Title: Ideal tetrahedra and their duals

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Abstract: Recently they have been generalized to other 3d homogeneous spaces, namely 3d anti de Sitter space and half-pipe space, a 3d homogeneous space with a degenerate metric.

We show that generalized ideal tetrahedra correspond to dual tetrahedra in 3d Minkowski, de Sitter and anti de Sitter space. They are those geodesic tetrahedra whose faces are all lightlike.

We investigate the geometrical properties of these dual tetrahedra in a unifi ed framework. We then apply these results to obtain a volume formula for generalised ideal tetrahedra and their duals, in terms of their dihedral angles and their edge lengths.

This is joint work with Dr Carlos Scarinci, KIAS.

Ideal tetrahedra and their duals

Emmy Noether Workshop:

The Structure of Quantum Space Time

Perimeter Institute for Theoretical Physics

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- joint work with Carlos Scarinci, KIAS
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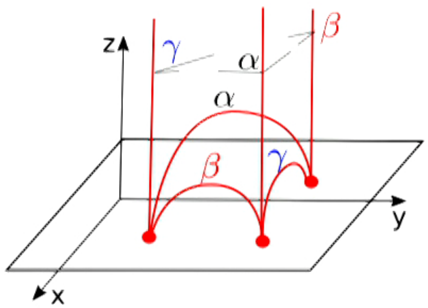
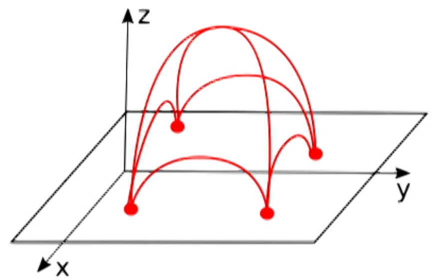
ideal hyperbolic tetrahedra

- **ideal hyperbolic tetrahedron**

  geodesic tetrahedron in $\mathbb{H}^3$ with vertices on $\partial_\infty \mathbb{H}^3 = \mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$





upper half-space model

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

$$\mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$$

- determined up to isometries by cross ratio $z = \dfrac{(v_1 - v_3)(v_2 - v_4)}{(v_1 - v_4)(v_2 - v_3)}$

- determined up to isometries by dihedral angles $\alpha, \beta, \gamma \in (0, \pi) \quad \alpha + \beta + \gamma = \pi$

  $\mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C}) \curvearrowright \partial_\infty \mathbb{H}^3$ ➪ vertices $\underbrace{v_1 = \infty,\ v_2 = 0,\ v_3 = 1,}_{\text{3-transitivity}}\ v_4 = z$

  $$z = \frac{\sin(\beta)}{\sin(\alpha)} e^{i\gamma}$$

- **volume of ideal tetrahedron**

$$\mathrm{vol}(\Theta) = \tfrac{1}{2}\mathrm{Cl}(\alpha) + \tfrac{1}{2}\mathrm{Cl}(\beta) + \tfrac{1}{2}\mathrm{Cl}(\gamma)$$

[Lobachevsky]
[Milnor]

$$\mathrm{Cl}(\theta) = -\int_0^\theta \log|2\sin(\tfrac{t}{2})|dt = \mathrm{Im}\,\mathrm{Li}_2(e^{i\theta}) = 2\Lambda(\tfrac{\theta}{2})$$

Clausen function — classical dilogarithm — Lobachevsky function

Why ideal tetrahedra?

- **hyperbolic structures via ideal tetrahedra** [Thurston]

  ideal triangulation of 3-manifold M (with boundary)
  + assignment of ideal tetrahedra satisfying gluing conditions

  ➪ hyperbolic structure on M, hyperbolic volume from volumes of ideal tetrahedra

- **generalisation** of ideal tetrahedra from $\mathbb{H}^3$ to HP_3 and AdS_3 [Danciger]

  ➪ analogous constructions and geometric transitions

  ➪ but: no volume formulas yet

Why duals? - applications in 3d gravity

- **in 3d:** vacuum solutions of Einstein equations have constant curvature
- no local gravitational degrees of freedom, global degrees of freedom from topology
- locally isometric to 3d de Sitter, Minkowski, anti-de Sitter space

3d gravity	duality	spaces with ideal tetrahedra
3d de Sitter space dS_3	⟷	3d hyperbolic space $\mathbb{H}^3$
3d Minkowski space M_3	⟷	3d half-pipe space HP_3
3d anti de Sitter space AdS_3	⟷	3d anti de Sitter space AdS_3

