Title: Fusion rules from entanglement

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Series: Perimeter Institute Quantum Discussions

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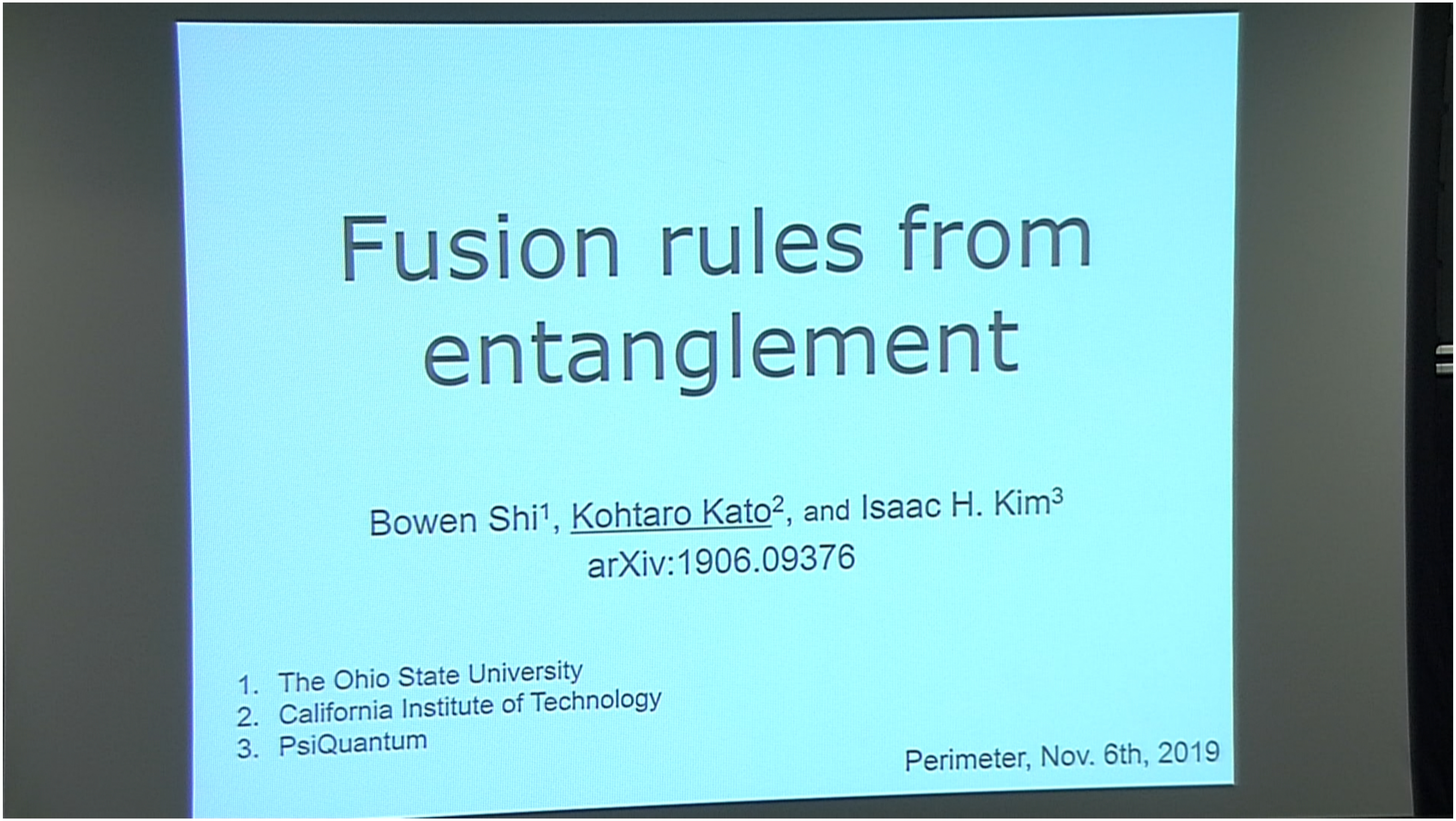
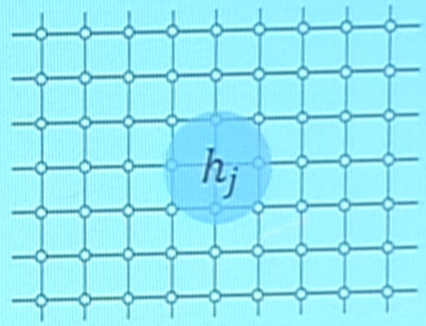
Abstract: Connections between 2D gapped quantum phases and the anyon fusion theory have been proven in various ways under different settings. In this work, we introduce a new framework connecting them by only assuming a conjectured form of entanglement area law for 2D gapped systems. We show that one can systematically define topological charges and fusion rules from the area law alone, in a well-defined way. We then derive the fusion rules of charges satisfy all the axioms required in the algebraic theory of anyons. Moreover, even though we make no assumption about the exact value of the constant sub-leading term of the entanglement entropy, this term is shown to be equal to the logarithm of the total quantum dimension of the anyon theory we defined. 


Fusion rules from entanglement
Bowen Shi[1], Kohtaro Kato[2], and Isaac H. Kim[3]
arXiv:1906.09376
1. The Ohio State University
2. California Institute of Technology
3. PsiQuantum
Perimeter, Nov. 6th, 2019

# Gapped quantum systems

- Gapped 2D local Hamiltonian $H$:

$$H = -\sum_j h_j \,,$$

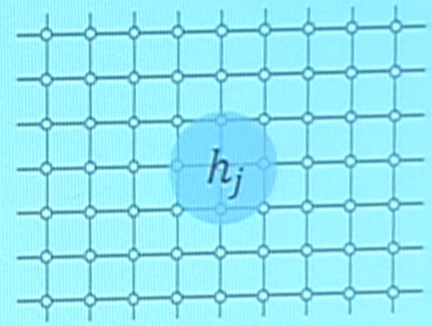


# Gapped quantum systems

➢ Gapped 2D local Hamiltonian $H$:

$$H = -\sum_j h_j\,, \quad E_1 - E_0 \geq \Delta > 0$$

for any system size.

