Title: Backreactions in Quantum Cosmological Perturbation Theory

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Abstract: Cosmological perturbation theory has a long tradition for describing the early phases of the Universe. As the observations of the CMB radiation suggest, it is reasonable, at least as a first approximation, to implement cosmological inhomogeneities as small perturbations around homogeneous and isotropic FRW solutions. In these approaches, backreactions between the inhomogeneities and the background are usually neglected. There is an ongoing debate about how and to which extend these backreactions affect the large scale structure of the Universe. Even at a purely classical level, there is no conclusive answer to this question yet.

In my talk, I am going to present a new systematic formalism for implementing backreactions in cosmological perturbation theory, in which both, the perturbations and the homogeneous degrees of freedom are considered as quantum degrees of freedom. As a more realistic theory of quantum fields on quantum cosmological space times, it can help to close the gap between a full theory of quantum gravity and symmetry-reduced models of quantum cosmology, and to confront these theories with observations. Our results show that quantum backreactions imply non-trivial corrections that are potentially phenomenologically significant.

Quantum Cosmological Backreactions

Space Adiabatic Perturbation Theory for Cosmological Perturbations

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joint work with Thomas Thiemann
based on arXiv:1906.08166, 1906.08194, 1909.07271

and with Jonas Neuser and Thomas Thiemann
based on arXiv:1906.08185

Perimeter Institute, November 14, 2019

Overview

1. The Status Quo
   - Quantum Cosmology
   - with Cosmological Perturbations
2. Review of the Born-Oppenheimer Approximation
   - for a molecular system
3. Backreactions with Space Adiabatic Perturbation Theory
   - a systematic generalization
4. Backreactions for Cosmological Perturbation Theory
   - Preparation of the canonical system
   - Results for second order backreactions
5. Outlook and Summary

Quantum Cosmology with Perturbations

- CMB measurements (e.g., by the Planck collaboration) suggest that **Symmetry-reduced models of Gravity or Quantum Gravity** give reasonable descriptions of the early Universe.
- **Numerous approaches to quantum cosmology**, such as string theory approaches (e.g., cyclic models[a]), euclidean[b] and lorentzian[c] (path integral) approaches, spinfoam cosmology[d], (canonical) loop quantum cosmology[e] and many more.
- **Observations indicate deviations** from this homogenous and isotropic picture.
- First step: Include inhomogeneities as **perturbative (quantum) fields on fixed cosmological backgrounds**.

[a] [Steinhardt, Turok '05]
[b] [Hartle, Hawking '83], [Hawking '84]
[c] [Feldbrugge, Lehners, Turok '17]
[d] [Rovelli, Vidotto '08], [Bianchi, Rovelli, Vidotto '10]
[e] [Bojowald '99, '00], [Ashtekar, Bojowald, Lewandowski '03], [Ashtekar, Pawlowski, Singh '06]

Quantum Cosmology with Perturbations

- Introduce perturbation fields for the metric and the matter fields on fixed cosmological backgrounds[a].
- **Gauge ambiguities**: Generic coordinate transformations in GR don't preserve split into homogeneous and inhomogeneous d.o.f.. Fictitious perturbations.
- Introduce **Gauge-invariant perturbation fields**, *e.g.*, Mukhanov-Sasaki variables[b] for the scalar sector.
- Transformations for the perturbative d.o.f. depend on the homogeneous d.o.f. **Canonical structure** of the whole system is broken.
- Mena Marugan *et al.*: Gauge-invariant scalar perturbations in (almost) canonical system, (*i.e.*, up to 2nd order in the perturbations) on cosmological backgrounds.[c]. Approaches for construction of gauge-invariant observables to higher orders.[d]

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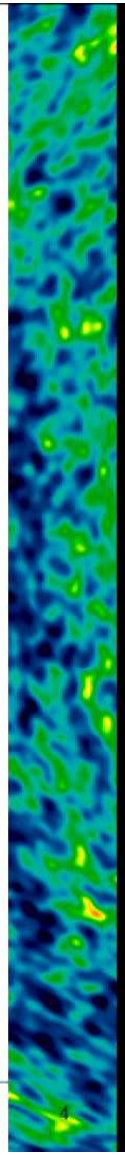
[a] [Lifshitz '46]

[b] [Sasaki '83], [Mukhanov '88]

[c] [Castello Gomar, Martin-Benito, Mena Marugan '15]

[d] [Dittrich, Tambornino '06 '07], [Brunetti, Fredenhagen, Hack, Pinamonti, Rejzner '16]



Backreactions in Cosmological Perturbation Theory

- Evaluation of perturbations on fixed (effective) quantum cosmological backgrounds.[1]
- No backreactions from perturbations onto the homogeneous sector. Discussion of backreactions for the classical case.[2]
- Backreactions in the quantum case, *i.e.,* for quantum mechanical homogeneous modes and perturbations, even less understood.

