Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 9

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Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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URL: http://pirsa.org/19110045

1707.08388 + 1810.00463
Ctj w/ D. Treumann.

Goal: Thm. $H^3(\mathbb{M}, U(1)) \cong \mathbb{Z}_{24} \oplus (\leq 4)$
(probably trivial)

$\mathbb{M}$ = Fischer-Griess monster is the largest sporadic simple gp.

$$\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 13^3 \cdot 11^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 10^{54}$$

$\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 13^3 \cdot 11^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 10^{54}$

remark: $\in \#SL_{13}(\mathbb{F}_2)$.

$H^1(\mathbb{M}, U(1)) = \hom(\mathbb{M}, U(1)) = \{*\}$ because $\mathbb{M}$ is simple

$H^2(\mathbb{M}, U(1)) = \{\text{central extensions}\} = \{*\}$ ~1980s.

$H^3(\mathbb{M}, U(1))$ $\exists$ 't Hooft anomalies of actions of $\mathbb{M}$ on (1+1)d QFTs

Method. work prime by prime. $H^\bullet(\mathbb{M}, U(1))$ is a finite ab. gp.

$\bigoplus_{p \mid \#\mathbb{M}} H^\bullet(\mathbb{M}; U(1))_{(p)}$ "p-part"; order is a power of p. s.t. $\chi(\#\mathbb{M}) = 0$ on this finite gp.

$\mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ or ...

Sylow theory: $\forall p$. $\exists\ S \subseteq M$ s.t. $\frac{\#M}{\#S}$ is coprime to p.

$H^\cdot(M; \mathcal{U}(1))_{(p)} \underset{\text{"transfer"}}{\overset{\text{rest}}{\rightleftarrows}} H^\cdot(S, \mathcal{U}(1))_{(p)}$

a sort of integral / push forward

$\circlearrowright$ = mult by $\frac{\#M}{\#S}$.

is an IO on $H^\cdot_{(p)}$.

rest is an injection onto a direct summand.
on (p)-parts.

rest is an injection onto a direct summand on (p)-parts.

Eg: $p \geq 17$ p-Sylow in M is a $\mathbb{Z}_p$. in fact $M \geq S = \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$.

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in PSL_2(\mathbb{F}_p).$$

$$H^3(M; \mathcal{U}(1))_{(p)} \subseteq H^3(S; \mathcal{U}(1))_{(p)}$$

$\uparrow$ compat w/ S.S.

$$\#M = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 13^{3} \cdot \underline{11^{2} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71} \sim 10^{54}$$

remark: $\in \#SL_{13}(\mathbb{F}_2)$.

$$E_2^{i,j} = H^i\left(\overbrace{\mathbb{Z}_{p^{\frac{1}{2}}}}^{\text{w. } (p)\text{-tors.}}; \overbrace{H^j(\mathbb{Z}_p; u(1))}^{\text{purely } (p)}\right) \Rightarrow H^{i+j}(S; u(1))_{(p)}$$

if $i > 0$, then H^i = "purely p-w. p-tors." $= 0$.

$E_2^{i,j} = \{i = 0 \text{ column}\}$.

$$H^j(\mathbb{Z}_p; \mathbb{Z}) = \begin{cases} \mathbb{Z}_p & \text{if } j \text{ odd} \\ 0 & \text{if } j \text{ even.} \end{cases}$$

$$H^{j+1}(\mathbb{Z}_p, \mathbb{Z}) = \mathbb{Z}[t] / \left(p^{t=0}\right)$$

$\mathbb{Z}_{p^{\frac{1}{2}}}$ acts faithfully on $\mathbb{Z}_p \cdot t$

acts by k^{th} power of the faithful action on $\mathbb{Z}_p \cdot t^i$

$H^3(M; \mathcal{U}(1))_{(p)} \subseteq H^3(S; \mathcal{U}(1))_{(p)}$

↑ compat w/ S.S.

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in PSL_2(\mathbb{F}_p)$

$\mathbb{Z}_p$

$\deg \leq p$ $PSL_2(\mathbb{F}_p)$

$\mathbb{F}_p^\times \cong \mathbb{Z}_{p-1}$

E_2 $\quad \mathcal{U}(1)$ $\quad \mathbb{Z}_p$

$E_\infty = H^*(S)$.

So $H^3(S; \mathcal{U}(1))_{(p)} = 0$.

So $H^3(M; \mathcal{U}(1))_{(p)} = 0$.

$p = 11$ use $S = (\mathbb{Z}_{11} \rtimes \mathbb{Z}_5^3) \subseteq M$

$S \subseteq M$

Push forward

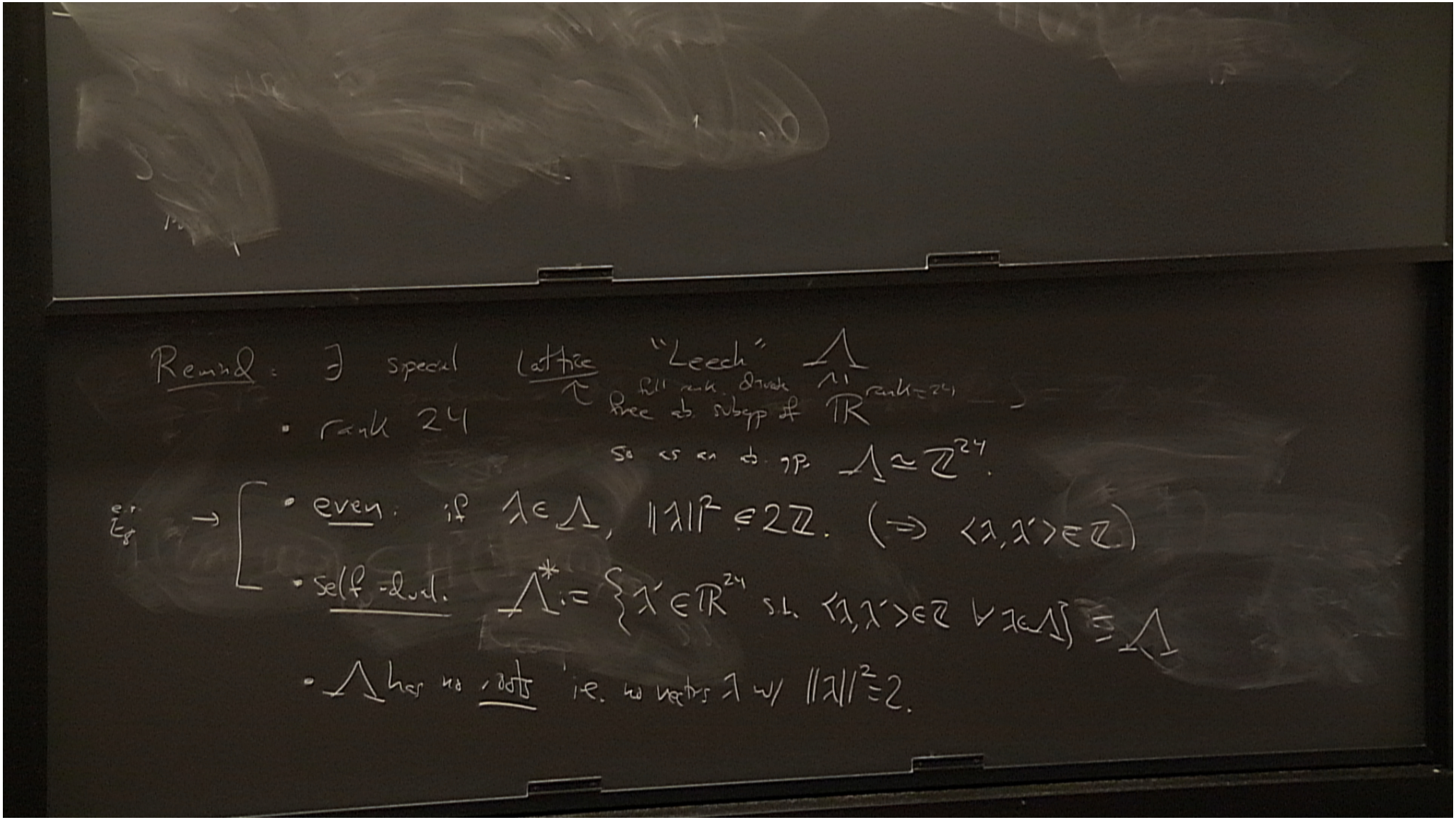
$$\#\mathbb{M} = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 13^{3}\cdot 11^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \sim 10^{54}$$

remark: $\in \#SL_{13}(\mathbb{F}_2)$.

The primes $p = 2, 3, 5, 7, 13$ are special. (cf. Conway–Norton Monstrous moonshine conjectures)

- there are non-conjugate cyclic subgps of $\mathbb{M}$ of order p
- p-Sylow in $\mathbb{M}$ is nonabelian.
- $p-1$ divides 24.

Why? I don't know. But I know the main ingredient.


Remind: ∃ special lattice "Leech" Λ
• rank 24
full rank discrete free ab. subgp of ℝ^24
rank = 24
So ∃ an ab. gp. Λ ≃ ℤ^24.
e.g. E8
• even: if λ ∈ Λ, ‖λ‖² ∈ 2ℤ. (⇒ ⟨λ, λ'⟩ ∈ ℤ)
• self-dual. Λ* := {λ' ∈ ℝ^24 s.t. ⟨λ, λ'⟩ ∈ ℤ ∀ λ ∈ Λ} = Λ
• Λ has no "holes" i.e. no vectors λ w/ ‖λ‖² = 2.

• self-dual. $\Lambda^* := \{\lambda' \in \mathbb{R}^{24} \text{ s.t. } \langle \lambda, \lambda' \rangle \in \mathbb{Z} \ \forall \lambda \in \Lambda\} = \Lambda$

• Λ has no roots i.e. no vectors λ w/ $\|\lambda\|^2 = 2$.

• $O(\Lambda) \subseteq O(\mathbb{R}^{24}) = C_0 = 2 C_1$ C_1 is sporadic simple.

Given even + self dual lattice, (1+1)d CFT

$\rightsquigarrow$ holomorphic bosonic

↑ no right movers

V_Λ $\sim$ EOM: holomorphic map from (1+1)d spacetime to $\mathbb{R}^{24}/\Lambda = T$

$c_L = \text{rank} = 24$

$c_R = 0$.

abelian WZW model.

• $T = \mathbb{R}^{24}/\Lambda$ acts by translations

• $O(\Lambda)$ acts by rotations.

∃ subtlety to lifting these classical symmetries to exact symmetries of non-split.

$\rightsquigarrow$ actually $\text{Aut}(V_\Lambda) \supseteq T \cdot O(\Lambda)$.

Some manifest automorphisms of V_Λ

for Leech, these are all the automorphisms

but for a general lattice, there are more

in particular: every root $\alpha \in \Lambda$ generates symmetry.