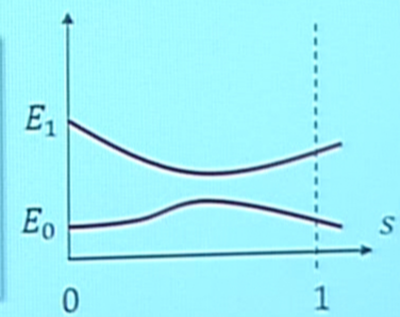



**Gapped q. phases** = Equivalence classes of ground states

$H_1|\psi_1\rangle = E_0^{(1)}|\psi_1\rangle$, $H_2|\psi_2\rangle = E_0^{(2)}|\psi_2\rangle$: ground states

$|\psi_1\rangle \sim |\psi_2\rangle$

$$\Leftrightarrow {}^{\exists}\, H(s) = -\sum_j h_j(s) \begin{cases} H(0) = H_1 \\ H(1) = H_2 \end{cases}$$



# Classifying gapped phases of matter

RESEARCH | REVIEW

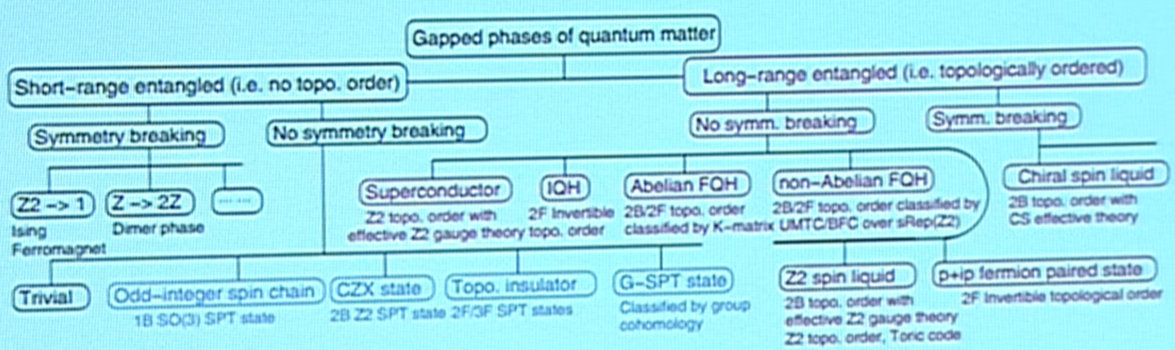



**Fig. 6. Zero-temperature phases of matter with energy gap.** Only gapped liquids are listed. The black entries are described by symmetry breaking. Colored entries are phases beyond symmetry breaking: the red ones have topological order, whereas the blue are product states with SPT order. 1B refers to a 1+1D bosonic system, 2F denotes a 2+1D fermionic system, etc. $G_H \to G_\Psi$ indicates that the symmetry group $G_H$ of the system (the Hamiltonian) is spontaneously broken down to the symmetry group $G_\Psi$ of the ground state. IQH, integer quantum Hall state; CS, Chern-Simons; UMTC, unitary modular tensor category; BFC, braided fusion category.

From
Xiao-Gang Wen,
"Choreographed entanglement dances: Topological states of quantum matter",
Science, 363, 6429, (2019).

# Classifying gapped phases of matter

- 1D gapped quantum system:

  Completely classified by the MPS formalism and beyond.
  [Chen, et.al `07] [Schuch, et.al '11] [Ogata, '19]…

  Symmetry-breaking (SB) + Symmetry protected topological (SPT) order

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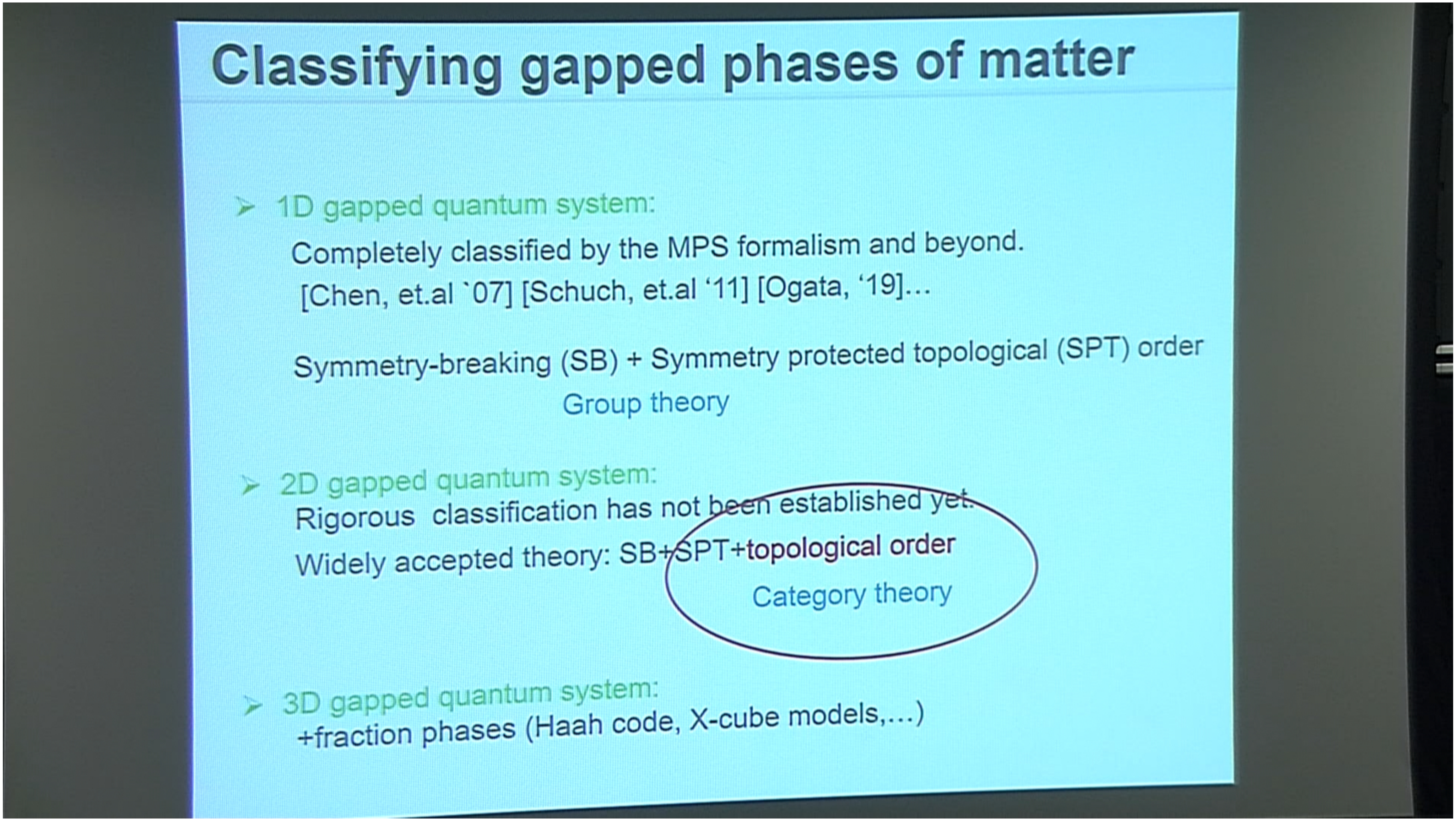
  Symmetry-breaking (SB) + Symmetry protected topological (SPT) order
  Group theory