New Approach for Quantum Backreactions

- Coupled quantum systems with very different rates of change of the respective subsystems offer approximative solutions by means of the Born-Oppenheimer approximation.[3]
- Idea for gravitation-matter systems: Define a small perturbative parameter $\varepsilon^2 := \frac{\kappa}{\lambda}$, where, $\kappa = 8\pi G$: gravitational coupling constant, λ: (standard model) matter coupling constants.

[1] [Feldbrugge, Lehners, Turok '17], [Bolliet et. al. '15], [S., Barrau et. al. '15]

[2] [Buchert et al. '03, '05, '15] , [Green, Wald '11, '12, '13, '14], [Kolb, Matarrese, Riotto '06]

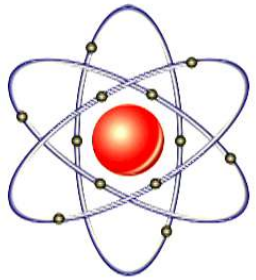
[3] [Born, Oppenheimer '27]

A Born-Oppenheimer-like Scheme[4]

Recall Born-Oppenheimer theory for molecules

- Exploit very small mass ratio of electrons and nuclei,

$$\varepsilon^2 := \frac{m_e}{M_N}$$



Molecular Hamiltonian in Schrödinger representation

$x \in \mathbb{R}^{3I}$: positions of the nuclei, $y \in \mathbb{R}^{3k}$: positions of the electrons.

$$\hat{H} = -\varepsilon^2 \frac{\triangle_x}{2m_e} - \frac{\triangle_y}{2m_e} + V_e(y) + V_{e,n}(x,y) + V_n(x) \tag{1}$$

$$=: -\varepsilon^2 \frac{\triangle_x}{2m_e} + \hat{H}_e(x) \tag{2}$$

Structure of the problem suggests to solve the eigenvalue problem for $\hat{H}_e(x)$ and define x-dependent projections, $P_n(x) := \psi_n(x;y)\langle\psi_n(x;y),\cdot\rangle$,

$$\hat{H}_e(x)\psi_n(x;y) = E_n(x)\psi_n(x;y), \quad \left[\hat{H}_e(x), P_n(x)\right] = 0 \ \ \forall x \in \mathbb{R} \tag{3}$$

[4][Born, Oppenheimer '27]

A Born-Oppenheimer-like Scheme

Define a projection on the full Hilbert space, $\mathscr{H} = L^2(\mathbb{R}^{3I}) \otimes L^2(\mathbb{R}^{3k}) = L^2(\mathbb{R}^{3I}, L^2(\mathbb{R}^{3k}))$ by means of a direct integral, $P_n := \int^{\oplus} \mathrm{d}x\, P_n(x)$[5]. Then,

$$\left[\hat{H}, P_n\right] = \mathscr{O}(\varepsilon), \tag{4}$$

i.e., the subspace of $\mathscr{H}$ associated with an energy band n of the electrons is almost invariant under the full dynamics.

Hamiltonian restricted to one of these electron subspaces,

$$\hat{H}_{\mathrm{eff},n} := P_n \hat{H} P_n \tag{5}$$

Dynamics not easy to extract: Perform unitary transformation $U : \mathscr{H} \to L^2(\mathbb{R}^{3I})$, e.g., by projecting on the electron subspace $P_n(x)$.

Define effective Hamiltonian for the nuclei including the backreactions on the simpler reference space,

$$\hat{H}^{\mathrm{ref}}_{\mathrm{eff},n} := U P_n \hat{H} P_n U^{\dagger} \in \mathscr{L}(L^2(\mathbb{R}^{3I})) \tag{6}$$

[5] [Teufel '03]

A Born-Oppenheimer-like Scheme

Result for molecular example,

$$\hat{H}^{\mathrm{ref}}_{\mathrm{eff},n} = -\frac{\triangle_x}{2M_{\mathrm{N}}} + E_n(x) + \mathscr{O}(\varepsilon^2). \tag{7}$$

Backreactions introduce potential energy due to electron-proton coupling.