- **3d gravity as toy model for quantum gravity**
  - relation to Chern-Simons theory [Witten]
  - many quantisation formalisms based on tetrahedra
    - state sum models or spin foams [Ponzano-Regge, Turaev-Viro, Barrett-Westbury,…]
    - Regge calculus
    - causal dynamical triangulations [Loll, Ambjørn,…]

  ➪ understand geometry of tetrahedra in 3d de Sitter, Minkowski, anti de Sitter space

  ➪ find nice tetrahedra as building blocks

  ➪ understand volumes

- **3d gravity via Teichmüller theory and hyperbolic geometry**
- causality assumptions ➪ full classification of spacetimes [Mess, Barbot, Benedetti-Bonsante,…]

$$\mathcal{M}_\Lambda(S) = \{\text{max. glob. hyp. Lorentz metrics of curvature } \Lambda \\ \text{with complete Cauchy surface } S\}/\mathrm{Diff}_0(S)$$

$$= \mathrm{T}^*\mathcal{T}(S) = \mathcal{ML}(S)$$ ~ phase space of 3d gravity

- Wick rotations relate spacetimes to domains in $\mathbb{H}^3$ [Benedetti-Bonsante]
- spacetimes via constructions from hyperbolic geometry: earthquake and grafting [Mess,…]
- description in terms of generalised shear coordinates [Scarinci-C.M.]

➪ mapping class group action on $\mathcal{M}_\Lambda(S)$ related to volume of ideal tetrahedra

3d geometries

3d Lorentzian spaces $\mathbb{X}_\Lambda$ | **3d ideal spaces $\mathbb{Y}_\Lambda$**

$\Lambda = 1$	dS_3	points		planes	$\mathbb{H}_3$	$(1,1,1)$
$\Lambda = 0$	M_3	lines	duality ↔	lines	HP_3	$(0,1,1)$
$\Lambda = -1$	AdS_3	planes		points	AdS_3	$(-1,1,1)$

Λ = curvature | Λ = signature

- **ambient space** $\mathbb{R}^4$ with bilinear form $\langle x, x\rangle_\Lambda = -x_1^2 + \Lambda x_2^2 + x_3^2 + x_4^2$

$$\mathrm{dS}_3 = \{x \in \mathbb{R}^4 \mid \langle x, x\rangle_\Lambda > 0\}/\mathbb{R}^\times$$
$$\mathrm{M}_3 = \{x \in \mathbb{R}^4 \mid x_2^2 > 0\}/\mathbb{R}^\times$$
$$\mathrm{AdS}_3 = \{x \in \mathbb{R}^4 \mid \langle x, x\rangle_\Lambda < 0\}/\mathbb{R}^\times$$

with induced metrics

$$\mathbb{H}_3 = \{y \in \mathbb{R}^4 \mid \langle y, y\rangle_\Lambda < 0\}/\mathbb{R}^\times$$
$$\mathrm{HP}_3 = \{y \in \mathbb{R}^4 \mid \langle y, y\rangle_\Lambda < 0\}/\mathbb{R}^\times \qquad \subset \mathbb{R}\mathrm{P}^3$$
$$\mathrm{AdS}_3 = \{y \in \mathbb{R}^4 \mid \langle y, y\rangle_\Lambda < 0\}/\mathbb{R}^\times$$

with induced metrics

ideal boundary

$$\partial_\infty \mathbb{Y}_\Lambda = \{y \in \mathbb{R}^4 \setminus \{0\} \mid \langle y, y\rangle_\Lambda = 0\}/\mathbb{R}^\times$$

- **duality**

$$[x] \in \mathbb{X}_\Lambda \quad \rightarrow \quad [x]^* = \{[y] \in \mathbb{Y}_\Lambda \mid \langle x, y\rangle_\Lambda = (1 - |\Lambda|)x_2 y_2\}$$
$$[y]^* = \{[x] \in \mathbb{X}_\Lambda \mid \langle x, y\rangle_\Lambda = (1 - |\Lambda|)x_2 y_2\} \quad \leftarrow \quad [y] \in \mathbb{Y}_\Lambda$$

unified description by matrices

- **generalised complex numbers** $\mathbb{C}_\Lambda = \mathbb{R}[\ell]/(\ell^2+\Lambda) = \begin{cases} \mathbb{C} & \Lambda = 1 \\ \text{dual numbers} & \Lambda = 0 \\ \text{hyperbolic numbers} & \Lambda = -1 \end{cases}$

  parametrisation $z = x + \ell y \in \mathbb{C}_\Lambda \quad \ell^2 = -\Lambda$

  analytic functions on $U \subset \mathbb{R}$ ➪ analytic continuation to $U \subset V \subset \mathbb{C}_\Lambda$