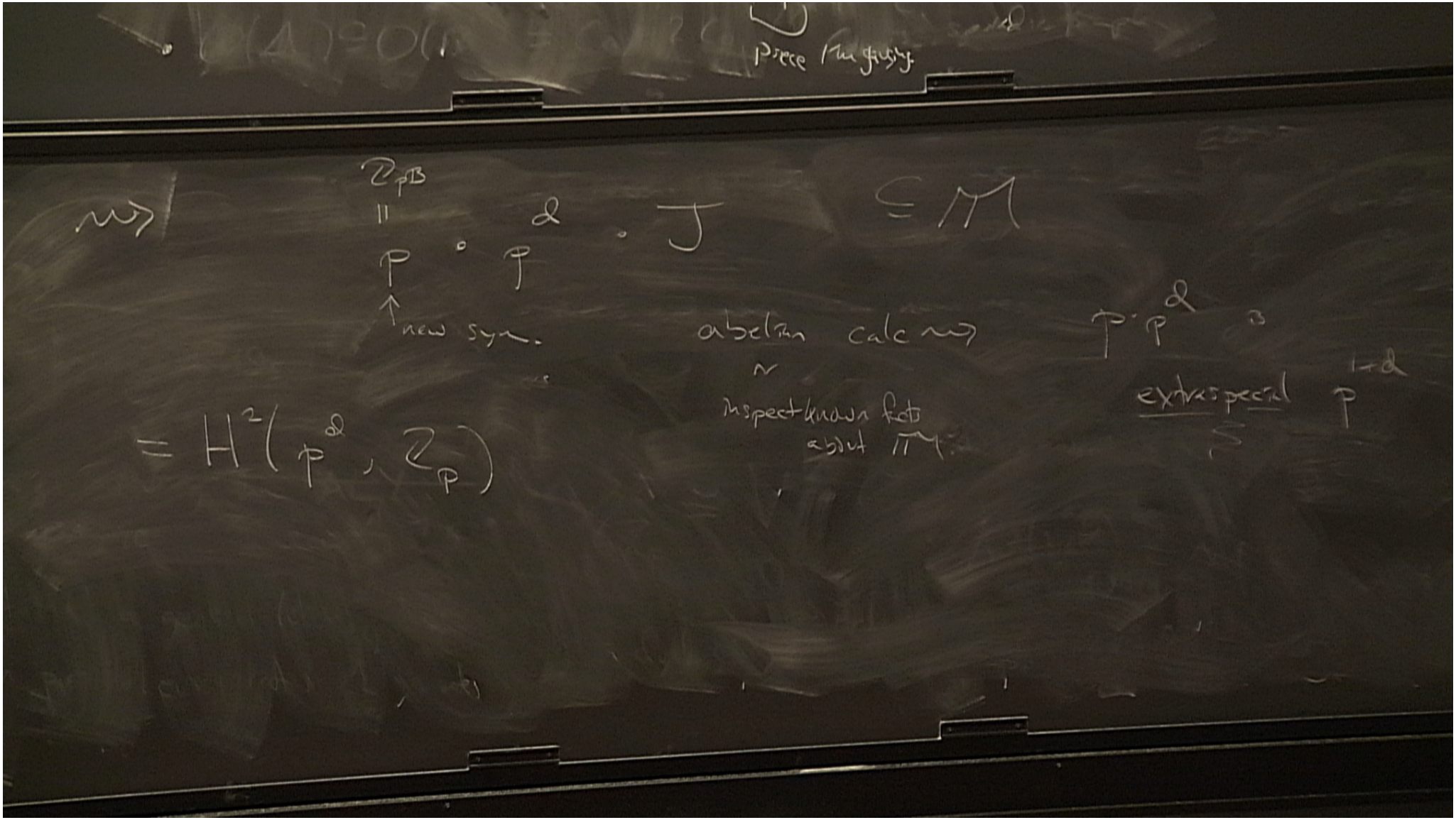
$$\#\mathbb{M} = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 13^{3}\cdot \underbrace{11^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71}_{\checkmark} \sim 10^{54}$$

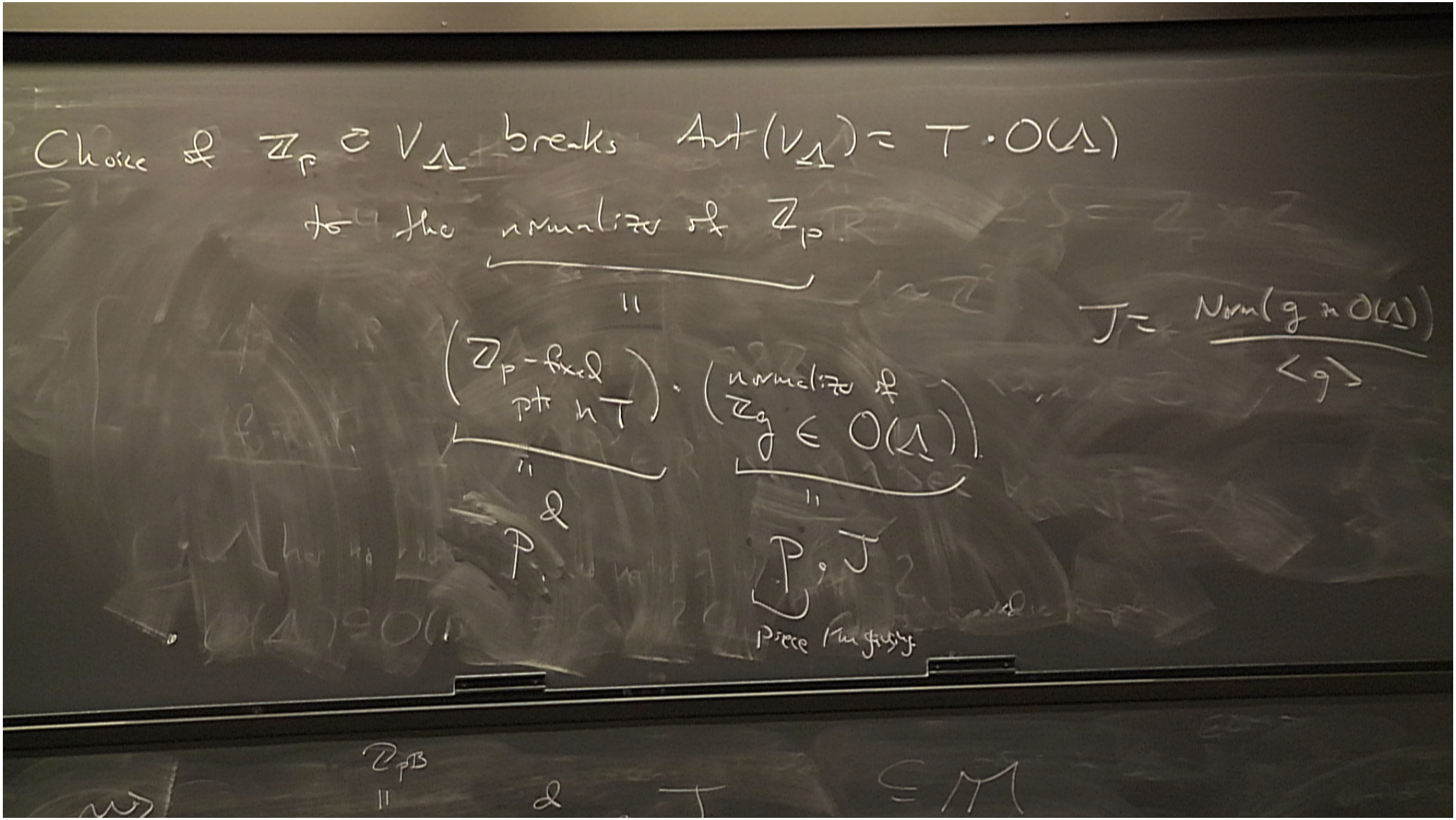
remark: $\in \#SL_{13}(\mathbb{F}_2)$.

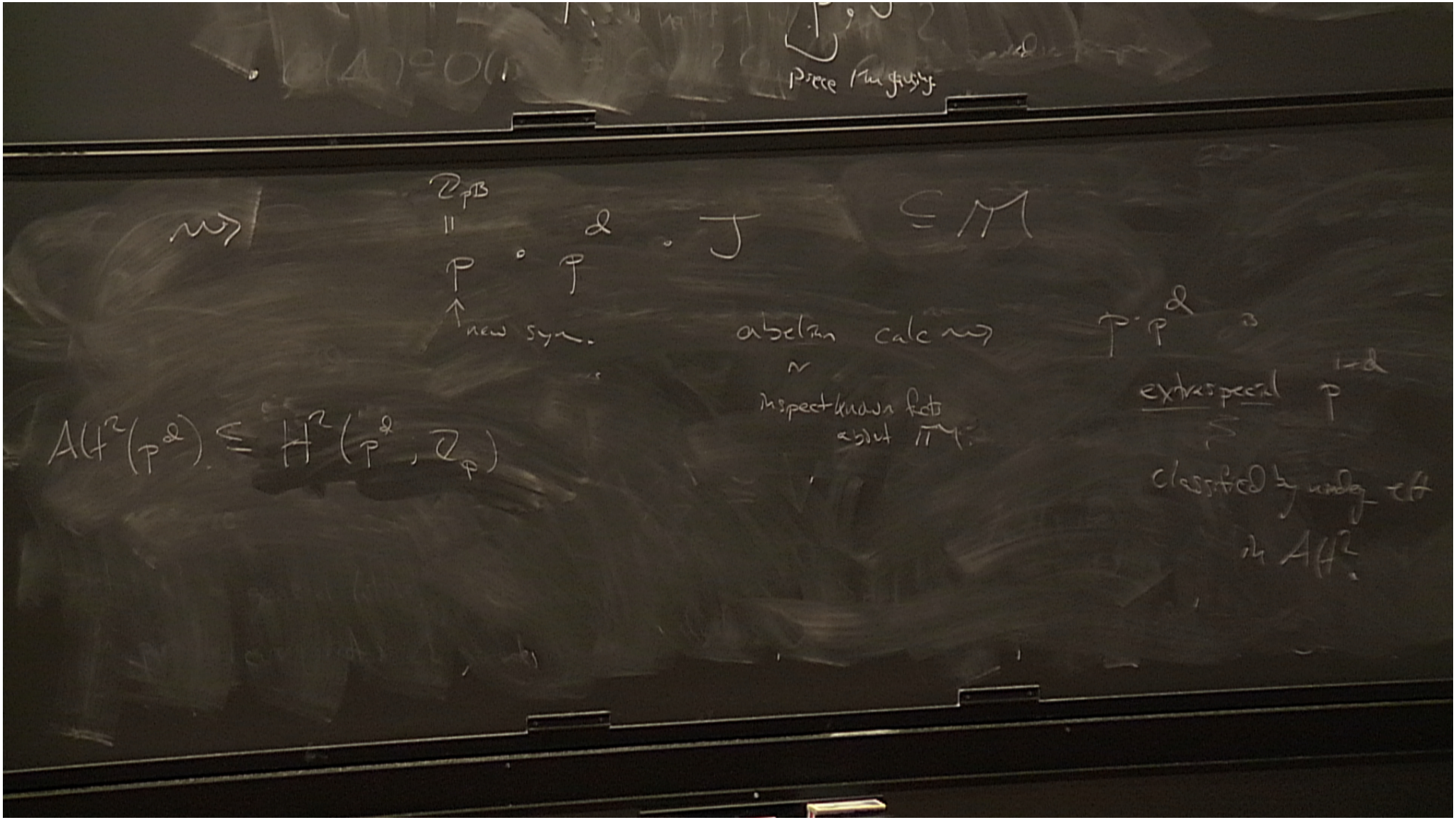
The primes $p = 2, 3, 5, 7, 13$ are special. (cf. Conway–Norton Monstrous moonshine conjectures)