Valid only for times $t \ll \varepsilon^{-1}\mathrm{t_e}$, but interesting dynamics for nuclei happens for $t > \varepsilon^{-1}\mathrm{t_e}$.

Improve on this by means of a perturbation theory with respect to ε.

The Space Adiabatic Theorem[6]

There exists (under certain assumptions on the structure of the system) a true projector, $\Pi_n \in \mathscr{L}(\mathscr{H})$, associated with a subspace of the "fast" subsystem, such that,

$$\left[\hat{H}, \Pi_n\right] = \mathscr{O}(\varepsilon^{\infty}). \tag{8}$$

[6] [Hövermann, Spohn, Teufel '01], [Panati, Spohn, Teufel '03]

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Space Adiabatic Perturbation Theory[7]

- Straightforward perturbative construction scheme for backreactions from the n-th eigenstate of the light (electron) systems on the heavy (nucleus) systems.
- Born-Oppenheimer approximation $\hat{=}$ 0^{th} order space adiabatic perturbation theory (SAPT).

Iterative Construction of

1) a projection operator (commutes with the Hamilton operator up to $\mathscr{O}(\varepsilon^{k+1})$ at k-th order): $\hat{\Pi}_n = \hat{\Pi}_{n,0} + \sum_{k=1}^{\infty} \varepsilon^k \hat{\Pi}_{n,k}$,
2) a unitary operator, $\hat{U}_n = \hat{U}_{j,0} + \sum_{k=1}^{\infty} \varepsilon^k \hat{U}_{n,k}$ and
3) an effective Hamilton operator for the heavy subsystem, $\hat{H}_{\text{eff},n} = \hat{H}_{\text{eff},n,0} + \sum_{k=1}^{\infty} \varepsilon^k \hat{H}_{\text{eff},n,k}$.

where n: quantun number of the light subsystem.

[7] [Panati, Spohn, Teufel '02, '03, '07]

Applications of Space Adiabatic Perturbation Theory

- **Wide range of applicability** since conditions on the structure of the system are mild: Local energy gap between the "fast" subspaces, tensor product structure of the Hilbert space, $\mathscr{H} = \mathscr{H}_{\mathrm{s}} \otimes \mathscr{H}_{\mathrm{f}}$.[8]
- Concrete analysis performed with **deformation** (or **phase space**) **quantum mechanics** (*i.e.* pseudo-differential calculus[9]) for the heavy subsystem.
- For example: Dirac equation with slowly varying external potentials, T-BMT equation for the spin dynamics of a relativistic particle in the semiclassical limit and backreaction onto the translational motion.[10]
- Formalism applicable to **constrained systems**. No time variable is needed (hence, the name...).

[8] [Panati, Spohn, Teufel '03], [Teufel '03]

[9] [Folland '89], [Hörmander '85], [Dimassi, Sjöstrand '99]

[10] [Teufel '03]

SAPT for Cosmological Perturbation Theory

Space adiabatic pertubation theory initially developed for quantum mechanical systems with a finite number of degrees of freedom.

Here: apply SAPT to cosmological perturbation theory, *i.e.*, a quantum field theory with infinitely many degrees of freedom.

Meet several challenges[11]

- Generic problems due to infinite number of degrees of freedom (*i.e.*, existence of infinitely many inequivalent representations for the field algebra).
- Occurence of indefinite mass squared values.
- Non-polynomial operators in the effective Hamilton constraint for the homogeneous sector.

[11] [S., Thiemann '19 I, IV]

Preparation of the Hamilton Constraint

- 4D-manifold $\mathscr{M} \cong \mathbb{R} \times \mathbb{T}^3$ where $\mathbb{T}^3$ has finite side lenghts L.
- Gravity with cosmological constant Λ, gravitational constant κ
  + Scalar field Φ with mass m_Φ and coupling constant λ.
  - Homogeneous sector: scale factor a, lapse N_0 and scalar field ϕ_0,
  - Perturbations of the spatial metric $(\alpha(x), \beta(x), h^{ij}(x))$, lapse $g(x)$, shift $k(x)$ and the scalar matter field $f(x)$.