- **involutions** on $\mathrm{Mat}(2, \mathbb{C}_\Lambda)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\circ = \begin{pmatrix} \bar{d} & -\bar{b} \\ -\bar{c} & \bar{a} \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}$

- **3d geometries**

3d Lorentzian spaces $\mathbb{X}_\Lambda$ — hermitian matrices / rescaling — 3d ideal spaces $\mathbb{Y}_\Lambda$

$\mathbb{X}_\Lambda =$
$\{X \in \mathrm{Mat}(2, \mathbb{C}_\Lambda) \mid X^\circ = X, \det X > 0\}/\mathbb{R}^\times$

$\mathbb{Y}_\Lambda =$
$\{Y \in \mathrm{Mat}(2, \mathbb{C}_\Lambda) \mid Y^\dagger = Y, \det Y > 0\}/\mathbb{R}^\times$

parametrisation

$\mathbb{R}^4 \to \{X \in \mathrm{Mat}(2, \mathbb{C}_\Lambda) \mid X^\circ = X\}$

$$x \mapsto \begin{pmatrix} x_2 + \ell x_4 & \ell(x_3 - x_1) \\ \ell(x_3 + x_1) & x_2 - \ell x_4 \end{pmatrix}$$

$\mathbb{R}^4 \to \{Y \in \mathrm{Mat}(2, \mathbb{C}_\Lambda) \mid Y^\dagger = Y\}$

$$y \mapsto \begin{pmatrix} y_1 + y_3 & y_4 + \ell y_2 \\ y_4 - \ell y_2 & y_1 - y_3 \end{pmatrix}$$

metrics

- metric on tangent space $T_1\mathbb{X}_\Lambda$

  $\langle x, x\rangle = -\det \mathrm{Im}(x)$

- metric on tangent space $T_1\mathbb{Y}_\Lambda$

  $\langle y, y\rangle = -\det y$

$$PSL(2,\mathbb{C}_\Lambda) = \begin{cases} PSL(2,\mathbb{C}) & \Lambda=1 \\ PSL(2,\mathbb{R})\times PSL(2,\mathbb{R}) & \Lambda=-1 \\ ISO(2,1) & \Lambda=0 \end{cases}$$

3d Lorentzian spaces $\mathbb{X}_\Lambda$ | **3d ideal spaces $\mathbb{Y}_\Lambda$**

$\mathrm{Isom}(\mathbb{X}_\Lambda) = \mathrm{PSL}(2,\mathbb{C}_\Lambda)$ **isometry group** $\mathrm{Isom}(\mathbb{Y}_\Lambda) = \mathrm{PSL}(2,\mathbb{C}_\Lambda)$

$G \rhd X = GXG^\circ$ **action of isometry group** $G \rhd Y = GYG^\dagger$

$x(t) = G\exp(\ell t T)G^\circ$ **geodesics** $y(t) = G\exp(\ell t T)G^\dagger$

$G \in \mathrm{PSL}(2,\mathbb{C}_\Lambda) \quad T \in \mathfrak{sl}(2,\mathbb{R})$

spacelike	$\det(T) < 0$	hyperbolic
lightlike	$\det(T) = 0$	parabolic
timelike	$\det(T) > 0$	elliptic

$X(t,s) = G\exp(t\ell X_1 + s\ell X_2)G^\circ$ **geodesic planes** $Y(t,s) = G\exp(t\ell Y_1 + s\ell Y_2)G^\dagger$

$X_1, X_2 \in \mathfrak{sl}(2,\mathbb{R})$ lin. independent

$\langle X_i, N\rangle = 0$

normal vector $N \in \mathfrak{sl}(2,\mathbb{R})$

$Y_1, Y_2 \in \mathfrak{sl}(2,\mathbb{R})$ lin. independent

$\langle Y_i, N\rangle = 0$

lightlike geodesic planes in $\mathbb{X}_\Lambda$ **duality** ideal boundary of $\mathbb{Y}_\Lambda$

