!

We know that B fluctuates. Perhaps the fluctuations will survive in S and be another confirmation of Everpresent Λ.

It would be interesting to investigate this.



Some other remarks on B

- The expressions for B have been generalized to $\mathbb{M}^d$ (F Dowker, L Glaser, 2013, CQG 30 195016; S Aslanbeigi, M Saravani, R D Sorkin, JHEP 1406 2014 024).

- Phenomenological studies of the nonlocality scale in B_k (A Belenchia, arXiv:1512.08485v1.)

- Boundary terms for the causal set action (M Buck, F Dowker, I Jubb, S Surya, CQG 32 2015 20, 205004).

Summary and Future Directions

- From scalar field theory on a causal set, we have learned lessons about nonlocality, using retarded operators, and how the spacetime volume enters the physics.
- Can we use entanglement entropy to learn about and model cosmological inhomogeneities?
- Could there be another signature of Everpresent Λ in the fluctuations of the causal set action? This may explain both why the mean value is zero and why it takes its small value.
- Many more interesting problems to explore in cosmology, involving the causal set action, SJ vacuum, and EE.