Procedure

1. Expand the action S up to 2^{nd} order in the perturbations.
2. Perform Legendre transformation and define associate conjugate momenta for
   - the homogeneous sector, (p_a, π_0)
   - the perturbations, $(\pi_\alpha(x), \pi_\beta(x), \pi_{ij}(x), \pi_f(x))$.

Hamilton constraint up to second order in the perturbations,

$$\mathscr{H} = N_0 \left[\mathscr{H}_0 + \mathscr{H}_2^{\mathrm{s}} + \mathscr{H}_2^{\mathrm{t}}\right] + g \cdot \mathscr{H}_g + k \cdot \mathscr{H}_k \tag{9}$$

- Need for gauge-invariant variables at the perturbative level.
- Generation of new secondary constraints.
- Hilbert-Schmidt condition is not satisfied.

Meeting the Hilbert-Schmidt Condition

Structure suggests to consider a simple rescaling of the fields by a,

$$\tilde{f} := a \cdot f, \quad \tilde{\pi}_f := \frac{\pi_f}{a}.$$

This breaks the canonical structure of the system.

Solution: More Generic Transformations for the System[12]

Restore the whole perturbative system and consider transformations which...

1. are **linear for the perturbation fields** χ, with $\tilde{\chi} = A(\phi_0, \pi_0, a, p_a)\,\chi$ s.t. the **gauge -invariant Mukhanov-Sasaki variable** v and $\mathscr{H}_g$, $\mathscr{H}_k$ become canonical variables,
2. restore the **canonical structure** of the full system (up to second order in the perturbations) by means of **transformations for the homogeneous variables** $(\phi_0, \pi_0, a, p_a) \to (\tilde{\phi}_0, \tilde{\pi}_0, \tilde{a}, \tilde{p}_a)$.
3. meet the **Hilbert-Schmidt condition**,
4. cancel terms that are not **densely defined on Fock space**.

[12] [Castello Gomar, Martin-Benito, Mena Marugan '15]

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SAPT for Quantum Cosmological Perturbation Theory

- Linear quantum constraints $\hat{\mathscr{H}}_g$ and $\hat{\mathscr{H}}_k$ can be solved naturally.
- Only remaining constraint: $\hat{\mathscr{H}}^{\mathrm{MS},h} := \hat{\mathscr{H}}_0 + \hat{\mathscr{H}}_2^{\mathrm{s}} + \hat{\mathscr{H}}_2^{\mathrm{t}} = 0$

Next step: Introduce the Adiabatic Perturbation Parameter ε

Adiabatic perturbation parameter for gravitation-matter system: $\varepsilon^2 := \frac{\kappa}{\lambda}$. Introduce ε and multiply the whole constraint by ε^2 and λ^{-1},

$$\hat{H}^{\mathrm{MS},h} = W\left[L^3\left(-\frac{1}{12}\frac{\varepsilon^4 p_a^2}{a} + \Lambda a^3 + \frac{\pi_0^2}{2a^3} + \frac{1}{2}\varepsilon^2 a^3 \phi_0^2\right)\right. \tag{12}$$
$$\left. + \frac{1}{2a}\int_{\mathbb{T}^3} \mathrm{d}^3x \left(\pi_\nu^2 + \nu\varepsilon^4\left(-\triangle + M_{\mathrm{MS}}^2\right)\nu + \frac{\pi^{ij}\pi_{ij}}{6} + h^{ij}\varepsilon^4\left(-3\triangle + (\varepsilon M_{\mathrm{T}})^2\right)h_{ij}\right)\right]$$

where W: Weyl quantization w.r.t. the whole system, and with effective Mukhanov-Sasaki and tensor masses,

$$M_{\mathrm{MS}}^2 = -\frac{\varepsilon^4 p_a^2}{18\,a^2} + \frac{7\,\varepsilon^2\,\pi_0^2}{2\,a^4} - 12\frac{a\,\phi_0\,\pi_0}{p_a} - 18\,\frac{\pi_0^4}{a^6\,p_a^2} + m_\Phi^2 a^2, \tag{13}$$

$$(\varepsilon M_{\mathrm{T}})^2 = \frac{\varepsilon^4 p_a^2}{6\,a^2} - \varepsilon^2\,m_\Phi^2 a^2 \phi_0 - 6\,\Lambda\,a^2 \tag{14}$$