= geodesic planes with lightlike but no timelike geodesics

$$\partial_\infty \mathbb{Y}_\Lambda = \mathbb{C}_\Lambda \mathrm{P}^1$$
$$= \{Y \in \mathrm{Mat}(2,\mathbb{C}_\Lambda)\setminus\{0\} \mid Y^\dagger = Y, \det Y = 0\}/\mathbb{R}^\times$$

generalised ideal tetrahedra

Def [Danciger]

- **ideal tetrahedron in $\mathbb{Y}_\Lambda$:** geodesic 3-simplex in $\mathbb{Y}_\Lambda$ with vertices in $\partial_\infty \mathbb{Y}_\Lambda$ that are pairwise connected by spacelike geodesics

Theorem [Danciger]

Ideal tetrahedron in $\mathbb{Y}_\Lambda$ is determined up to isometries by generalised cross ratio $z \in \mathbb{C}_\Lambda^\times$ or generalised dihedral angles $\alpha, \beta > 0$

Proof:

$\mathrm{PSL}(2, \mathbb{C}_\Lambda) \curvearrowright \partial_\infty \mathbb{Y}_\Lambda$ ⇨ vertices $v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $v_4 = \begin{bmatrix} z\bar{z} & z \\ \bar{z} & 1 \end{bmatrix}$

parametrization $z = -\dfrac{s_\Lambda(\beta)}{s_\Lambda(\alpha)} e^{\ell(\alpha+\beta)}$ with $s_\Lambda(x) = \begin{cases} \sin(x) & \Lambda = 1 \\ x & \Lambda = 0 \\ \sinh(x) & \Lambda = -1 \end{cases}$

Theorem [Scarinci, C.M.]

volume of generalised ideal tetrahedron Θ

$$\mathrm{vol}(\Theta) = \tfrac{1}{2}\mathrm{Cl}_\Lambda(2\alpha) + \tfrac{1}{2}\mathrm{Cl}_\Lambda(2\beta) - \tfrac{1}{2}\mathrm{Cl}_\Lambda(2(\alpha+\beta))$$

with generalised Clausen function

$$\mathrm{Cl}_\Lambda(\theta) = -\int_0^\theta dt \log|2 s_\Lambda(\tfrac{t}{2})|$$

duals of generalised ideal tetrahedra

3d Lorentzian spaces $\mathbb{X}_\Lambda$

- lightlike geodesic plane $v^* \subset \mathbb{X}_\Lambda$
- spacelike geodesic $v^* \cap w^* \subset \mathbb{X}_\Lambda$
- intersection point $u^* \cap v^* \cap w^* \in \mathbb{X}_\Lambda$

duality ⟷

3d ideal spaces $\mathbb{Y}_\Lambda$

- vertex $v \in \partial_\infty \mathbb{Y}_\Lambda$
- spacelike edge between $v, w \in \partial_\infty \mathbb{Y}_\Lambda$
- face of ideal tetrahedron with vertices $u, v, w \in \partial_\infty \mathbb{Y}_\Lambda$

Def [Scarinci, C.M.]

- **lightlike tetrahedron in $\mathbb{X}_\Lambda$:** geodesic 3-simplex in $\mathbb{X}_\Lambda$ whose faces are lightlike geodesic planes

Theorem [Scarinci, C.M.]

- generalised ideal tetrahedra in $\mathbb{Y}_\Lambda$ are dual to lightlike tetrahedra in $\mathbb{X}_\Lambda$
- lightlike tetrahedra determined up to isometries by two real parameters $\alpha, \beta > 0$

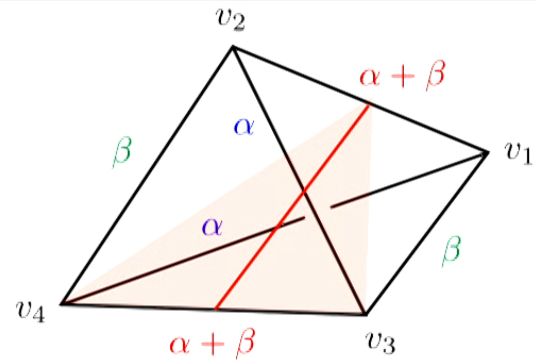
Proof: $\mathrm{PSL}(2, \mathbb{C}_\Lambda) \curvearrowright \mathbb{X}_\Lambda$ ⇨ vertices $v_1 = \begin{bmatrix} e^{\ell\alpha} & -2\ell s_\Lambda(\alpha) \\ 0 & e^{-\ell\alpha} \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} e^{\ell\beta} & 0 \\ 2\ell s_\Lambda(\beta) & e^{-\ell\beta} \end{bmatrix}$