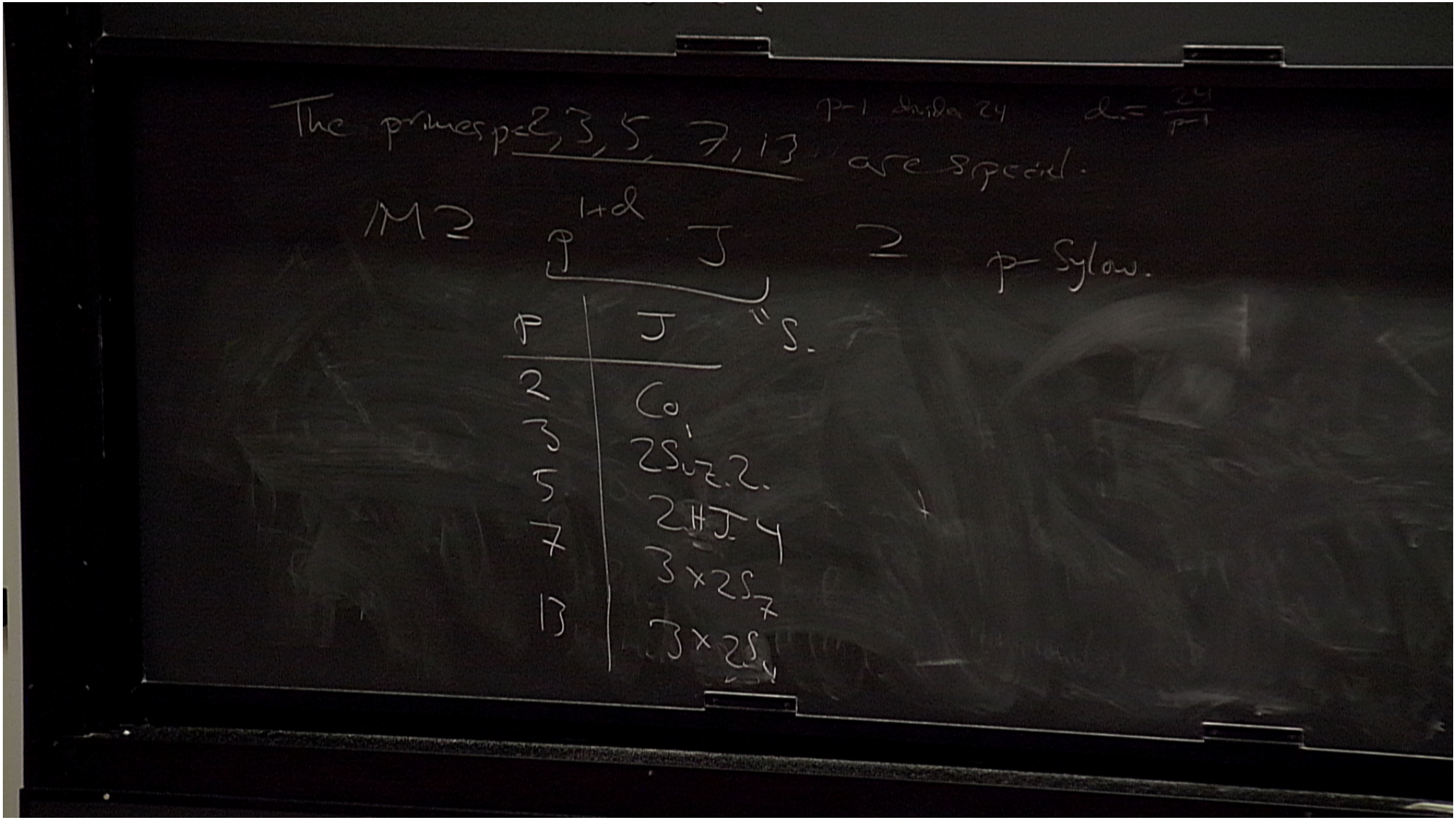
- $\exists\, g \in O(\Lambda)$ of order p s.t. $g \subset \mathbb{R}^{24}$ has no (nonzero) fixed pts
  Unique up to conjugacy.
- $\mathbb{Z}_p \subseteq O(\Lambda)$ max this $\mathbb{Z}_p$ lifts in a unique-up-to-conjugacy way to $\mathbb{Z}_p \subseteq \mathrm{Aut}(V_\Lambda)$.

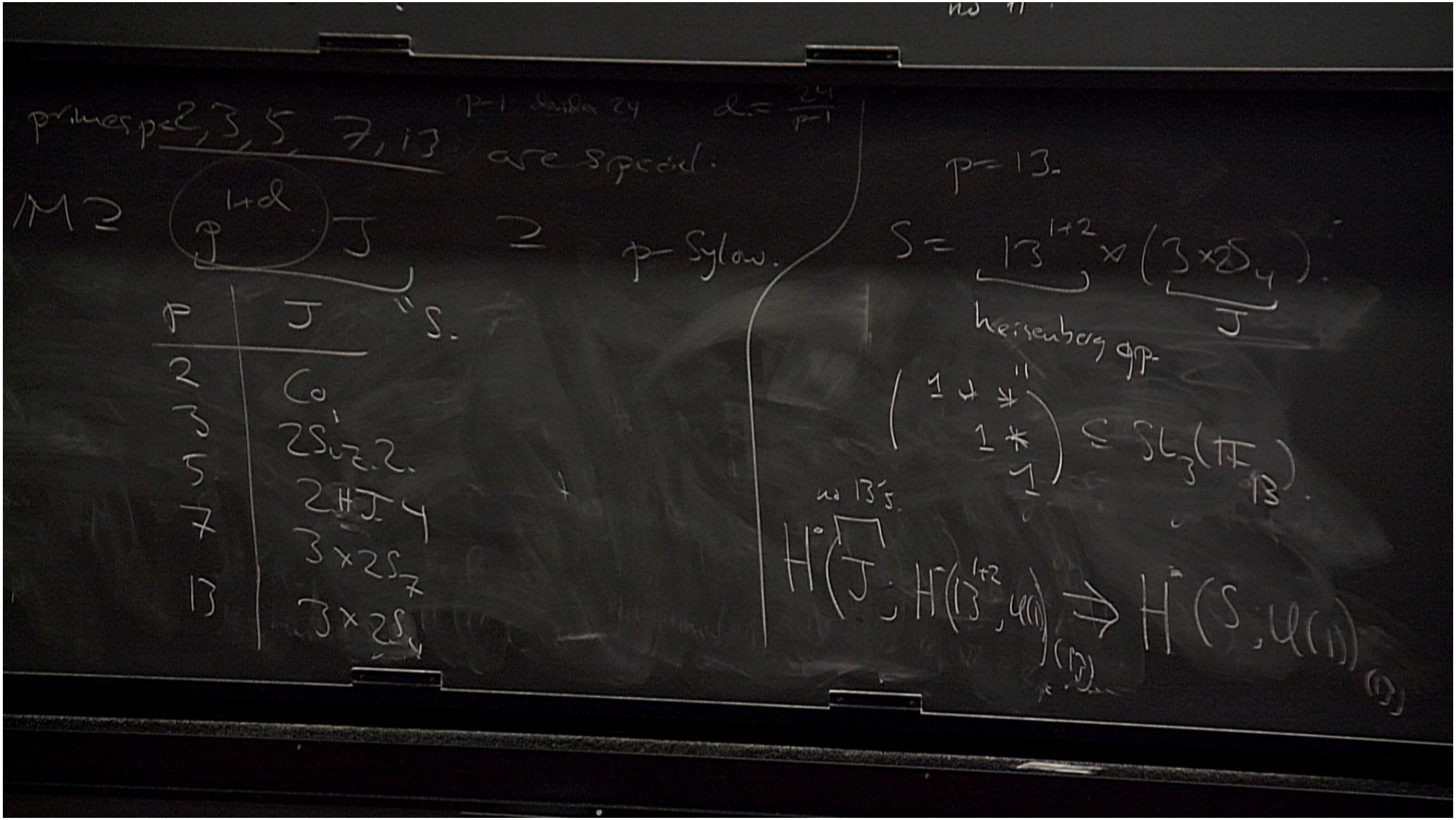
In fact, to a 1-dimensional dss auto.

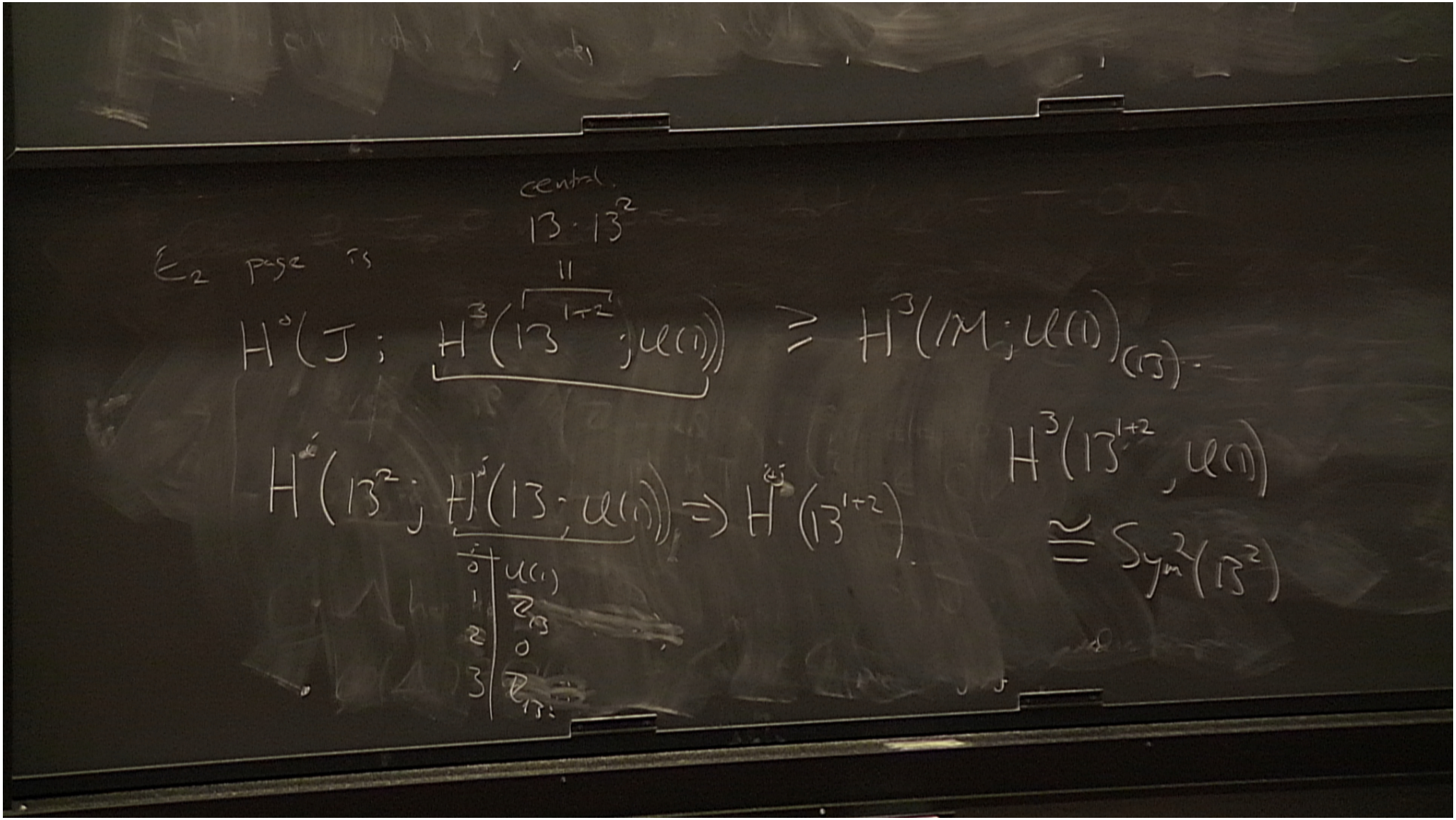

Piece rigidity
$\mathbb{Z}_{pB}$
$\cong$
$P \cdot P^{d} \cdot J \subseteq \Pi$
new sym.
abelian calc
$\sim$
inspect known fcts about Π
$P \cdot P^{d} \cong$
extraspecial P^{1+d}
$= H^2(P^d, \mathbb{Z}_p)$