SAPT for Quantum Cosmological Perturbation Theory

- **Technical realization: Deformation quantization**[13] with respect to the homogeneous degrees of freedom. Therefore, define a rescaling,

$$\varepsilon^2 p_a \to p_a, \quad \varepsilon \pi_0 \to \pi_0. \tag{15}$$

- Perform a **Wigner-transformation** W_{hom}^{-1} of the operators on the full Hilbert space w.r.t. the homogeneous d.o.f.,

$$h^{\text{MS},h}(a, p_a, \phi_0, \pi_0) = W_{\text{hom}}^{-1}\left[\hat{H}^{\text{MS},h}\right] \in S(\varepsilon; \Gamma, \mathscr{L}(\mathscr{F}_s)) \tag{16}$$

  is a function on the homogeneous phase space Γ with values in the space of linear operators on the perturbation (Fock) Hilbert space, $\mathscr{L}(\mathscr{F}_s)$.

- The operator product transforms into a **Moyal product** "$\star_\varepsilon$" on $S(\varepsilon; \Gamma, \mathscr{L}(\mathscr{F}_s))$. E.g., Weyl-ordered star-product for functions f, g on Γ,

$$f \star_\varepsilon g = f g + \frac{i\varepsilon}{2}\{f, g\}_{(\phi_0, \pi_0)} + \frac{i\varepsilon^2}{2}\{f, g\}_{(a, p_a)} - \frac{\varepsilon^2}{8}\sum_{i,j,k,l} \Pi_\phi^{ij}\Pi_\phi^{kl}(\partial_i\partial_k f)(\partial_j\partial_l g) + \mathscr{O}(\varepsilon^3)$$

  where $\Pi_\phi \in \Lambda^2(T\Gamma)$ is the Poisson bi-vector restricted to the inflaton phase space.

[13] [Groenewold '46], [Moyal '49], [Kontsevich '97]

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SAPT for Quantum Cosmological Perturbation Theory

SAPT step by step

1. **Wigner-transform the Hamilton constraint** operator $\hat{H}^{\mathrm{MS},h}$ with respect to the homogeneous d.o.f. $\rightarrow$ function on Γ with values in the perturbation field Fock space $\mathscr{L}(\mathscr{F}_{\mathrm{s}})$.

   Note: States in $\mathscr{F}_{\mathrm{s}}$ depend parametrically on (ϕ_0, π_0, a, p_a).

2. **Take a *discrete* eigenstate** $e_n \in \mathscr{F}_{\mathrm{s}}$ and construct the projector, $\Pi_{0,n} := e_n \langle e_n, \cdot \rangle_{\mathscr{F}_{\mathrm{s}}} \in \mathscr{L}(\mathscr{F}_{\mathrm{s}})$.

3. **Determine the Wigner-transformed symbol functions** as a perturbation series in ε,
   - $\Pi_{(k)} := \sum_{l=0}^{k} \varepsilon^l \Pi_l$ must satisfy, $h \star_\varepsilon \Pi_{(k)} - \Pi_{(k)} \star_\varepsilon h = \mathscr{O}(\varepsilon^{k+1})$.
   - $U_{(k)} := \sum_{l=0}^{k} \varepsilon^l U_l$ must satisfy, $U_{(k)} \star_\varepsilon \Pi_{(k)} \star_\varepsilon U^*_{(k)} - \Pi_{\mathrm{R}} = \mathscr{O}(\varepsilon^{k+1})$, where Π_{R} is the reference projection on the simpler subspace.
   - $h^{\mathrm{MS},h}_{\mathrm{eff},(k)} = U \star_\varepsilon \Pi \star_\varepsilon h^{\mathrm{MS},h} \star_\varepsilon \Pi \star_\varepsilon U^* \big|_k$

SAPT for Quantum Cosmological Perturbation Theory

Effective Hamilton Constraint at 0^{th} Order (Born-Oppenheimer)

Let for simplicity, $V = 0$ and $\Lambda = 0$, use original variables and rescale by a constant.

$$\hat{H}^{\mathrm{MS},h}_{\mathrm{eff},0} = W_{\mathrm{hom}} \left[L^3 \left(-\frac{\varepsilon^2 p_a^2}{12a} + \frac{\pi_0^2}{2a^3} \right) + \frac{1}{a} \sum_{\vec{k}} \sqrt{\vec{k}^2 + M^2_{\mathrm{MS}}} \cdot n_{\mathrm{MS},\vec{k}} + \frac{1}{6a} \sum_{\vec{k},\mu} \sqrt{18\vec{k}^2_{\mu} + 6(\varepsilon M_{\mathrm{T}})^2} \cdot n_{\mathrm{T},\vec{k},\mu} \right]$$

where the sums run over $\frac{2\pi}{L}\mathbb{Z}^3 \setminus 0$, and μ is the polarization label (odd or even) for the tensor modes.