$$v_3 = \begin{bmatrix} e^{\ell(\alpha+\beta)} & 0 \\ 0 & e^{-\ell(\alpha+\beta)} \end{bmatrix} \quad v_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

duality ⇨ generalised ideal tetrahedron in standard parametrisation

properties of lightlike tetrahedra

[Scarinci, C.M.]



- edges are spacelike
- opposite edges have equal length $\alpha, \beta, \alpha+\beta$
- distinguished pair of longest edges - length $\alpha+\beta$
- dihedral angles of ideal tetrahedron $\leftrightarrow$ edge lengths of lightlike tetrahedron
- midpoints of longest edges connected by timelike geodesic of maximal arclength
- geodesics in tetrahedron between other pairs of opposite edges are spacelike
- midpoint of longest edge and endpoints of opposite edge form equilateral triangle with edge lengths $\alpha+\beta$ and $\frac{1}{2}|\alpha-\beta|$
- for $\Lambda = 1, 0$: lightlike tetrahedra $\overset{1:1}{\longleftrightarrow}$ triples of pairs of non-coplanar spacelike geodesics
- for $\Lambda = -1$: lightlike tetrahedra $\overset{1:1}{\longleftrightarrow}$ triples of pairs of non-coplanar spacelike geodesics subject to additional condition

volumes of lightlike tetrahedra

Theorem [Scarinci, C.M.]

volume of lightlike tetrahedron Θ with edge lengths $\alpha, \beta, \alpha+\beta$

$$\mathrm{vol}(\Theta) = \frac{1}{2\Lambda}\left(\mathrm{Cl}_\Lambda(2\alpha) + \mathrm{Cl}_\Lambda(2\beta) - \mathrm{Cl}_\Lambda(2(\alpha+\beta))\right)$$

$$+\frac{1}{\Lambda}\left(\alpha \log|2s_\Lambda(\alpha)| + \beta \log|2s_\Lambda(\beta)| - (\alpha+\beta)\log|2s_\Lambda(\alpha+\beta)|\right)$$

$$= \frac{1}{\Lambda}\mathrm{vol}(\Theta^*) - \frac{1}{\Lambda}\left(\alpha \cdot \mathrm{Cl}'_\Lambda(2\alpha) + \beta \cdot \mathrm{Cl}'_\Lambda(2\beta) - (\alpha+\beta)\cdot \mathrm{Cl}'_\Lambda(2(\alpha+\beta))\right)$$