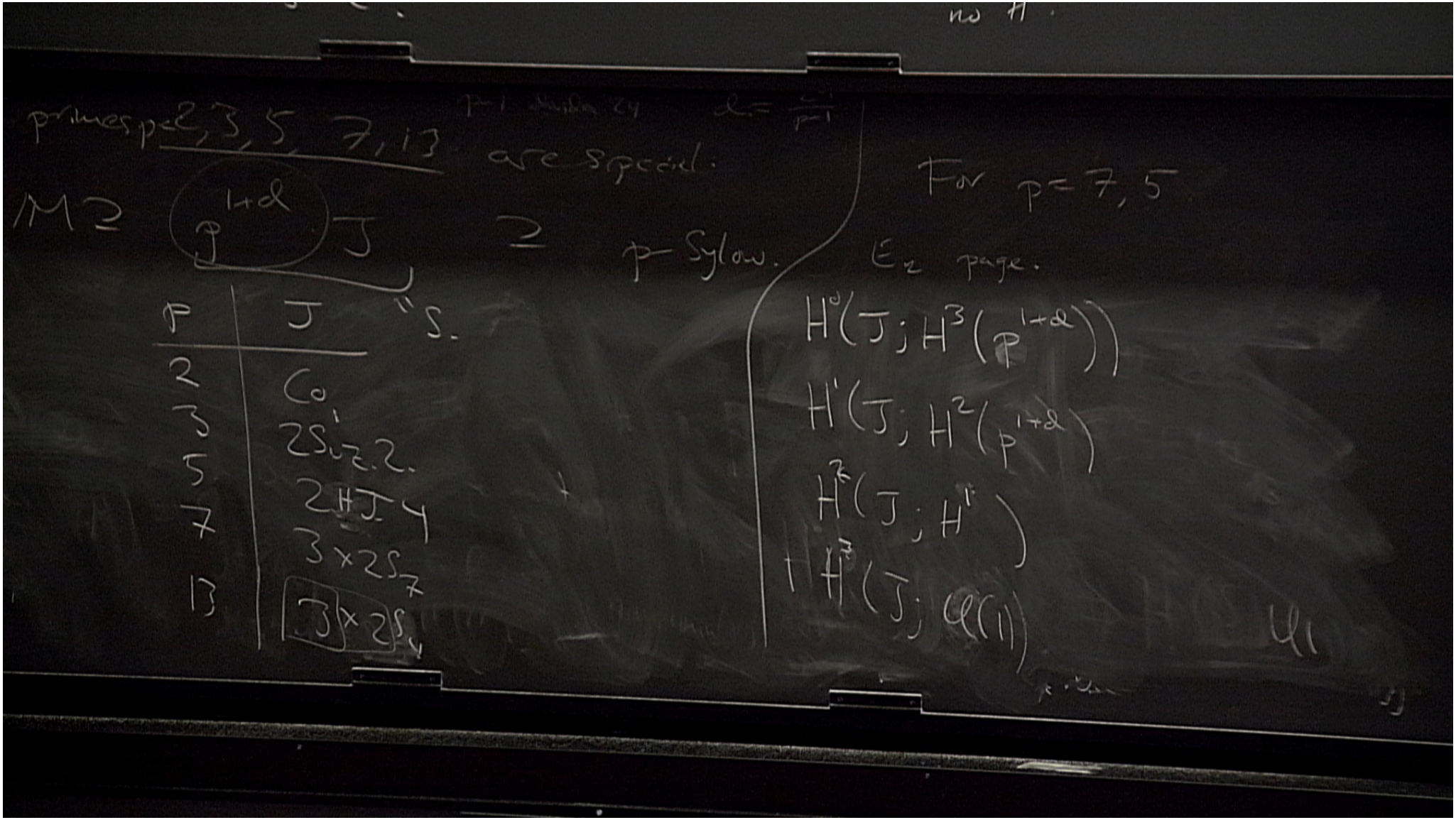

Choice of $\mathbb{Z}_p \subset V_\Lambda$ breaks $Aut(V_\Lambda) = T \cdot O(\Lambda)$
to the normalizer of $\mathbb{Z}_p$
$J = \frac{Norm(g \subset O(\Lambda))}{\langle g \rangle}$
$(\mathbb{Z}_p$-fixed pt in $T) \cdot (\text{normalizer of } \mathbb{Z}_g \in O(\Lambda))$
P & P, J
Piece that fixes

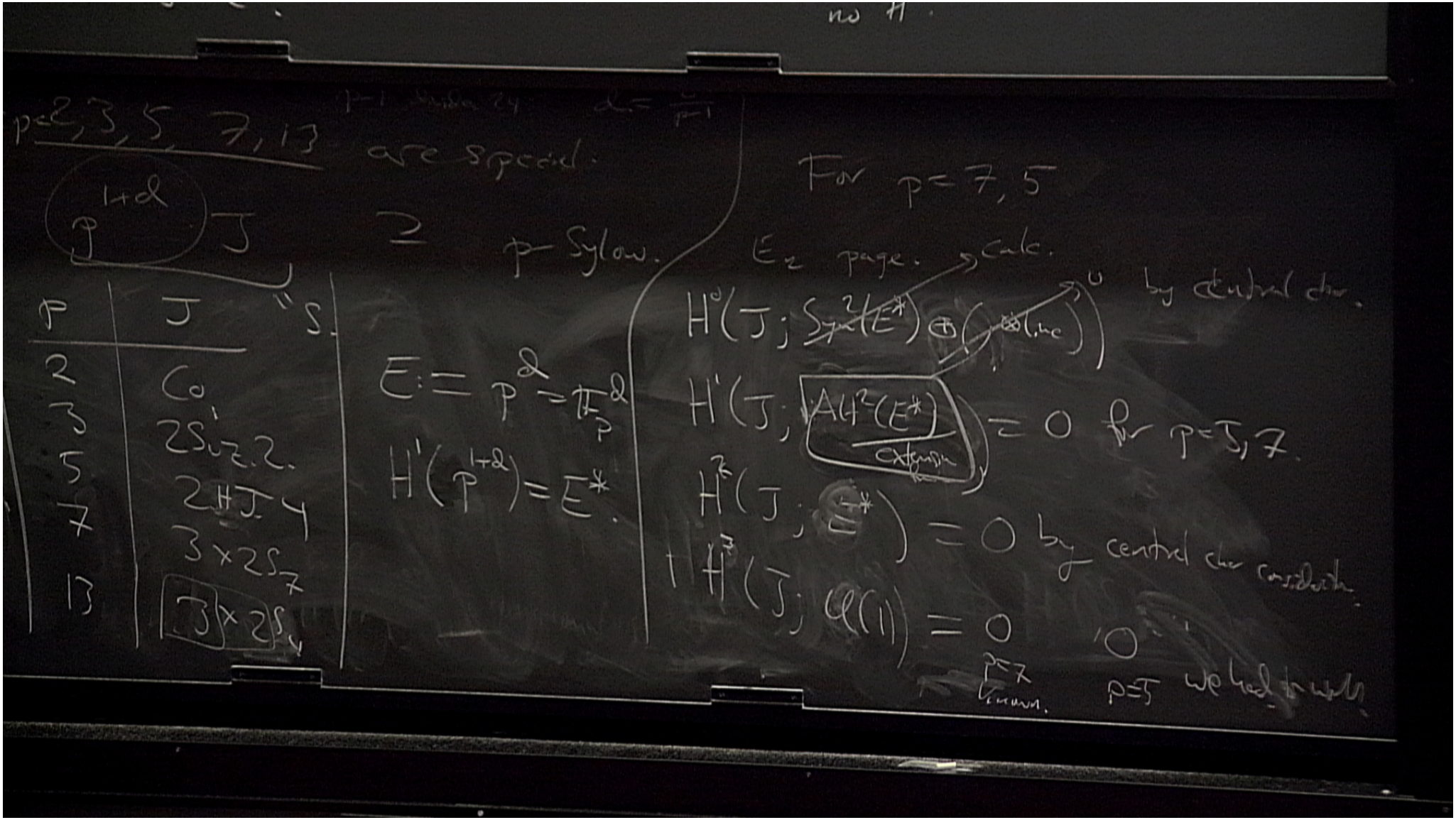

Piece I'm getting
new sym.
abelian calc
inspect known facts
about
extra speed
classified by


The primes p=2, 3, 5, 7, 13 are special.
p-1 divides 24 d = 24/(p-1)
M2
1+d
p J 2 p-Sylow.
"s.
p | J
2 | C0.
3 | 2S3 x 2.
5 | 2 # J. 4
7 | 3 x 2S7
13 | 3 x 2S4


primes p ≤ 2, 3, 5, 7, 13 are special.
p-1 divides 24
d = 24/(p-1)
M2
1+d
J
2
p-Sylow.
p | J
2 | C0
3 | 2S3 ≀ Z2
5 | 2HJ.4
7 | 3 × 2S7
13 | 3 × 2S4
p = 13.
S = 13^{1+2} ⋊ (3 × 2S4)
J
Heisenberg gp
(1 * *; 1 *; 1) ⊆ SL3(F13)
≅ 13'5.
H0(J; H(13^{1+2}, U(1))) ⇒ H(S; U(1))
(13)
(3)


central
$13 \cdot 13^2$
E_2 page is
$H^0(J; H^3(13^{1+2}; \mathcal{U}(1))) \cong H^3(M; \mathcal{U}(1))_{(13)}$
$H^3(13^{1+2}, \mathcal{U}(1))$
$\cong Sym^2(13^2)$
$H^i(13^2; H^j(13; \mathcal{U}(1))) \Rightarrow H^{i+j}(13^{1+2})$
i
0 $\mathcal{U}(1)$
1 $\mathbb{Z}_{13}$
2 0
3 $\mathbb{Z}_{13}$


primes $p \le 2, 3, 5, 7, 13$ are special.
M_2 $(p^{1+2}) J$ 2
p-Sylow.
For $p = 7, 5$
E_2 page.
$H^0(J; H^3(p^{1+2}))$
$H^1(J; H^2(p^{1+2}))$
$H^2(J; H^1)$
$H^3(J; \mathbb{Q}(1))$
p	J
2	Co_1
3	$2S_{12}.2$
5	$2HJ.4$
7	$3 \times 2S_7$
13	$3 \times 2S_4$


p=2, 3, 5, 7, 13
are special.
$(P^{1+2})J$
2 p-Sylow.
$E := P^2 = \Phi_P^2$
$H^1(P^{1+2}) = E^*$.
For p=7, 5
E_2 page. J-calc.
$H^0(J; \mathrm{Sym}^2(E^*) \otimes (\det)^{-1})$ by central char.
$H^1(J; \wedge^2(E^*)) = 0$ for p=5, 7.
extension
$H^2(J; E^*) = 0$ by central char condition.
$H^3(J; \mathbb{Q}(1)) = 0$
p=7 known. p=5 we had to work.
p | J
2 | C_3
3 | $2S_4 \cong 2.A_4$
5 | $2 \times J_2 . 4$
7 | $3 \times 2S_7$
13 | $3 \times 2S_4$

$\#SL_{13}(\mathbb{F}_2)$.