- Standard FRW + homogeneous inflaton Hamilton constraint (extendible to $V \neq 0$ and $\Lambda \neq 0$)
- plus typical Klein-Gordon like (finite) energy contributions from the Mukhanov-Sasaki and the tensor modes of the *n*-th energy band.

Note: Only finitely many $n_{\mathrm{T},\vec{k},\mu}$, $n_{\mathrm{MS},\vec{k}}$ non-vanishing!

SAPT for Quantum Cosmological Perturbation Theory

Effective Hamilton Constraint at 2nd Order (Backreaction)

Computations involve derivatives of e_n w.r.t. (ϕ_0, π_0, a, p_a),

$$\frac{\partial e_n}{\partial \phi_0} =: \sum_m (\mathscr{A}_{\phi_0})_n^m \, e_m \tag{17}$$

$$= \sum_{\vec{k}} \left(\alpha^{\phi_0}_{n_{\mathrm{MS},\vec{k}}}(\phi_0, \pi_0, a, p_a) \delta^m_{\{..,n_{\mathrm{MS},\vec{k}}-2,..\}} + \beta^{\phi_0}_{n_{\mathrm{MS},\vec{k}}}(\phi_0, \pi_0, a, p_a) \delta^m_{\{..,n_{\mathrm{MS},\vec{k}}+2,..\}} + ... \right) e_m \tag{18}$$

and similarly $\mathscr{A}_{\pi_0}$, $\mathscr{A}_a$, $\mathscr{A}_{p_a}$ (connections). Evaluation for the model with $V = 0$, $\Lambda = 0$ and restriction to the phase space regions for which $M^2_{\mathrm{MS}} > 0$, $M^2_{\mathrm{T}} > 0$,

$$\hat{H}^{\mathrm{MS},h}_{\mathrm{eff},2} = W_{\mathrm{hom}} \left[\sum_{\vec{k}} \frac{2n_{\mathrm{MS},\vec{k}} + 1}{(\vec{k}^2 + M^2_{\mathrm{MS}})^{5/2}} \, h_{\mathrm{MS}}(\phi_0, \pi_0, a, p_a) + \sum_{\vec{k},\mu} \frac{2n_{\mathrm{T},\vec{k},\mu} + 1}{(18\vec{k}^2_{\mu} + 6(\varepsilon M_{\mathrm{T}})^2)^{5/2}} \, h_{\mathrm{T}}(\phi_0, \pi_0, a, p_a) \right]$$

SAPT for Quantum Cosmological Perturbation Theory

Result: Effective Hamilton constraint for the homogeneous d.o.f. with backreactions from the scalar and tensor modes (from one of the (possibly degenerate) subspaces of the relevant eigenspace, n).

$$\hat{H}^{\mathrm{MS},h}_{\mathrm{eff},(2)} = W_{\mathrm{hom}}\Bigg[L^3\left(-\frac{\varepsilon^2}{12}\frac{p_a^2}{a}+\frac{1}{2}\frac{\pi_0^2}{a^3}\right)+\frac{1}{a}\sum_{\vec{k}}\sqrt{\vec{k}^2+M_{\mathrm{MS}}^2}\cdot n_{\mathrm{MS},\vec{k}}+\frac{1}{6a}\sum_{\vec{k},\mu}\sqrt{18\vec{k}_\mu^2+6(\varepsilon M_{\mathrm{T}})^2}\cdot n_{\mathrm{T},\vec{k},\mu}$$
$$-\sum_{\vec{k}}\frac{2n_{\mathrm{MS},\vec{k}}+1}{(\vec{k}^2+M_{\mathrm{MS}}^2)^{5/2}}\cdot\frac{81}{4}\frac{\pi_0^{12}}{a^{19}p_a^6}\left(1+\frac{\varepsilon^2a^2p_a^2}{2\pi_0^2}c+\frac{\varepsilon^4a^4p_a^4}{4\pi_0^4}\right)^2$$
$$-\sum_{\vec{k},\mu}\frac{2n_{\mathrm{T},\vec{k},\mu}+1}{(18\vec{k}_\mu^2+6(\varepsilon M_{\mathrm{T}})^2)^{5/2}}\cdot\frac{3}{2}\varepsilon^8\frac{p_a^2\pi_0^4}{a^{11}}\Bigg] \qquad (19)$$