- 2D gapped quantum system:

  Rigorous classification has not been established yet.
  Widely accepted theory: SB+SPT+topological order
  Category theory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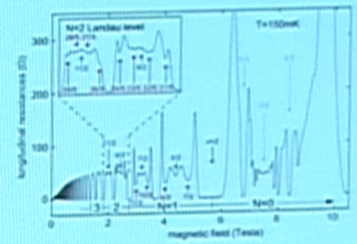
Category theory
3D gapped quantum system:
+fraction phases (Haah code, X-cube models,...)

# Topologically ordered phases

*Non-trivial gapped q. phases* with the following properties:

- ✓ No local order parameter
- ✓ Topology-dependent degeneracy of ground states
- ✓ Anyonic excitations
- ✓ Gapless boundary (sometimes)




well-explained by **Topological Quantum Field Theory (TQFT)**

→ Topological orders are distinguished by algebraic theory of anyons

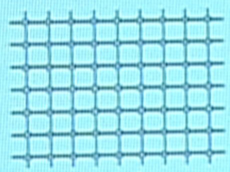
**Modular tensor category (MTC)**

How do we **prove** it for generic gapped systems?

# Question

**Microscopic descriptions**

$H, |\psi_{gs}\rangle, \ldots$



QM

?

**Anyon theory**

$\mathcal{L} = \{1, a, b, c, \ldots\}, \; a \times b = \sum_c N_{ab}^c \, c$


Category theory

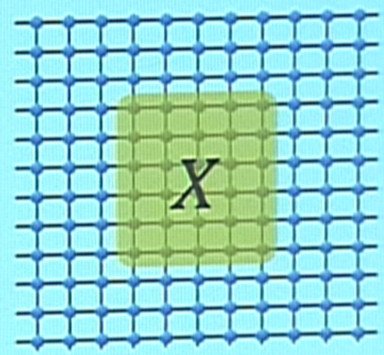
- Established in broad classes of models (case-by-case)
  (toric code, quantum double model, Levin-Wen model, FQHE,…)
- Tensor network approaches
  [Sahinoglu, et. al, ‘14]: MPO-injective PEPS description
- Operator-algebraic approaches
  [Haag, ‘96],…: algebraic QFT approach (relativistic & continuum)
  [Naaijkens, Fiedler ‘15]: infinite spin lattice w/Haag duality & splitting property

# Area law

Gapped ground states are believed to obey an area law

**Entanglement entropy**

$$S(X)_\rho := -\mathrm{Tr}\rho_X \log_2 \rho_X$$




**Area law (weak form)**

$$S(X)_\rho \le O(|\partial X|)$$

- Proven for 1D gapped systems [Hastings, '07].
- Complete proof has not yet been obtained for 2D or higher dimensions.

**Observations:** **Area law in 2D (strong form)**

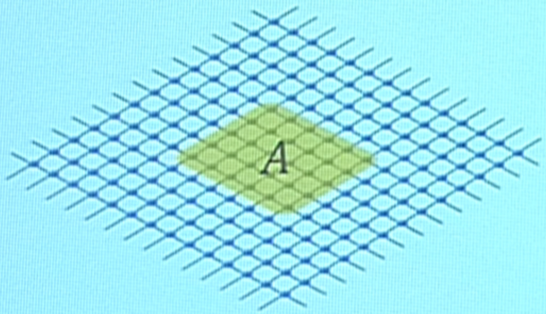
$$S(X)_\rho = \alpha|\partial X| - \gamma + o(1)$$

# Our work

## Quantum information theoretical approach

**Area law of entanglement**

$$S(A)_\rho = \alpha|\partial A| - \gamma$$




without Hamiltonian!

**Anyon models**

This talk

$$\mathcal{L} = \{1, a, b, c, \ldots\}$$

$$a \times b = \sum_c N_{ab}^c \, c$$

On-going

Intuitively clear, simple assumption without any Hamiltonian.

# Anyon theory

Charges, fusion rules and F and R-matrices specify an anyon theory.

- **Charges (Superselection sectors)**

  Possible types of quasiparticles (anyons)

  $$\mathcal{L} = \{1, a, b, c, \dots\}$$

- **Fusion rules**

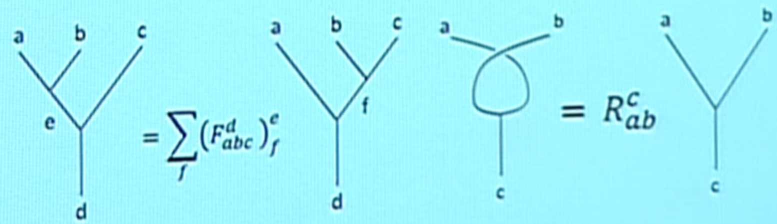
  Possible total charges of two charges

  cf.) spin $\frac{1}{2} \times \frac{1}{2} = 0 + 1$

  $$a \times b = \sum_c N_{ab}^c \, c$$

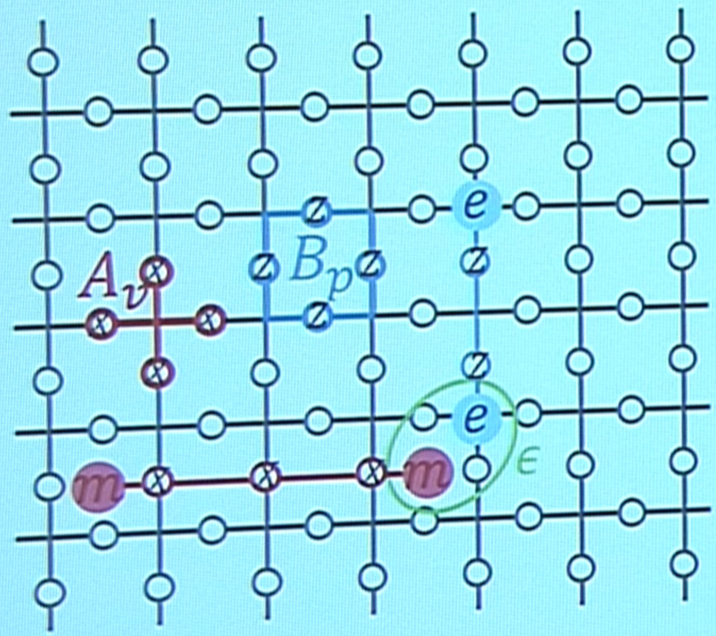
- ***F* and *R* matrices**

  Specify braiding statistics



# Example: Toric code

$$H = -\sum_{v\in +} A_v - \sum_{p \in \square} B_p \qquad [A_v, A_{v'}] = [B_p, B_{p'}] = [A_v, B_p] = 0$$




$$\begin{cases} A_v|\psi\rangle = |\psi\rangle, & \forall v \\ B_p|\psi\rangle = |\psi\rangle, & \forall \mathrm{p} \end{cases}$$