$\times \times \times \times$ done, no H^3.

$p = 2, 3, 5, 7, 13$ are special.

$\left(p^{1+2}\right) J$ ⊇ p-Sylow.

p	J
2	C_0
3	$2S_{U_2(2)}$
5	$2HJ.4$
7	$3 \times 2S_7$
13	$J \times 2S_4$

$E := p^2 = \mathbb{F}_p^2$

$H^1(p^{1+2}) = E^*$.

For $p = 7, 5$

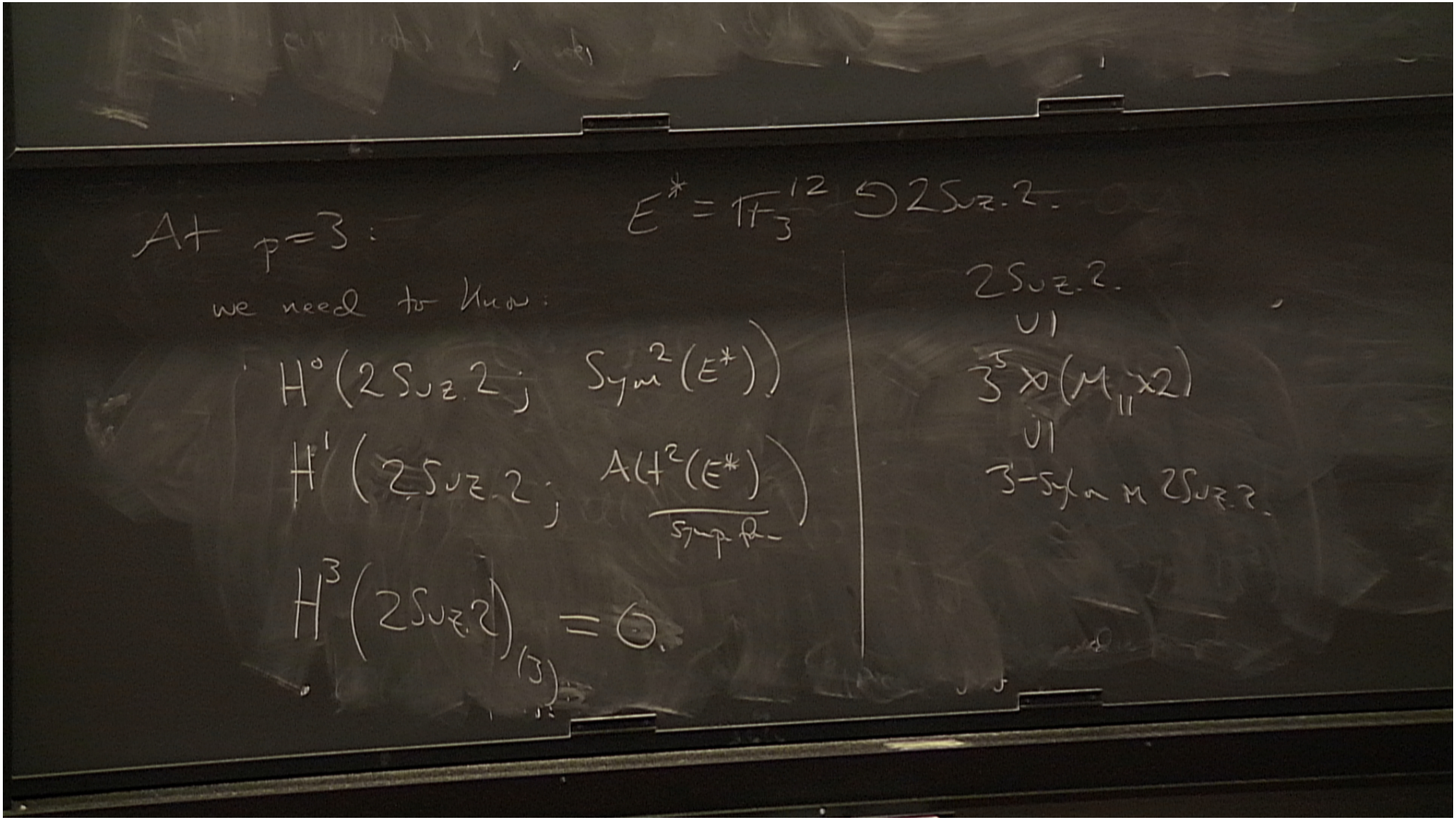
E_2 page. ⇒ Calc.

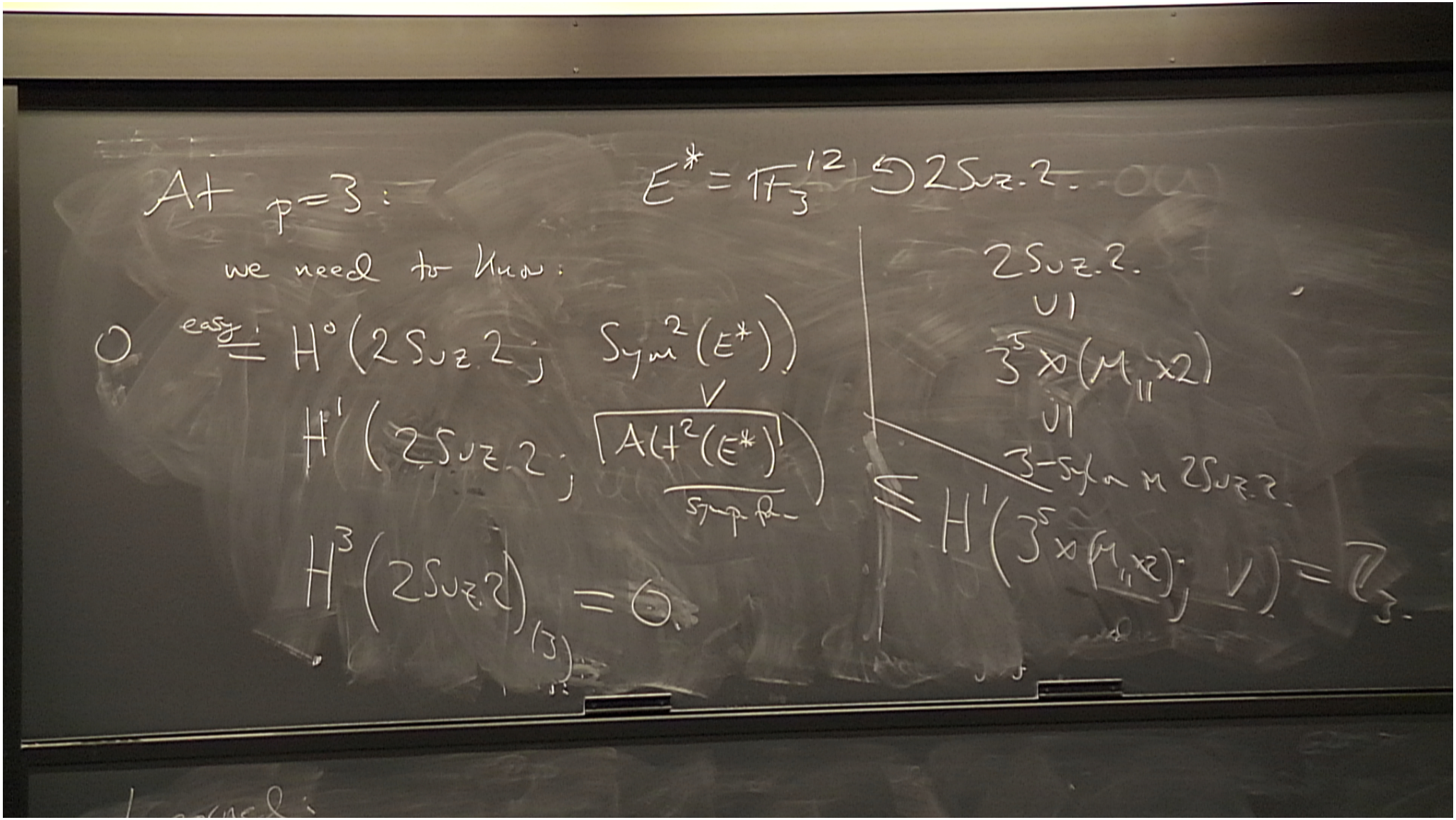
$H^0(J; \mathrm{Sym}^2(E^*) \otimes (\det)^{-1})$ by central char.

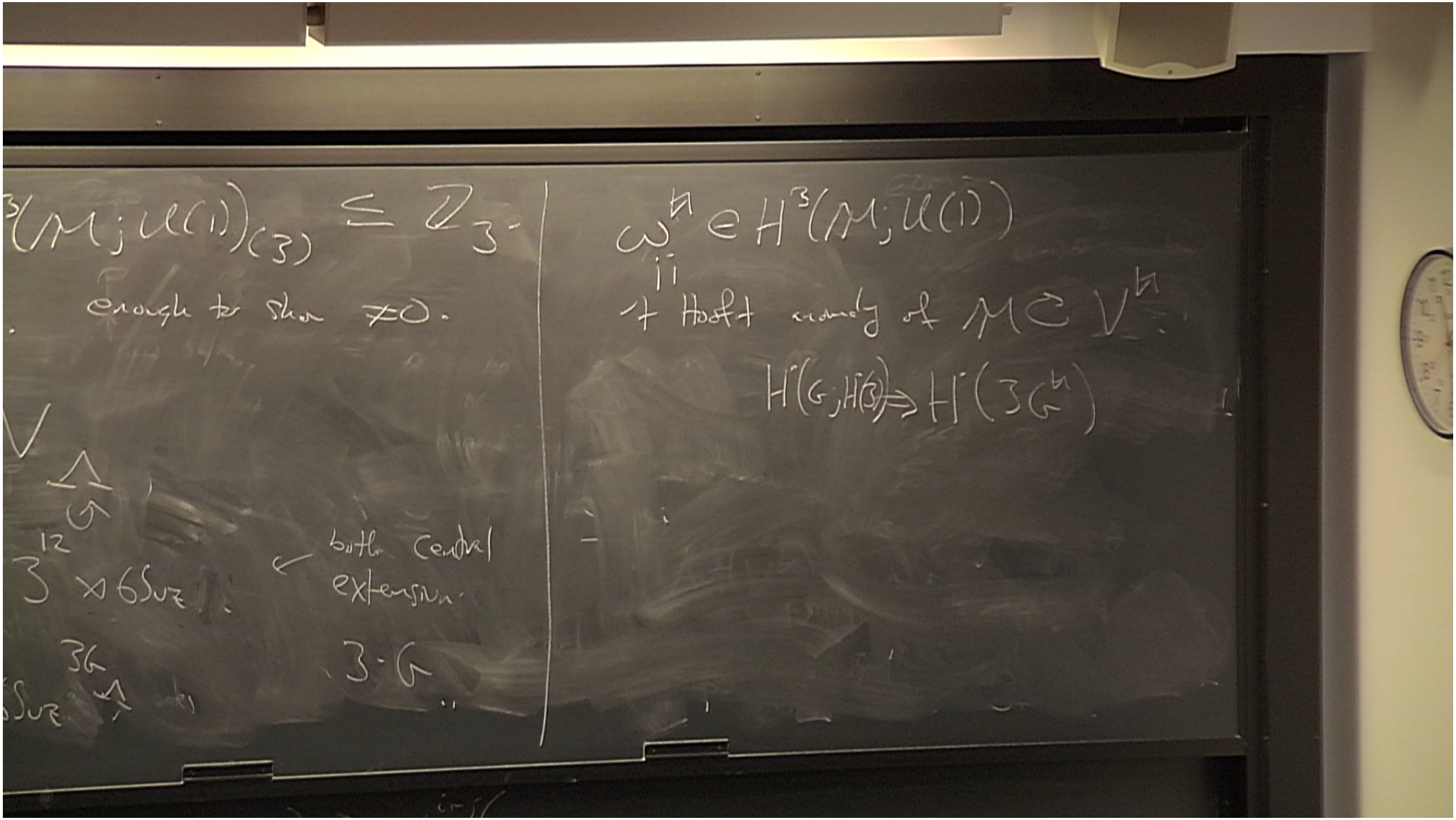
$H^1(J; \wedge^2(E^*)) = 0$ for $p = 5, 7$.