where c: numerical constant.

Even if the perturbations are switched off (*i.e.*, if all quantum numbers $\left(n_{\mathrm{MS},\vec{k}}, n_{\mathrm{T},\vec{k},\mu}\right)$ vanish), **Casimir-type backreactions** remain!

SAPT for Quantum Cosmological Perturbation Theory

Analysis of the Constraint: Non-Polynomial Form

Effective Hamiltonian has contributions with non-polynomial functions of *both*, the configuration *and* the momentum variables.

Consider for example the bare Casimir-term,

$$\hat{H}^{\mathrm{MS},h}_{\mathrm{eff},(2)} = W_{\mathrm{hom}} \left[L^3 \left(-\frac{\varepsilon^2}{12}\frac{p_a^2}{a} + \frac{1}{2}\frac{\pi_0^2}{a^3} \right) - \sum_{\vec{k}} \frac{1}{(\vec{k}^2 + M_{\mathrm{MS}}^2)^{5/2}} \cdot \frac{81}{4}\frac{\pi_0^{12}}{a^{19}p_a^6} \left(1 + \frac{\varepsilon^2 a^2 p_a^2}{2\pi_0^2} c + \frac{\varepsilon^4 a^4 p_a^4}{4\pi_0^4} \right)^2 \right.$$

$$\left. - \sum_{\vec{k},\mu} \frac{1}{(18\vec{k}_\mu^2 + 6(\varepsilon M_{\mathrm{T}})^2)^{5/2}} \cdot \frac{3}{2}\varepsilon^8 \frac{p_a^2 \pi_0^4}{a^{11}} \right]$$

Possible Solutions

- Estimates of the sum by means of integrals.
- Find dense set of vectors which is invariant under any of the considered operators.[14]

[14] [Thiemann, in prep.]

SAPT for Quantum Cosmological Perturbation Theory

There are regions in the homogeneous phase space Γ for which,

$$M_{\mathrm{MS}}(\phi_0, \pi_0, a, p_a)^2 < 0, \quad (\varepsilon M_{\mathrm{T}})(\phi_0, \pi_0, a, p_a)^2 < 0$$

$\rightarrow$ **Tachyonic instabilities!** Already for standard Mukhanov-Sasaki model.

Avoidance of Tachyonic Instabilities

- In the model presented above: Perform a canonical embedding such that the positivity of the masses becomes manifest (restriction of the phase space).
- A priori unjustified, but positive mass region contains at least the kernel of the classical homogeneous constraint

Further Improvement

- Explore the space of transformations $A(\phi_0, \pi_0, a, p_a)$ in order to obtain positive definite M^2_{MS} and M^2_{T} *everywhere* in phase space.
- Restrict to wave vectors $\vec{k}$, for which $(\vec{k}^2 + M^2_{\mathrm{MS}})$ and $(18\vec{k}^2_{\mu} + 6(\varepsilon M_{\mathrm{T}})^2)$ strictly positive.

Outlook

Backreactions in Quantum Cosmology

- Explore the space of possible (almost) canonical transformations for obtaining positive definite mass squares, M^2_{MS} and M^2_{T}.[15]
- Treat the Hamilton constraint classically in order to learn for the quantum analysis.
- First analyse simpler models, e.g., inflaton zero mode backreaction on homogeneous geometry or deparmetrised scalar field models.[16]
- Analyse the solutions of the full Hamilton constraint with backreactions: Find a set of dense invariant vectors. Evaluate the possible effects on the bounce dynamics in quantum cosmology.
- Numerical evaluation in order to compare to recent cosmological data (e.g., effects on inflationary dynamics, generation of primordial scalar and tensor power spectra for the modified background.[17]