**Charge set:** $\mathcal{L} = \{1, e, m, \epsilon\}$

**Fusion rules:**

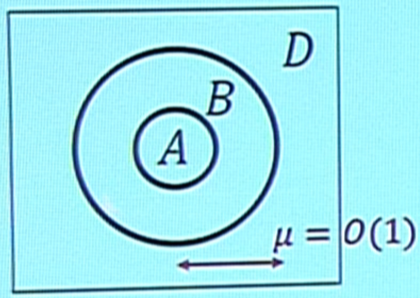
$1 \times x = x \qquad x \times x = 1 \qquad (x = 1, e, m, \epsilon)$

$\mathrm{e} \times m = \epsilon \qquad m \times \epsilon = e \qquad \epsilon \times e = m$

# Our axioms (precise ver.)

**Assumption**: ∃a reference state $\rho$ satisfying the two axioms

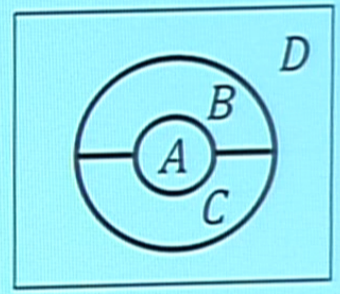
Axiom **A0**:




$$S(A|B)_\rho + S(A)_\rho = 0$$

$\rightarrow \rho_{AD} = \rho_A \otimes \rho_D$

Axiom **A1**:




$$S(A|B)_\rho + S(A|C)_\rho = 0$$

$\rightarrow \rho_{ABD}, \rho_{ACD}$ are q. Markov chains

- Follows from the area law: $S(A)_\rho = \alpha|\partial A| - \gamma$
- One could consider $\approx 0$ instead of the exact conditions
- Can be extended to larger regions by iterations.

$* S(A|B) \coloneqq S(AB) - S(B)$

# Information convex

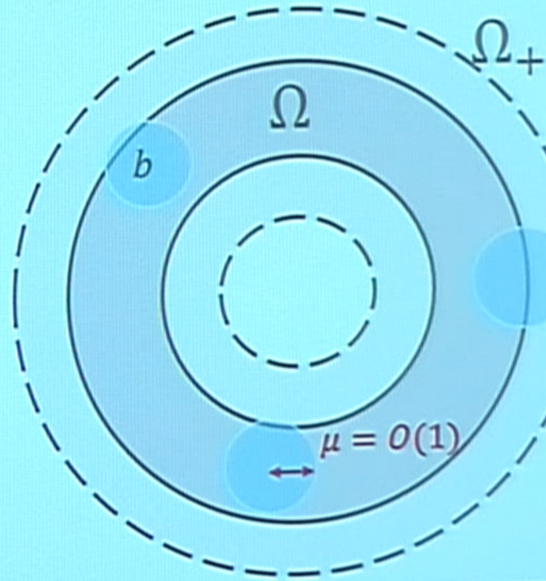
A key concept to define the anyon theory without Hamiltonian

$\tilde{\Sigma}(\Omega) \coloneqq \{\sigma_{\Omega_+} \mid \sigma_b = \rho_b \ \forall\, b\text{: ball}\}$ locally indistinguishable from $\rho$

**Information convex [Shi,`18]:**

$$\Sigma(\Omega) \coloneqq \{\sigma_\Omega = \mathrm{Tr}_{\Omega_+ \setminus \Omega}\sigma_{\Omega_+} \mid \sigma_{\Omega_+} \in \tilde{\Sigma}(\Omega)\}$$

- $\Sigma(\Omega)$ is a convex set.
- Hamiltonian-free definition.
- If $\Omega$ is a disc, then $\Sigma(\Omega) = \{\rho_\Omega\}$.
  Similar to the topological order condition [Michalakis, Zwolak, ‘13].
- Similar concepts are in [Haah,’16],[KK, Naaijkens’18] (operator-algebraic).



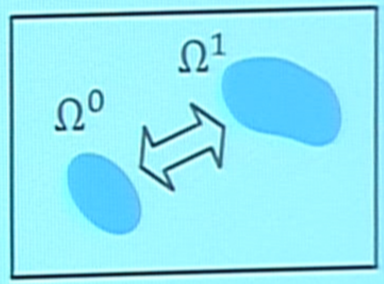
# Isomorphism theorem

How does the structure of $\Sigma(\Omega)$ depends on $\Omega$?

**Theorem 1**

If $\Omega^0$ and $\Omega^1$ are connected by local deformations $\{\Omega^t\}$, there is a bijective CPTP-map

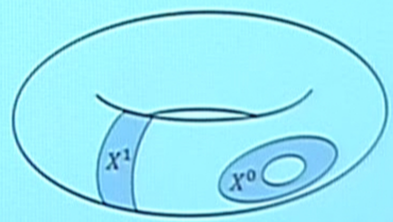
$$\Phi_{\{\Omega^t\}}: \Sigma(\Omega^0) \to \Sigma(\Omega^1)$$




- The isomorphism preserves the entropy difference

$$S(\sigma) - S(\omega) = S\big(\Phi(\sigma)\big) - S\big(\Phi(\omega)\big).$$

- For annuli, $\Phi_{\{\Omega^t\}} = \Phi_{\{\Omega^s\}}$ if two paths $\{\Omega^t\}, \{\Omega^s\}$ connected smoothly.
  - → They are possibly different on e.g. a torus.