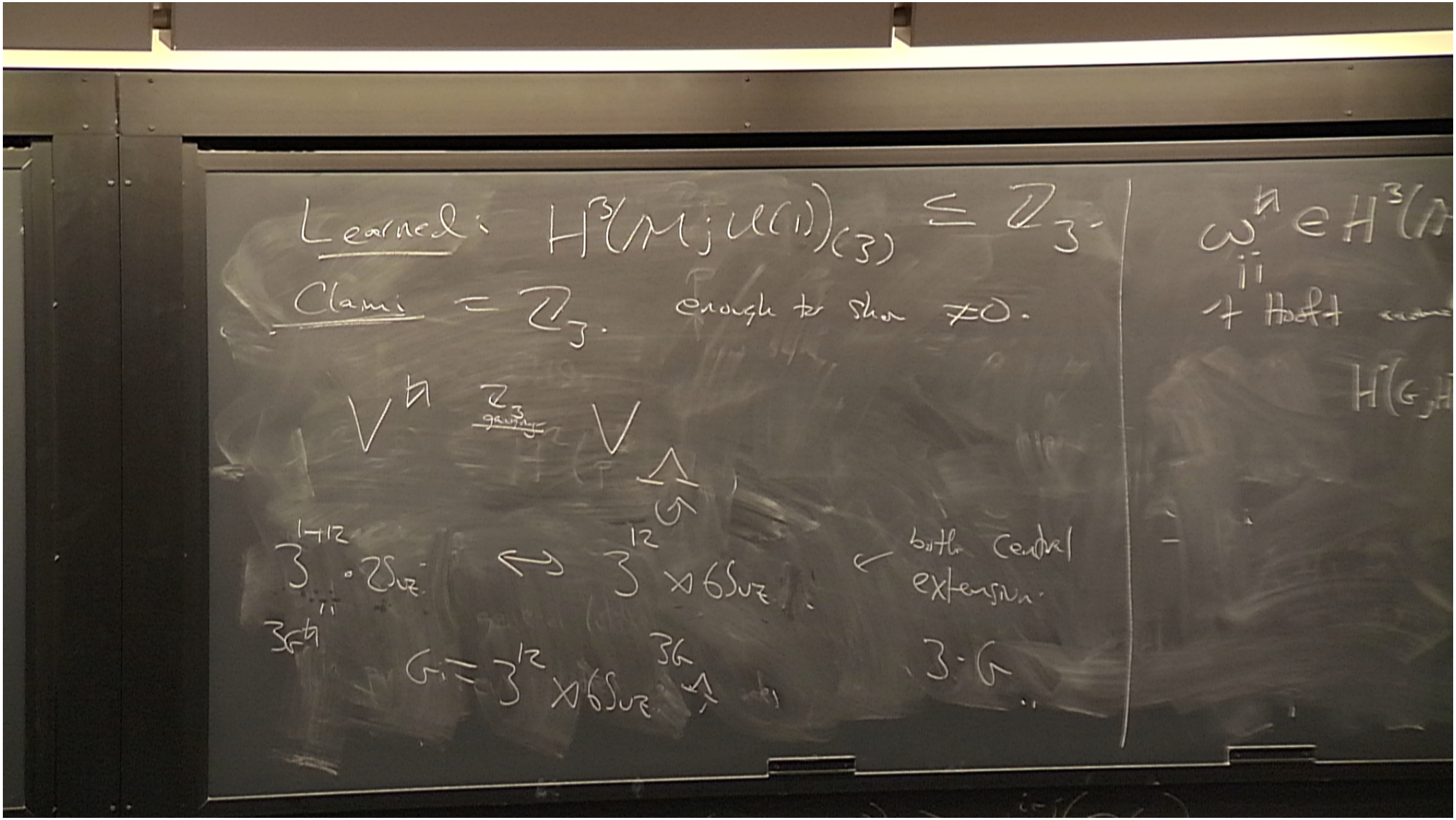
$H^2(J; E^*) = 0$ by central char considerations.

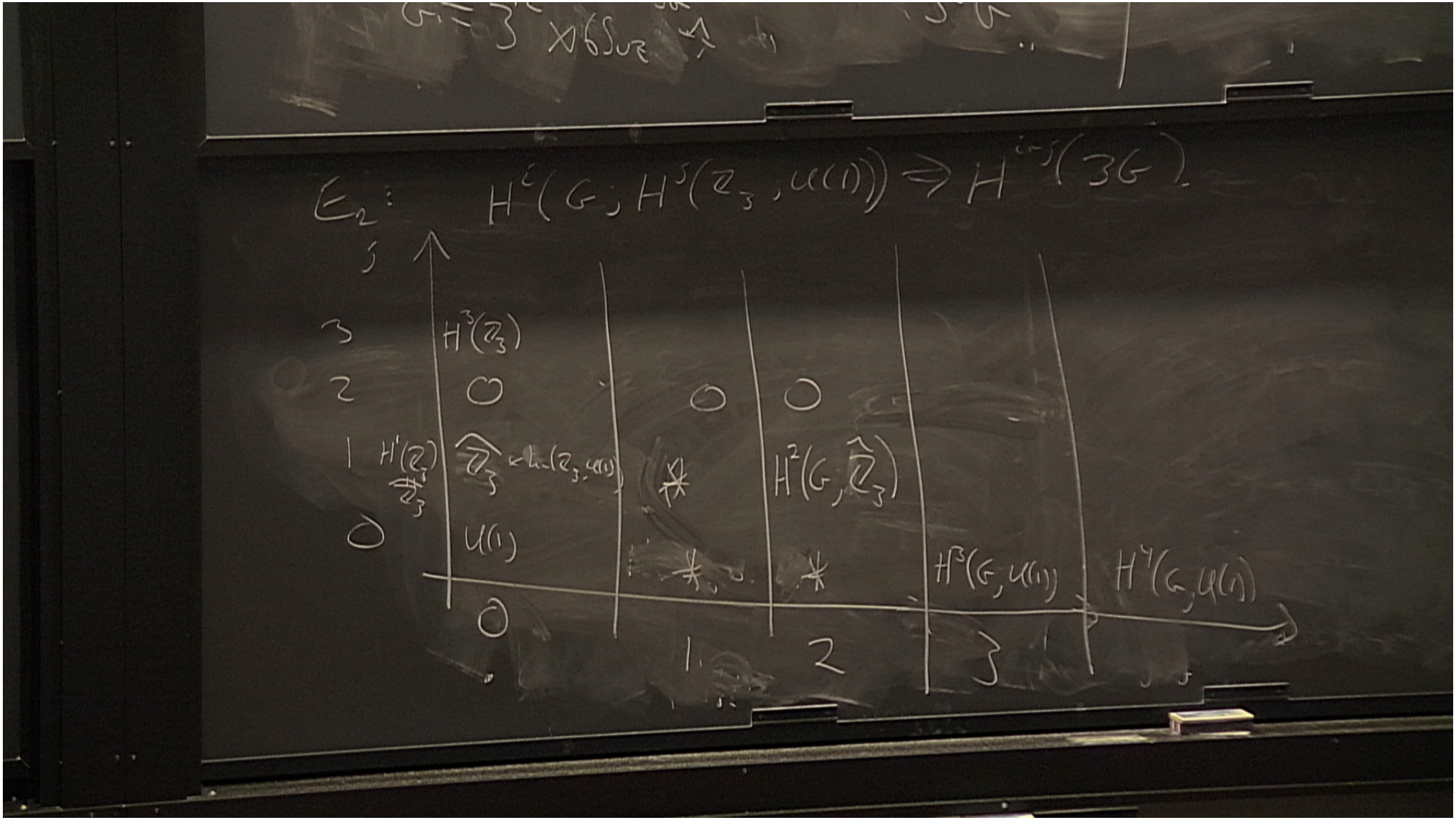
$H^3(J; \mathbb{Q}(1)) = 0$

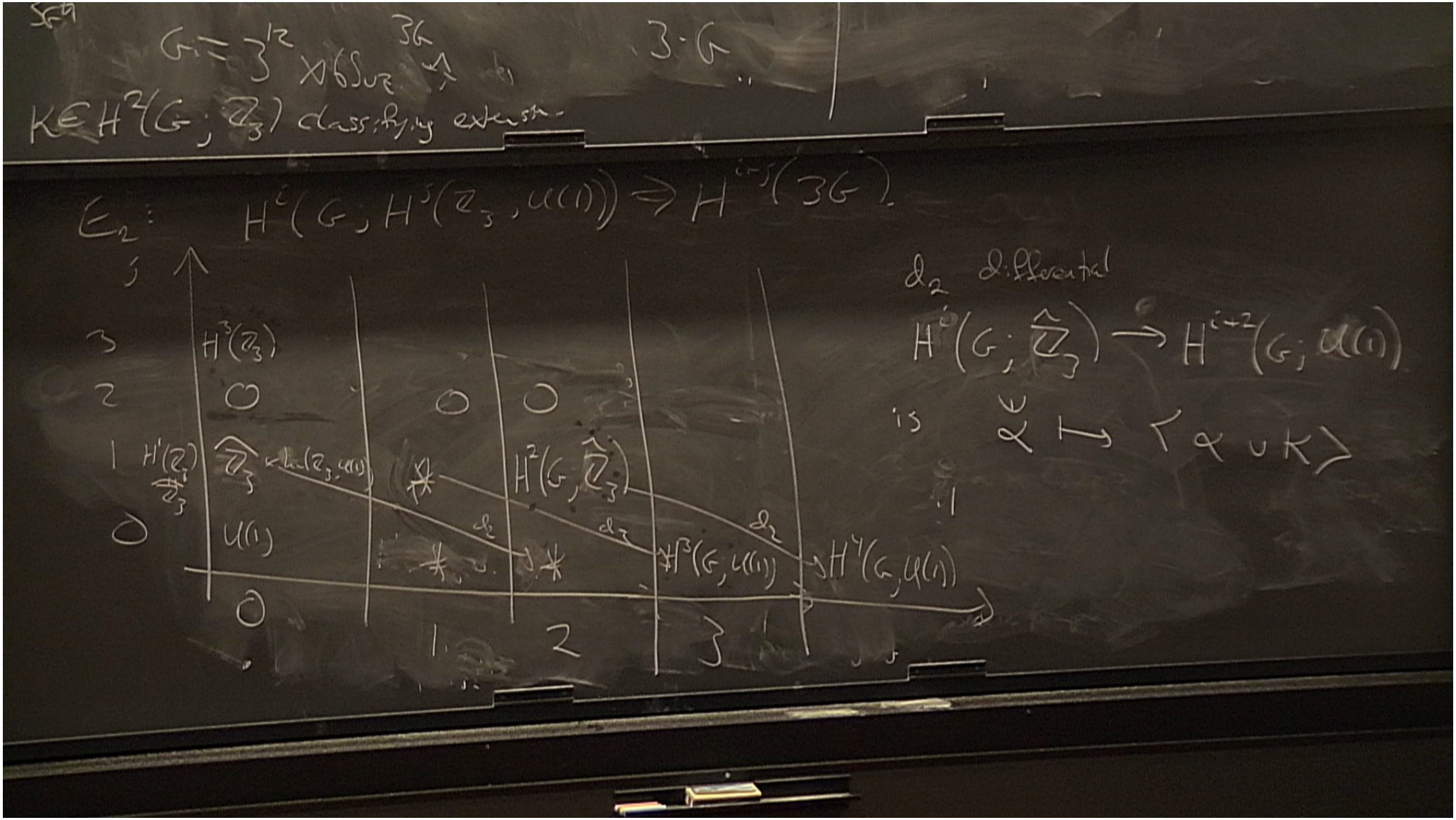

At p=3:
$E^* = \mathcal{T}_{\mathbb{F}_3}^{12} \ominus 2S_{UZ.2.}$
we need to know:
$H^0(2S_{UZ.2.};\ Sym^2(E^*))$
$H^1(2S_{UZ.2.};\ \underbrace{Alt^2(E^*)}_{Sym\ R.})$
$H^3(2S_{UZ.2.})_{(3)} = 0.$
$2S_{UZ.2.}$
U1
$3^5 \times (M_{11} \times 2)$
U1
3-Sylow of $2S_{UZ.2.}$