Schläfli formula

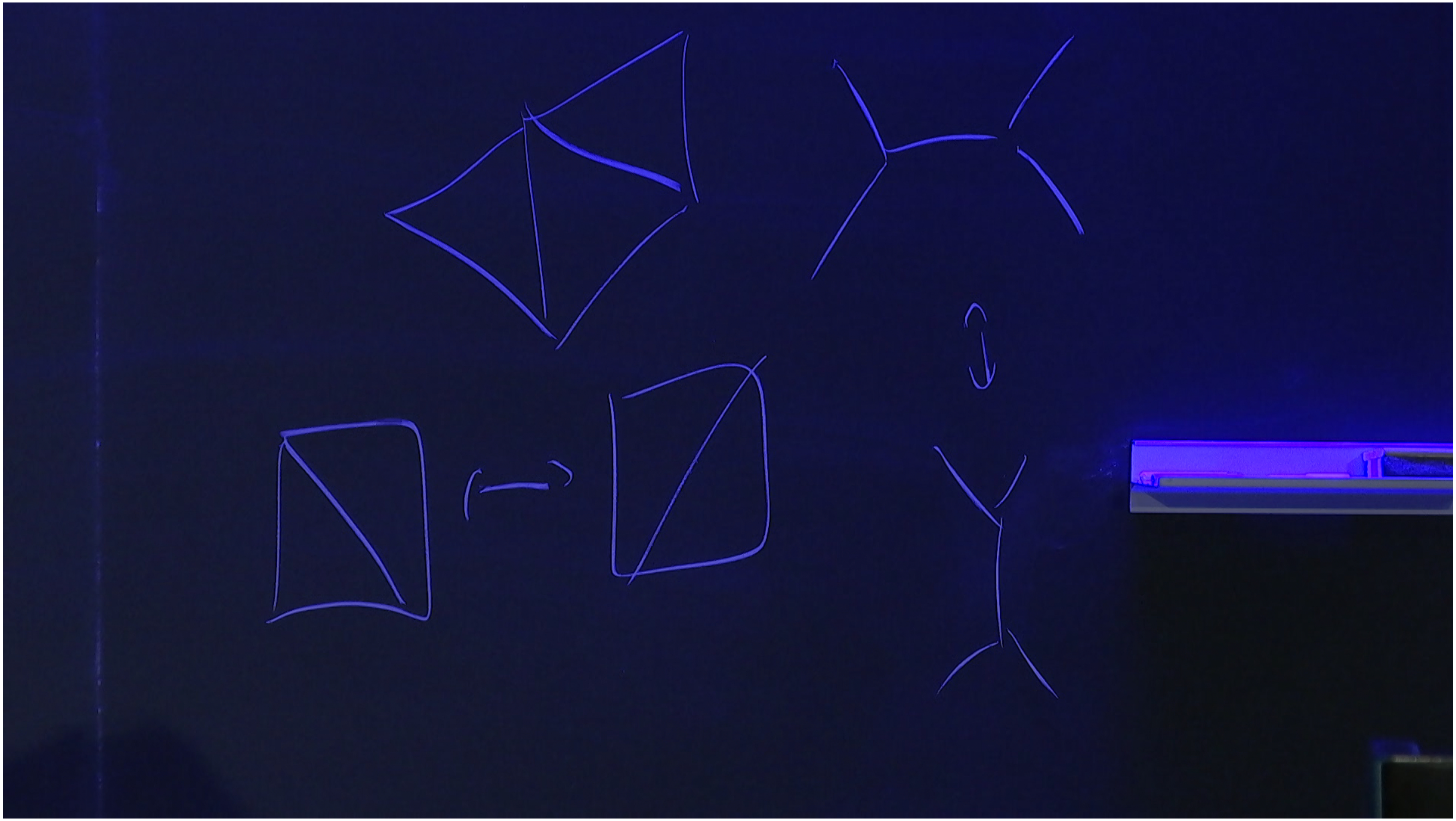
with generalised Clausen function

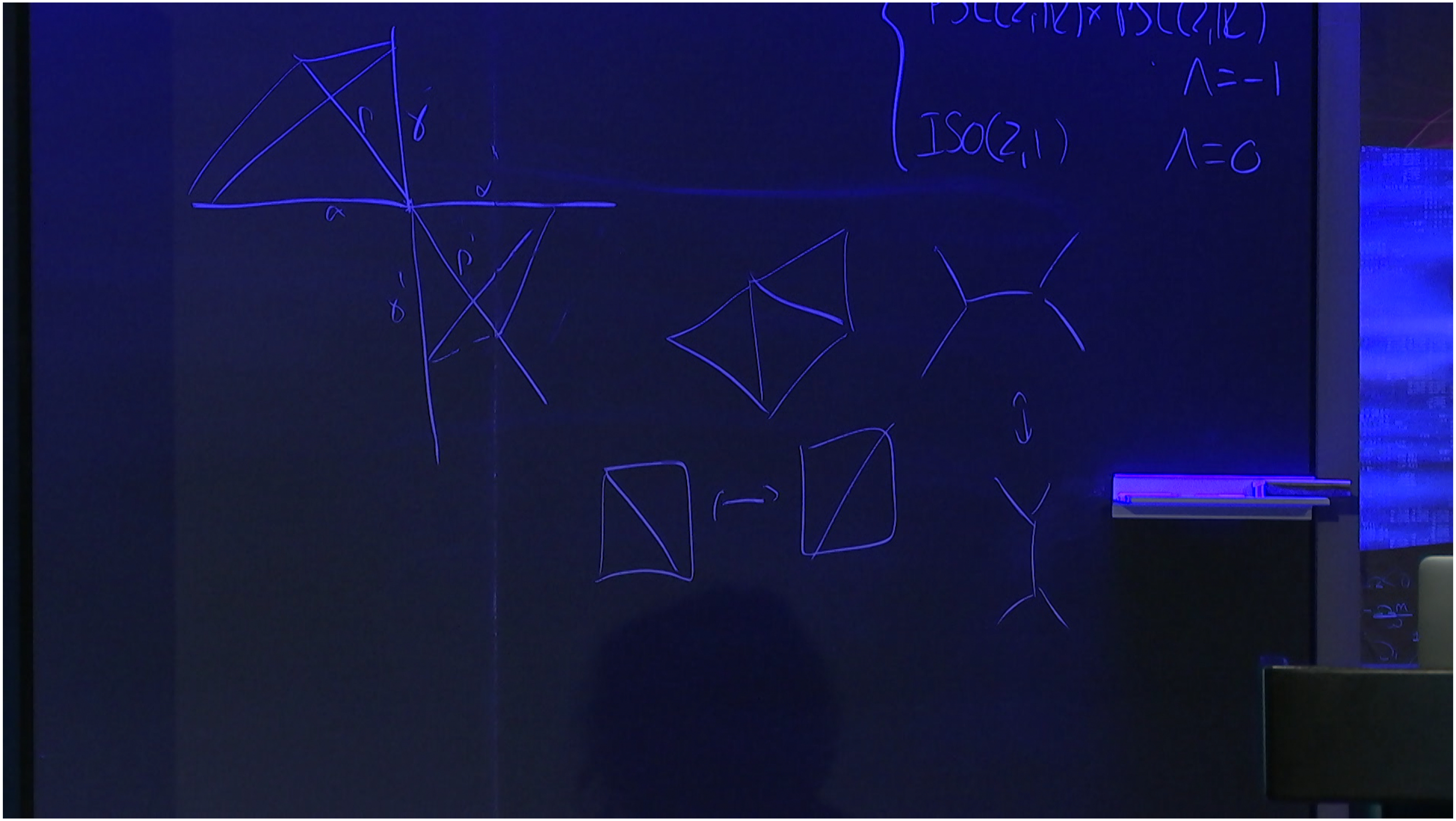
$$\mathrm{Cl}_\Lambda(\theta) = -\int_0^\theta dt \log|2s_\Lambda(\tfrac{t}{2})|$$

power series development

$$\mathrm{vol}(\Theta) = \sum_{k=1}^{\infty} \frac{4^k(-1)^{k-1}\Lambda^{k-1}B_{2k}}{(2k+1)!}\sum_{j=1}^{k}\binom{k+1}{j}\alpha^j\beta^{k+1-j} = \frac{1}{3}\alpha\beta(\alpha+\beta) + O(\Lambda)$$

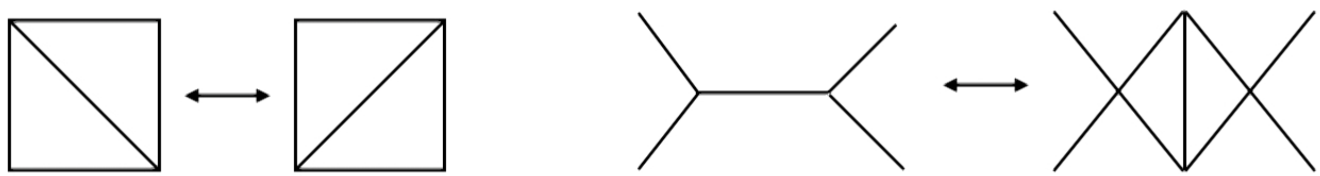
B_{2k} Bernoulli numbers




PSL(2,ℝ) × PSL(2,ℝ)
Λ = -1
ISO(2,1)
Λ = 0

Outlook and Conclusions

- **unified description of generalised ideal tetrahedra and dual lightlike tetrahedra**
- **simple, unified volume formulas**
- **applications ?**
- mapping class group action on moduli spaces $\mathcal{M}_\Lambda(S)$ of maximal globally hyperbolic 3d spacetimes of constant curvature
  - description in terms of data on triangulations (generalised shear coordinates)
  - mapping class group actions: by gluing (degenerate) ideal or lightlike tetrahedra



- in higher dimensions

Lorentzian spaces		**spaces with ideal boundary**
de Sitter space dS_n	duality ↔	hyperbolic space $\mathbb{H}^n$
Minkowski space M_n		half pipe space HP_n
anti de Sitter space AdS_n		anti de Sitter space AdS_n
lightlike n-simplices		**ideal n-simplices**