# A key lemma

**Merging Lemma [KK, Furrer, Murao '16]:**

Consider a set of states $\mathcal{S} = \{\rho_{ABC}\}$ and $\sigma_{BCD}$ such that $\rho_{BC} = \sigma_{BC}$ and

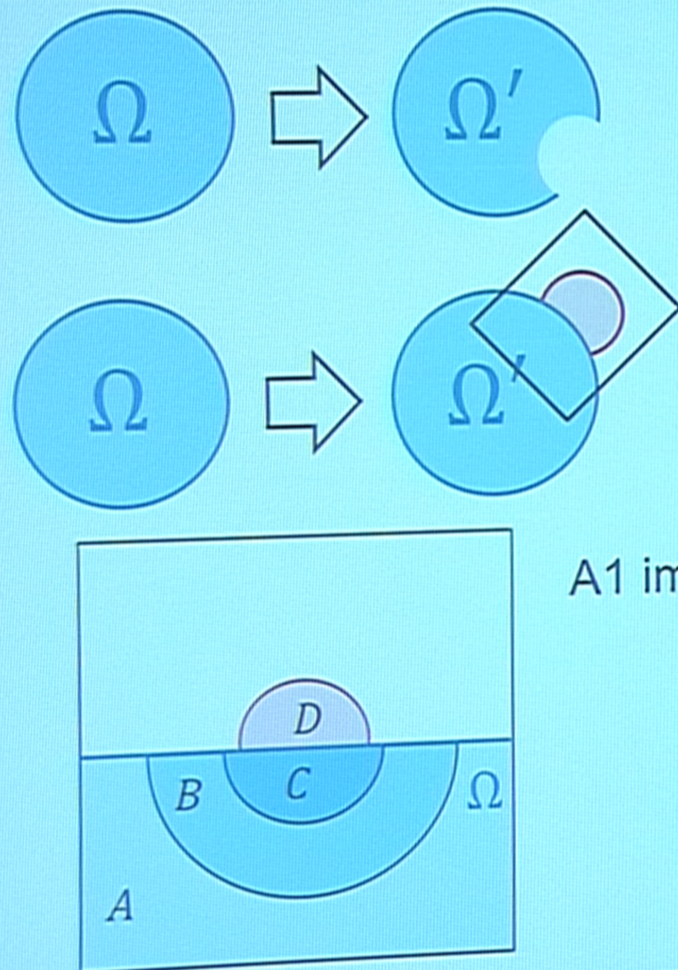
$$I(A:C|B)_\rho = I(B:D|C)_\sigma = 0, \forall \rho \in \mathcal{S}. (*)$$

Then, there exists a unique set of `merged' states $\{\tau^\rho_{ABCD}\}$ which satisfy

1. $\tau^\rho_{ABC} = \rho_{ABC}$, $\tau^\rho_{BCD} = \sigma_{BCD}$.
2. $I(A:D|BC)_\tau = 0$.
3. $S(\tau^\rho_{ABCD}) - S\left(\tau^{\rho'}_{ABCD}\right) = S(\rho_{ABC}) - S(\rho'_{ABC})$.

- $\tau^\rho_{ABCD} = \mathcal{E}^\sigma_{C\to CD}(\rho_{ABC})$, $\mathcal{E}^\sigma$ is the Petz recovery map [Petz, '03].
- A special version of Quantum Marginal Problem (in QMA [Liu, et. al '07]).

# Proof sketch (Isomorphism theorem)




The partial trace: $\Sigma(\Omega) \rightarrow \Sigma(\Omega')$

$\sigma_{\Omega} \in \Sigma(\Omega)$

$\rho_{BCD}$: the reference state

A1 implies $I(A:C|B)_{\sigma} = I(B:D|C)_{\rho} = 0$.

$\rightarrow \exists \tau_{ABCD}$ $\begin{cases} \tau_{ABC} = \sigma_{ABC} \\ \tau_{BCD} = \rho_{BCD} \end{cases}$

$\rightarrow \cdots \rightarrow \tau_{ABCD} \in \Sigma(\Omega D)$

# Charges / Superselection sectors

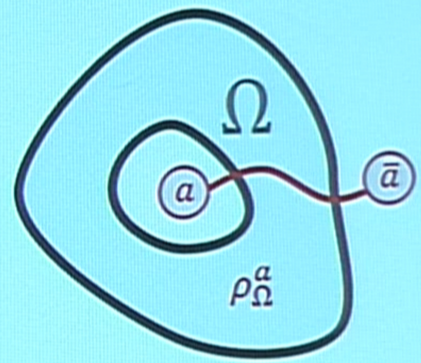
We define charges via the information convex on an annulus

**Theorem 2**

If $\Omega$ is an annulus, there is a finite set of labels $\mathcal{L}$ such that

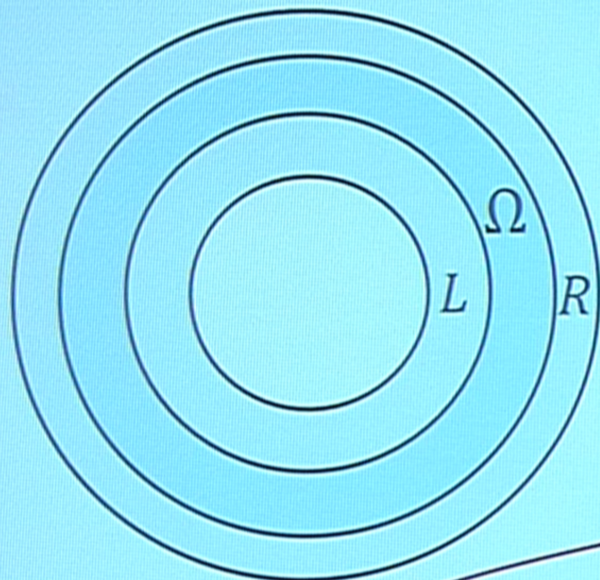
$$\sigma_\Omega = \bigoplus_{a\in\mathcal{L}} p_a \rho_\Omega^a, \qquad \forall \sigma_\Omega \in \Sigma(\Omega).$$

$\rho_\Omega^a$ is independent of the choice of $\sigma_\Omega$.




- Intuitively, each $\rho_\Omega^a$ corresponds to the reduced state of an excited state with a fixed charge pair.
- The reference state $\rho_\Omega$ is an extreme point, the “vacuum” $\rho_\Omega^1 \equiv \rho_\Omega$.
- For each $a \in \mathcal{L}$, we can show there is an anti-charge $\bar{a} \in \mathcal{L}$.

# Proof sketch




Lemma:
Any extreme point $\sigma \in \Sigma(\Omega)$ satisfies

$$I(L{:}R)_\sigma = 0.$$

Pick two extreme points $\sigma_\Omega,\ \sigma'_\Omega \in \Sigma(\Omega)$.