[15] [S., Thiemann '19 I, IV]

[16] [Neuser, S., Thiemann '19 II], [S., Thiemann '19 (III)]

[17] [Martineau, Barrau, S. '17], [Lesgourges '11], [S., Barrau et. al. '15])

Outlook

- Compute higher orders in the space adiabatic perturbative scheme and improve the estimates. Check the convergence of the formal power series.
- Find relation of our results (linear order cosmological perturbation theory) to higher order gauge-invariant formalisms.[18]

Further Applications of Space Adiabatic Perturbation Theory

- Transition to full quantum gravity (*i.e.*, no split into homogeneous and inhomogeneous degrees of freedom).[19]
- Apply space adiabatic perturbation theory to quantum mechanical micro-engine model in order to understand the energy transition from a "fast" bosonic system coupled to a heavy particle moving in a gravitational potential.[20]

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[18] [Dittrich, Tambornino '06 '07], [Brunetti, Fredenhagen, Hack, Pinamonti, Rejzner '16]

[19] [Stottmeister, Thiemann '15 (I),(II),(III)]

[20] [Anglin, Eichmann, S., work in progress]

Summary

- Application of systematic **space adiabatic perturbation theory** to the **cosmological setting** for including backreactions from the perturbations onto the homogeneous sector. [S., Thiemann '19 I]
- Computation of non-trivial backreactions up to second (space adiabatic) order for,
  - **homogeneous geometry with backreactions from zero mode of the inflaton field**, [Neuser, S., Thiemann '19 II]
  - **homogeneous geometry with backreactions from an inflaton *field*** (deparametrised model with additional dust fields), [S., Thiemann '19 III]
  - **homogeneous geometry and inflaton with backreactions from gauge-invariant Mukhanov-Sasaki and tensor perturbations**. [S., Thiemann '19 IV]
- Non-trivial backreactions from perturbations onto the homogeneous and isotropic sector need to be evaluated (analytically or numerically).
- Computation of backreaction effects on the dynamics of the homogeneous sector need a variety of mathematical tools.
- Derive a **quantum analog of the BKL conjecture**.
- **Direct comparison with observational data possible**. [S., Barrau et. al. '15], [Martineau, Barrau, S. '17]

SAPT for Quantum Cosmological Perturbation Theory

Result: Effective Hamilton constraint for the homogeneous d.o.f. with backreactions from the scalar and tensor modes (from one of the (possibly degenerate) subspaces of the relevant eigenspace, n).

$$\hat{H}^{\mathrm{MS},h}_{\mathrm{eff},(2)} = W_{\mathrm{hom}}\Bigg[L^3\left(-\frac{\varepsilon^2}{12}\frac{p_a^2}{a}+\frac{1}{2}\frac{\pi_0^2}{a^3}\right)+\frac{1}{a}\sum_{\vec{k}}\sqrt{\vec{k}^2+M_{\mathrm{MS}}^2}\cdot n_{\mathrm{MS},\vec{k}}+\frac{1}{6a}\sum_{\vec{k},\mu}\sqrt{18\vec{k}_\mu^2+6(\varepsilon M_{\mathrm{T}})^2}\cdot n_{\mathrm{T},\vec{k},\mu}$$
$$-\sum_{\vec{k}}\frac{2n_{\mathrm{MS},\vec{k}}+1}{(\vec{k}^2+M_{\mathrm{MS}}^2)^{5/2}}\cdot\frac{81}{4}\frac{\pi_0^{12}}{a^{19}p_a^6}\left(1+\frac{\varepsilon^2a^2p_a^2}{2\pi_0^2}c+\frac{\varepsilon^4a^4p_a^4}{4\pi_0^4}\right)^2$$
$$-\sum_{\vec{k},\mu}\frac{2n_{\mathrm{T},\vec{k},\mu}+1}{(18\vec{k}_\mu^2+6(\varepsilon M_{\mathrm{T}})^2)^{5/2}}\cdot\frac{3}{2}\varepsilon^8\frac{p_a^2\pi_0^4}{a^{11}}\Bigg] \qquad (19)$$

where c: numerical constant.

Even if the perturbations are switched off (*i.e.*, if all quantum numbers $\left(n_{\mathrm{MS},\vec{k}}, n_{\mathrm{T},\vec{k},\mu}\right)$ vanish), **Casimir-type backreactions** remain!