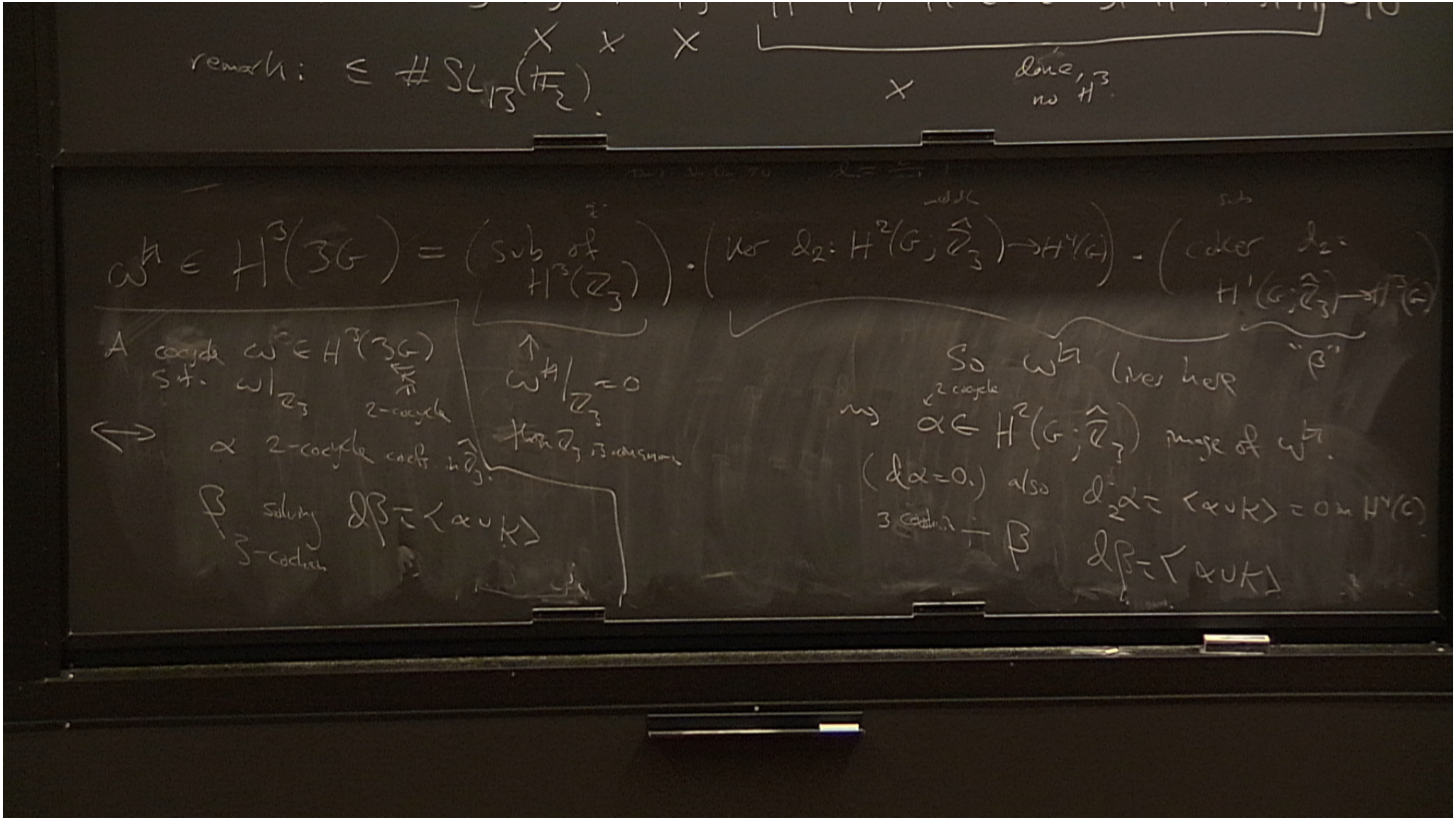

At p=3:
$E^* = TF_3^{12} \circlearrowleft 2S_{UZ.2.}$
we need to know:
$0 \overset{\text{easy}}{=} H^0(2S_{UZ.2.}; Sym^2(E^*))$
$H^1(2S_{UZ.2.}; \Lambda^2(E^*))$
Symp. Pr.
$H^3(2S_{UZ.2.})_{(3)} = 0.$
$2S_{UZ.2.}$
U1
$3^5 \times (M_{11} \times 2)$
U1
3-Sylow of $2S_{UZ.2.}$
$= H^1(3^5 \times (M_{11} \times 2); 1) = \mathbb{Z}_3.$

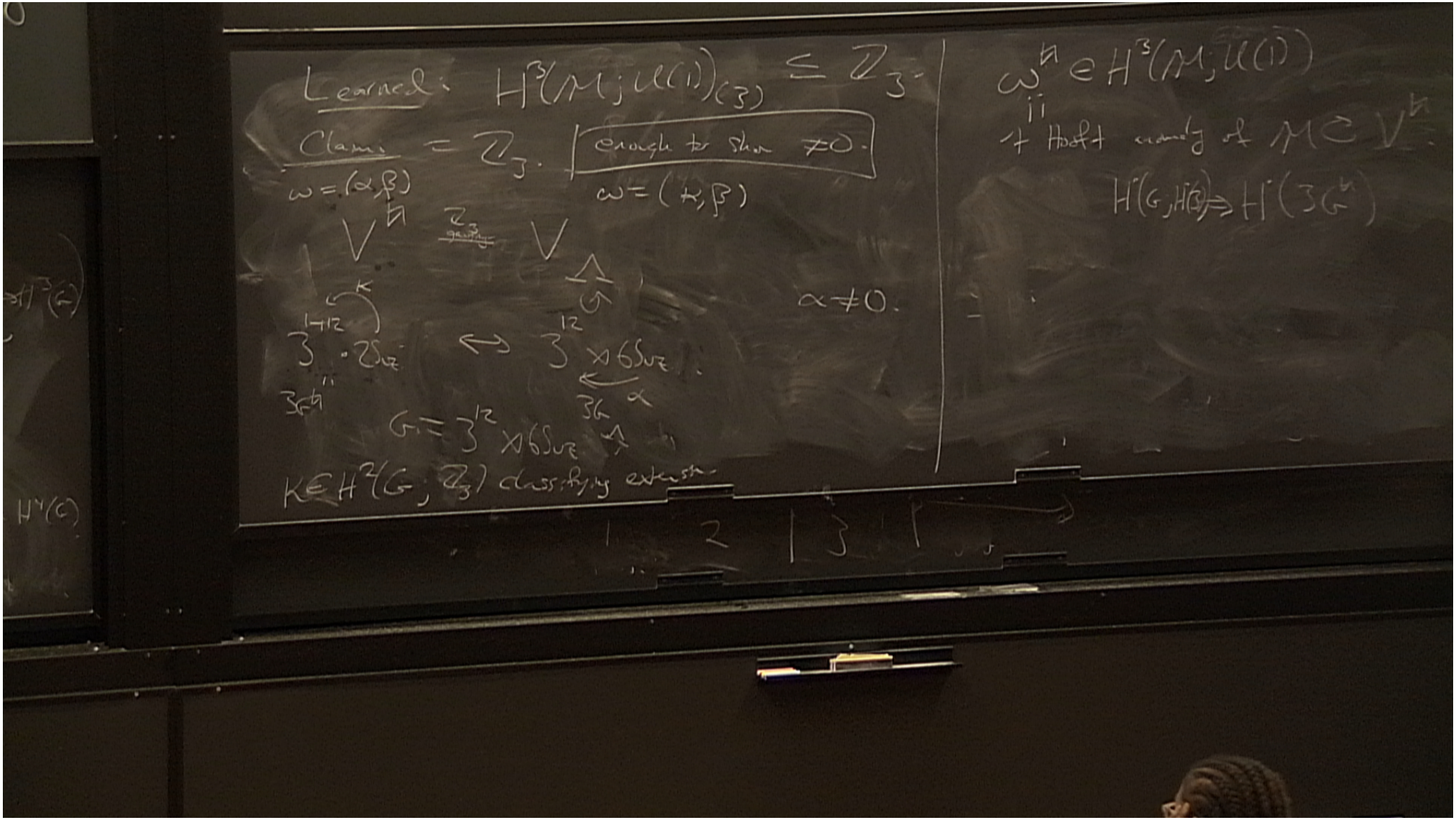

$H^3(M;U(1))_{(3)} \subseteq \mathbb{Z}_3$
enough to show $\neq 0$.
ω_{ij} $\in H^3(M;U(1))$
't Hooft anomaly of $M \subset V$
$H^*(G;H(\mathbb{Z}))$
$H^*(3G)$
V
G
12
$3 \times 6S_{UZ}$
both central extensions
$3 \cdot G$
$3G$
$6S_{UZ}$