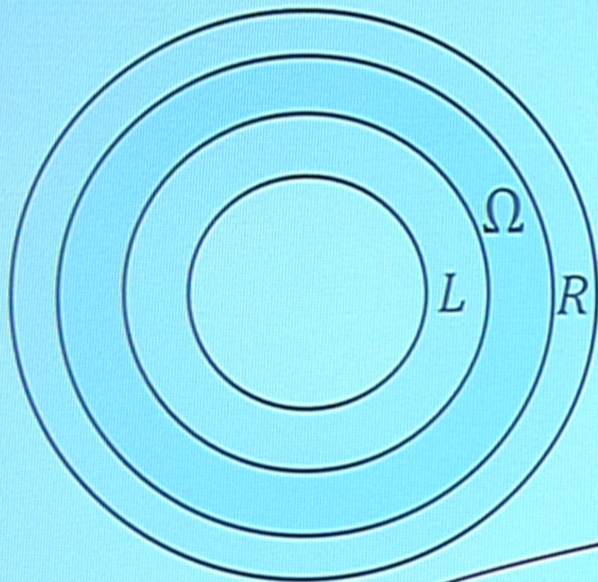
By the isomorphism theorem,

$$\Sigma(\Omega) \cong \Sigma(L) \cong \Sigma(R).$$

$$F(\sigma_\Omega, \sigma'_\Omega) \leq F(\sigma_L \otimes \sigma_R, \sigma'_L \otimes \sigma'_R) = F(\sigma_L, \sigma'_L) \cdot F(\sigma_R, {\sigma_R}')$$

$$F(\sigma_\Omega, \sigma'_\Omega) = F(\sigma_\mathrm{L}, \sigma'_L) = F(\sigma_R, \sigma'_R)$$

# Proof sketch

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$$F(\sigma_\Omega, \sigma'_\Omega) = F(\sigma_L, \sigma'_L) = F(\sigma_R, \sigma'_R)$$

$$\rightarrow F(\sigma_\Omega, \sigma'_\Omega) = 0 \text{ or } 1 \qquad (0 \leq F \leq 1)$$

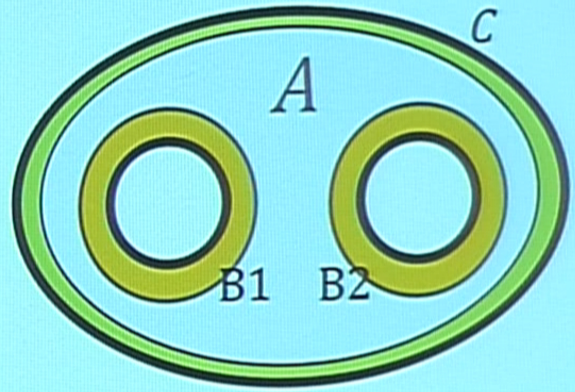
# Fusion rules

To define fusion rules, we consider a 2-hole disk

$$\Omega = ABC$$

If $\Omega$ is a 2-hole disk, then

$$\sigma_\Omega = \bigoplus_{a,b,c} p_{abc}\sigma_\Omega^{(a,b,c)}, \quad \forall \sigma_\Omega \in \Sigma(\Omega).$$




$$\sigma_\Omega^{(a,b,c)}: \sigma_{B_1}^{(a,b,c)} = \rho_{B_1}^a, \ \sigma_{B_2}^{(a,b,c)} = \rho_{B_2}^b$$

$$\sigma_C^{(a,b,c)} = \rho_C^c$$

Now $\sigma_\Omega^{(a,b,c)}$ depends on $\sigma_\Omega$.

**Theorem 3**

$\left\{\sigma_\Omega^{(a,b,c)}\right\} \cong$ a state space on a finite Hilbert space $V_{ab}^c$.

We *define* the fusion multiplicity by $N_{ab}^c \coloneqq \dim V_{ab}^c$

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^c c$$

# Proof sketch

Pick an extreme point $\omega^{(a,b,c)} \in \left\{\sigma_{\Omega}^{(a,b,c)}\right\}$

Purify it $\omega_{\Omega}^{(a,b,c)} \to |\varphi\rangle_{\Omega E}$

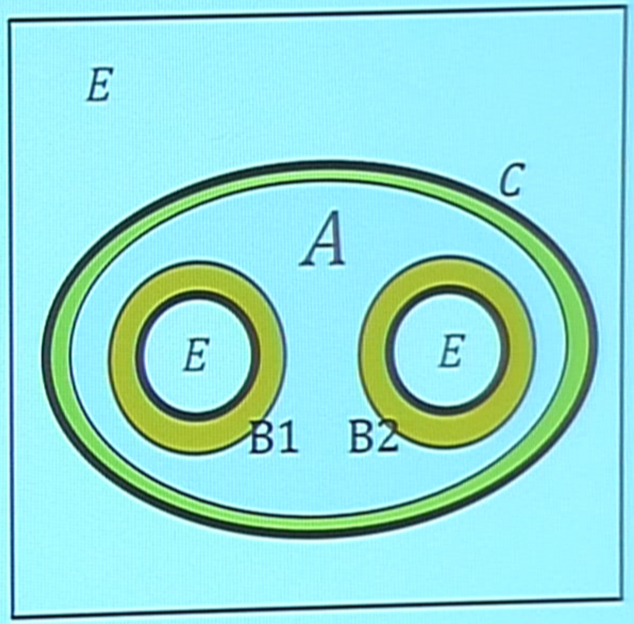
We merge all element in $\left\{\sigma_{\Omega}^{(a,b,c)}\right\}$ with $\varphi_{BCE}$

⇩

$\mathcal{S}_{\Omega E} :=$ the set of merged states

- extreme points in $\Sigma(\Omega) \to$ pure states
- Superpositions/mixtures in $\mathcal{S}_{\Omega E} \to$ elements in $\Sigma(\Omega)$

$\to \mathcal{S}_{\Omega E} \cong \mathcal{S}(V_{ab}^{c})$, the state space of a finite-dimensional Hilbert space $V_{ab}^{c}$



# Axiomatic properties of fusion rules

In the anyon theory, the fusion multiplicities $N_{ab}^{c}$ must satisfy the following rules.

1. $N_{ab}^{c} = N_{ba}^{c}$: commutativity of fusion rules
2. $N_{a1}^{c} = \delta_{a,c}$: vacuum
3. $N_{ab}^{1} = \delta_{b,\bar{a}}$: anticharge
4. $N_{ab}^{c} = N_{\bar{a}\bar{b}}^{\bar{c}}$: charge-anticharge duality
5. $\sum_i N_{ab}^{i} N_{ic}^{d} = \sum_j N_{aj}^{d} N_{bc}^{j}$ : associativity

**Theorem 4**

$N_{ab}^{c}$ in our definition satisfies all the properties.

# Proof sketch

Ex.) $N_{a1}^{c} = \delta_{a,c}$

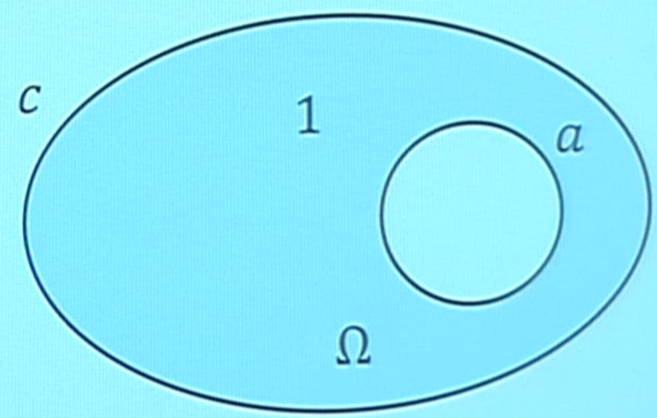
Consider a 2-hole disk with $(1, a, c)$.

$$\sigma_{\Omega}^{(1,a,c)} \in \Sigma(\Omega)$$

One can merge the vacuum sector with a disk $D$.

$\Omega' = \Omega \cup D$

$$\sigma_{\Omega}^{(1,a,c)}, \rho_D \to \tau_{\Omega'}$$




$$\tau_{\Omega'} \in \Sigma(\Omega')$$

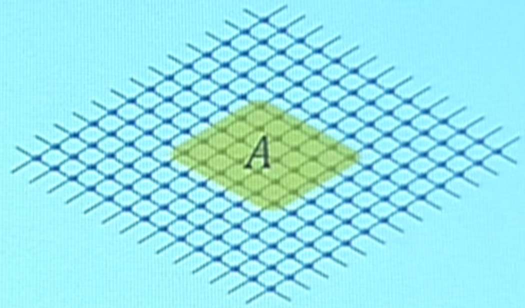
$\Omega'$ is an annulus and $\tau_{\Omega'}$ is in $a$ –sector.

$$\tau_{\Omega'} = \rho_{\Omega'}^{a} \to c = a.$$

# Topological Entanglement Entropy [1/2]

The area law states that

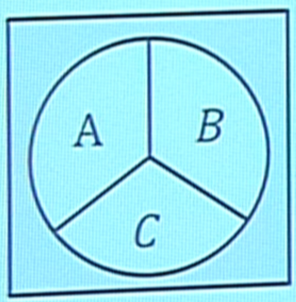
$$S(A)_\rho = \alpha|\partial A| - \gamma\,.$$



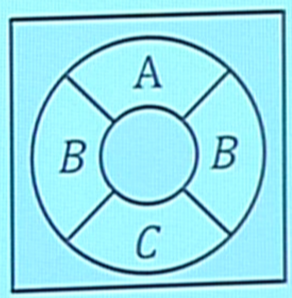

$\gamma$: **topological entanglement entropy**
[Kitaev, Preskill, '06] [Levin, Wen '06]

What's the value of $\gamma$?

$$S_{\text{topo}} := S(AB)_\rho + S(BC)_\rho + S(CA)_\rho - S(A)_\rho - S(B)_\rho - S(C)_\rho - S(ABC)_\rho$$



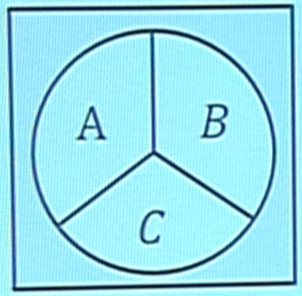

$S_{\text{topo}} = \gamma$




$S_{\text{topo}} = 2\gamma$

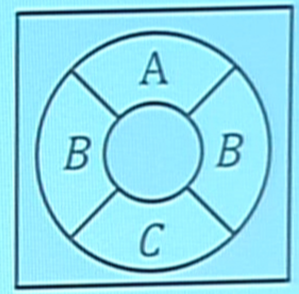
# Topological Entanglement Entropy [2/2]

KP partition




$S_{\text{topo}} = \gamma$

LW partition




$S_{\text{topo}} = 2\gamma$

**Quantum dimension:** $d_a d_b = \sum_{c \in \mathcal{L}} N_{ab}^{c} d_c\,, \qquad d_a \in \mathbb{R}_{\geq 1}$

- For Levin-Wen models or TQFT, it was shown that

$$\gamma = \log\mathcal{D},\ \mathcal{D} := \sqrt{\textstyle\sum_{a \in \mathcal{L}} d_a^2}\,.$$

**Theorem 5**

$S_{\text{topo}} = \log\mathcal{D}$ for KP partition, $S_{\text{topo}} = \log\mathcal{D}^2$ for LW partition.

# Proof sketch: entropic contribution

$$2f(a) \coloneqq S(\rho_\Omega^a) - S(\rho_\Omega^1)$$

Merging $\rho_{LM}^a, \rho_{MR}^b \to \sigma_Y^{a\times b}$

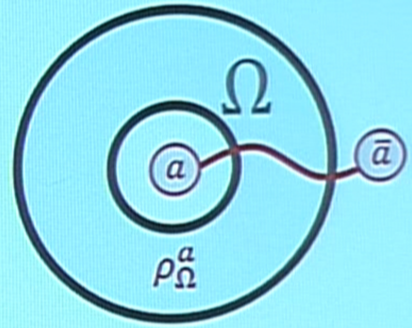
Since $\sigma_Y^{a\times b}$ is a quantum Markov state,

$$S\left(\sigma_Y^{a\times b}\right) - S(\sigma_Y^1) = 2f(a) + 2f(b).$$

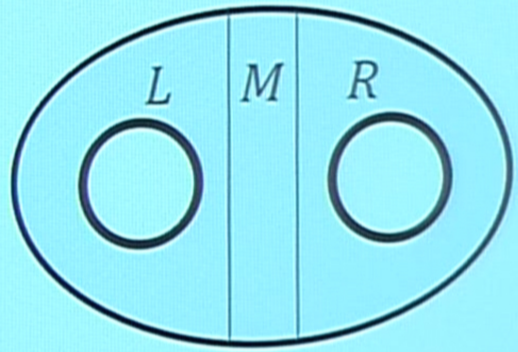
Also, $\sigma_Y^{a\times b}$ has the maximum entropy in $\Sigma(Y)$:

$$S\left(\sigma_Y^{a\times b}\right) - S(\sigma_Y^1) = f(a) + f(b) + \log\left(\sum_c N_{ab}^c 2^{f(c)}\right)$$

$$\to 2^{f(a)} 2^{f(b)} = \sum_c N_{ab}^c 2^{f(c)}$$




$Y = LMR$



# Proof sketch: topo. ent. Entropy (LW)

$\rho_{LR}, \rho_{MR}$ :the reduced states of the reference state

Merging $\rho_{LR}, \rho_{MR} \rightarrow \tau_{\Omega} \neq \rho_{\Omega}^{1}$ !

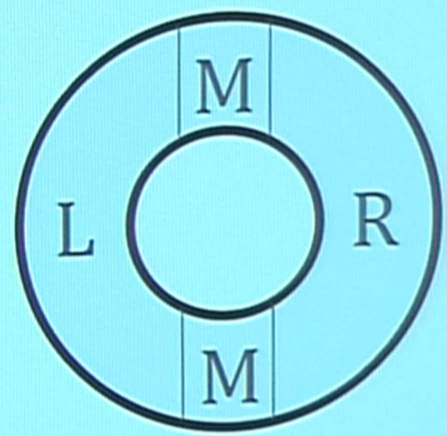
Actually, $\tau_{LMR}$ is the maximum entropy state in $\Sigma(\Omega)$.

- $\sigma_{\Omega} = \oplus p_a \rho_{\Omega}^{a} \ \forall \sigma_{\Omega} \in \Sigma(\Omega)$
- $S(\rho_{\Omega}^{a}) - S(\rho_{\Omega}^{1}) = \log d_a^2$

$$\rightarrow \tau_{\Omega} = \sum_a \frac{d_a^2}{\mathcal{D}^2} \rho_{\Omega}^{a}.$$

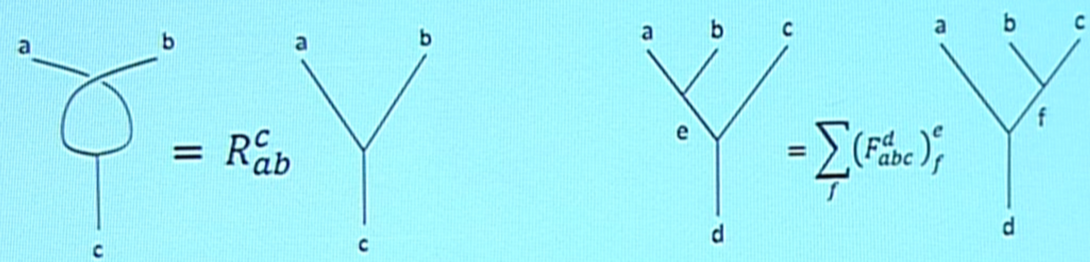
- $S(LMR)_{\tau} = S(LM)_{\tau} + S(MR)_{\tau} - S(M)_{\tau}$

$$\rightarrow S_{\text{topo}} = S(\tau_{\Omega}) - S(\rho_{\Omega}^{1}) = \log \mathcal{D}^2.$$




$\Omega = LMR$

# Open question 1: R- and F-matrices




**How to define R and F-matrices ?**

**Problem:** hard to see $U(1)$ factors in the information convex (mixed states)

[Kawagoe, Levin '19]: A way to extract these matrices unambiguously.

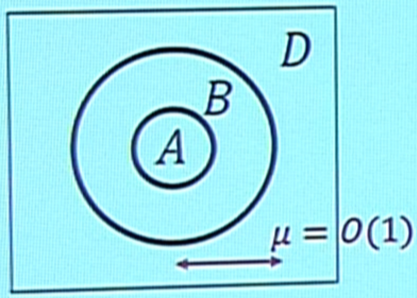
**Another topological quantities?**

[Shi,'19]: Extend our methods to show unitarity of S-matrix.

# Open question 2: stability problem

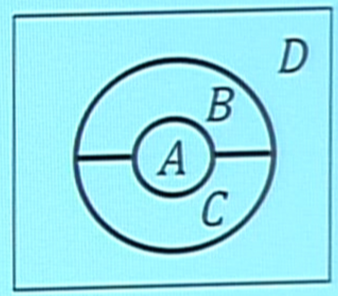
To claim we derive properties of a *phase*, they should be the same for any two systems in the same phase.

Axiom **A0**:




$$S(A|B)_\rho + S(A)_\rho = 0$$

Axiom **A1**:




$$S(A|B)_\rho + S(A|C)_\rho = 0$$

**Problem:** A1 is not true in some cases (even approximately)!

States with spurious topological entanglement entropy

# Open question 2: stability problem

## Spurious topological entanglement entropy

**[Bravyi '08]:** There exists a ground state which is **not** in any topologically ordered phase **($\mathcal{D} = 1$)**, and has $I(A: C|B)_\rho > 0$ for particular $ABC$.

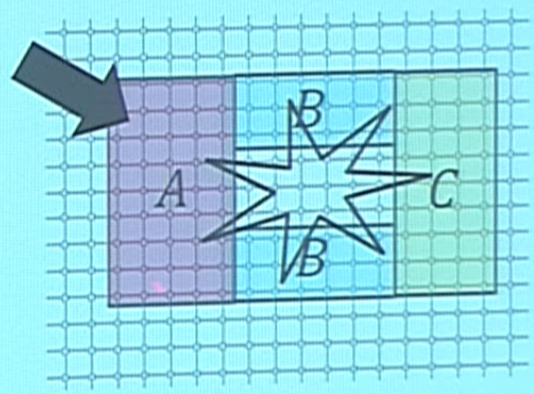
1D cluster state embedded in 2D lattice

- Known as a ground state in 1D $Z_2 \times Z_2$ SPT Phase

**[William, '18]:** homogeneous 2D model

- Known as a ground state in $Z_2 \times Z_2$ Subsystem SPT Phase

- **[KK, Brandao, in preparation]:** There is a model without such SPT but has spurious contribution




$S_{topo} \neq 2\log\mathcal{D}$

***SPT phase:** nontrivial under a **symmetry constraint** (trivial if we ignore symmetry)

# Summary

We derive [a part of] the anyon theory from the area law (axioms A0, A1).

- Charges and fusion rules are defined via the structures of the information convex for various regions.
- We proved these concepts satisfy necessarily requirements.
- We show that the sub-leading term of the area law must be $\log \mathcal{D}$.

**Next step:** Defining R and F matrices in a consistent way.

**Challenge:** Show the stability by e.g. weakening Axiom A1 to include systems with spurious topo. EE.

**Thank you!**