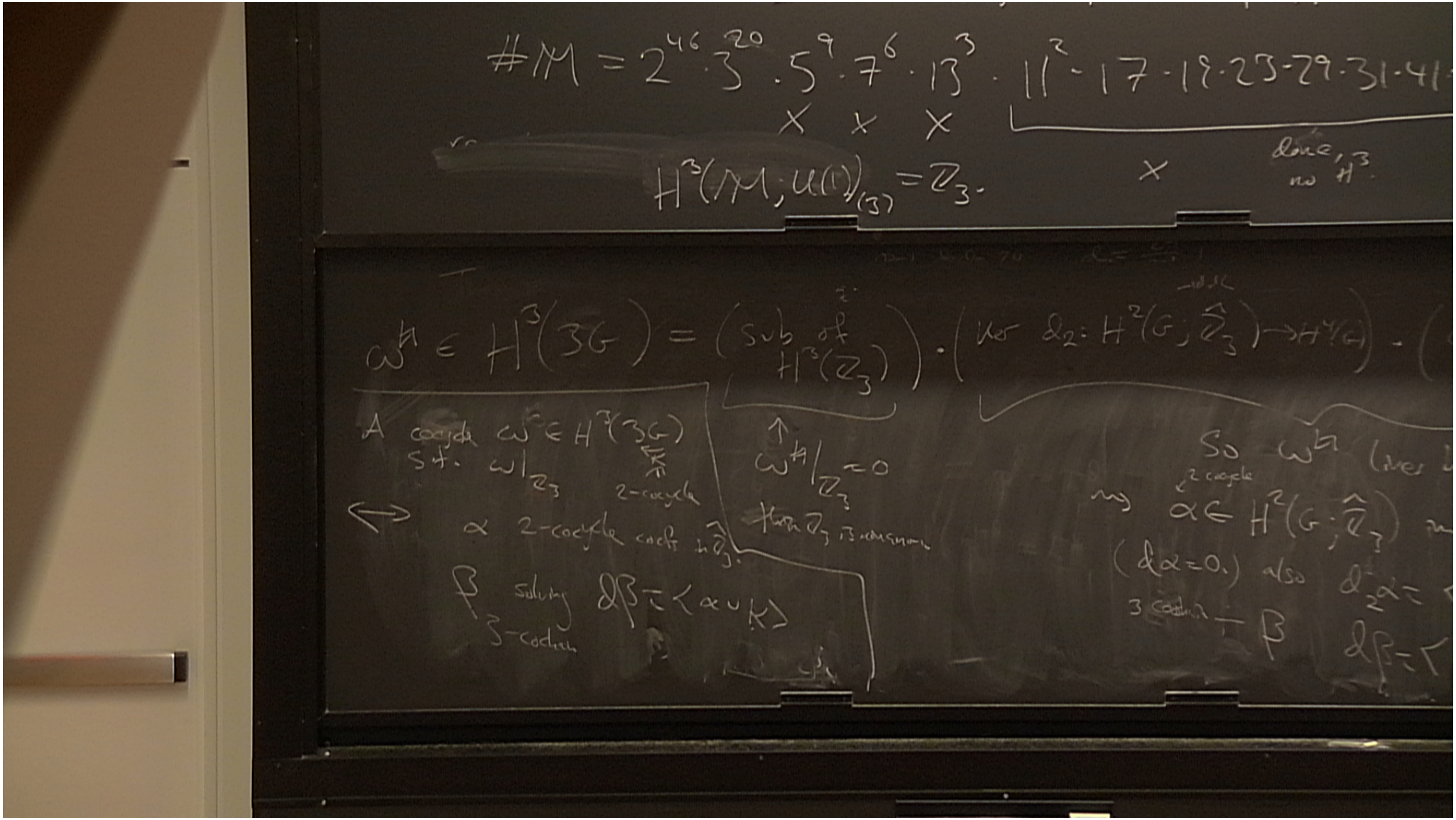

Learned: $H^3(M;U(1))_{(3)} \subseteq \mathbb{Z}_3$.
Claim: $= \mathbb{Z}_3$. enough to show $\neq 0$.
V $\mathbb{Z}_3$ gauging V
G
$3^{1+2}_{+} \cdot 2S_{U_2}$ $\leftrightarrow$ $3^{1+2}_{+} \rtimes 6S_{U_2}$ ← both central extensions
$3G$
$G = 3^{1+2}_{+} \rtimes 6S_{U_2}$ $3G$ $3 \cdot G$
$\omega_3 \in H^3(M$
't Hooft
$H(G,$

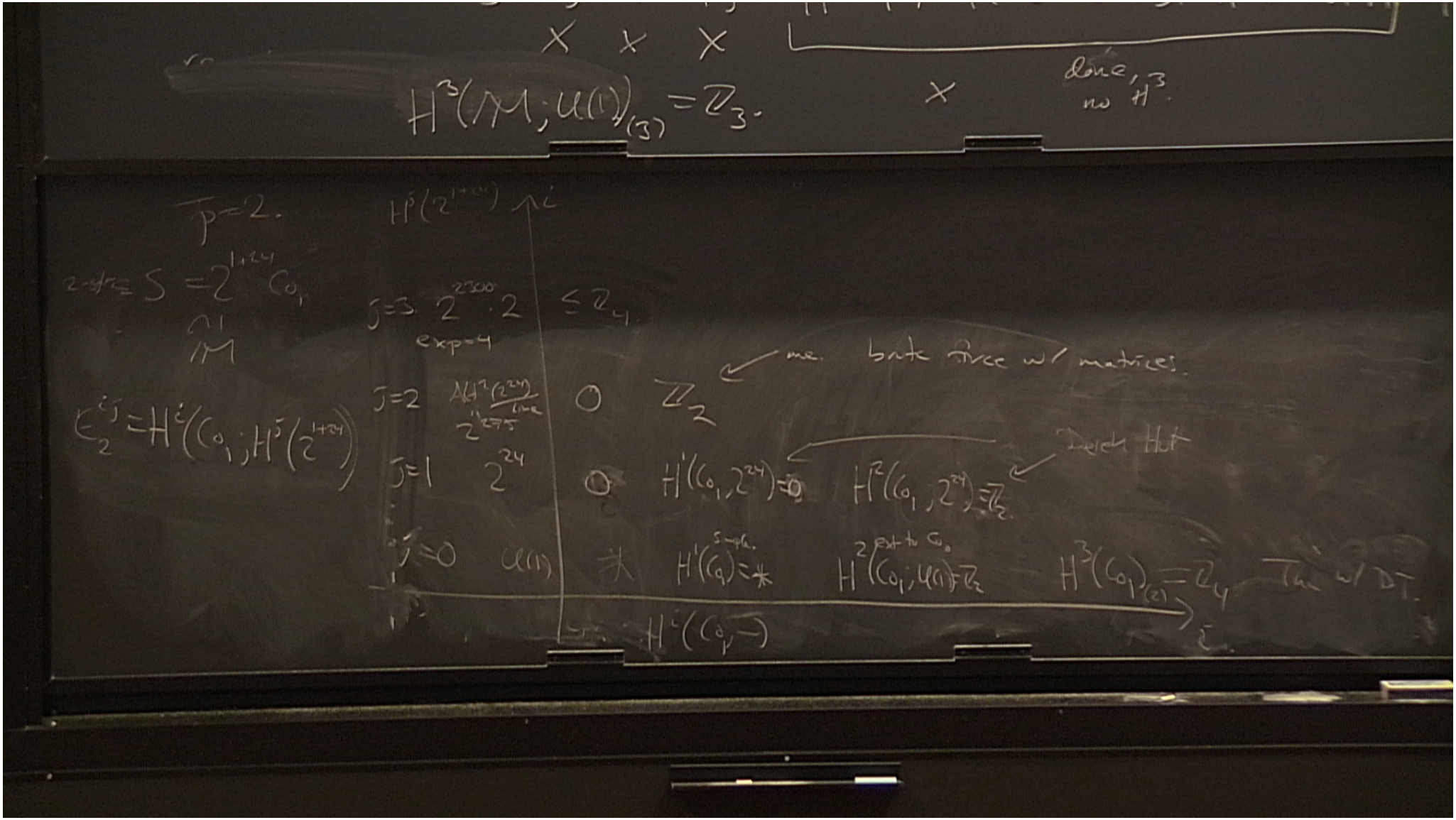

E_2: $H^i(G; H^j(\mathbb{Z}_3, U(1))) \Rightarrow H^{i+j}(3G)$
j
3
2
1
0
$H^3(\mathbb{Z}_3)$
0
$H^1(\mathbb{Z}_3; \mathbb{Z}_3)$
$\mathbb{Z}_3$
$U(1)$
0
0
0
$H^2(G, \mathbb{Z}_3)$
$H^3(G, U(1))$
$H^4(G, U(1))$
1
2
3

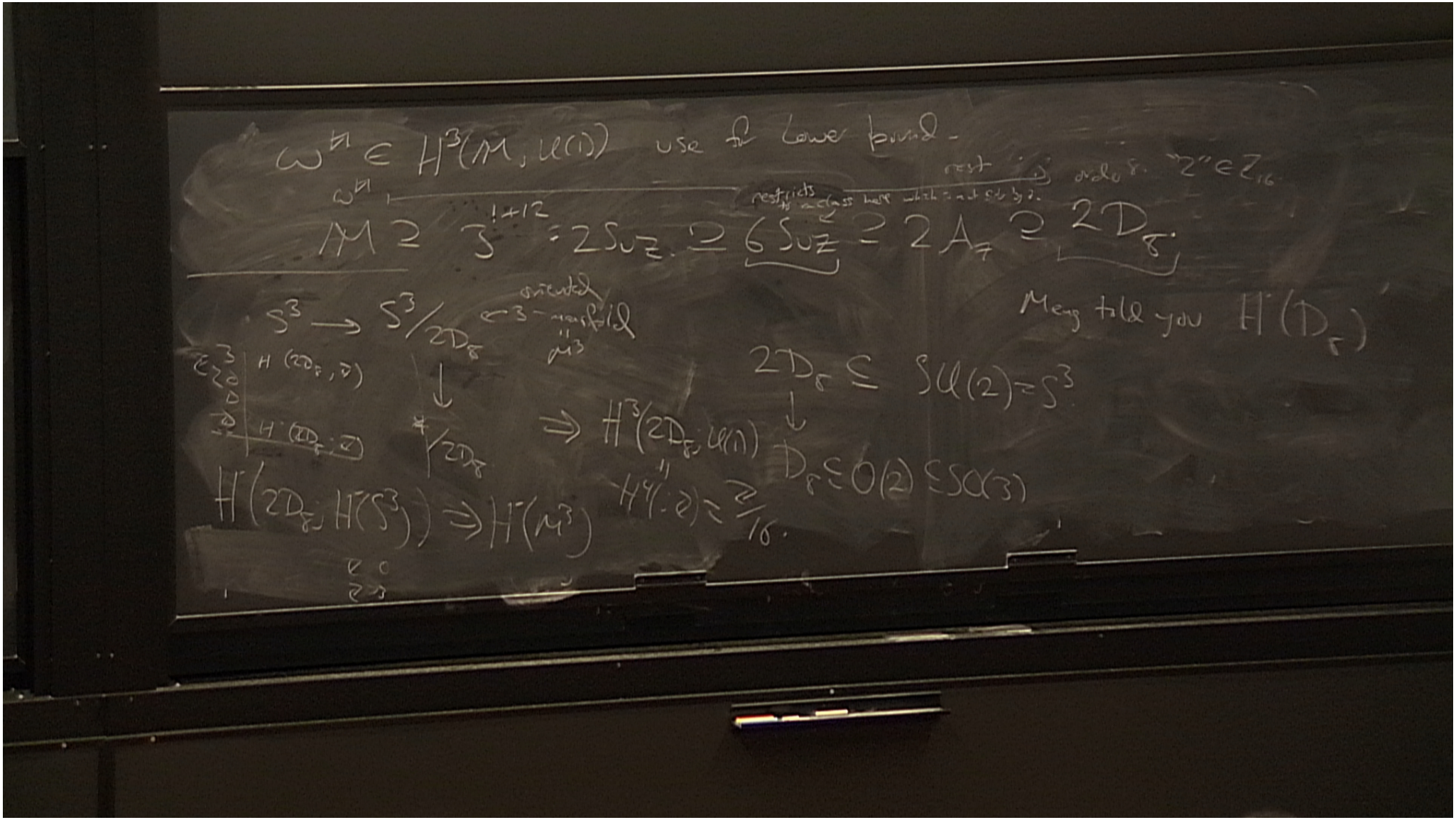

$G = 3^{12} \times SU(2)$
$3G$
$3 \cdot G$
$K \in H^2(G; \mathbb{Z}_3)$ classifying extension
E_2: $H^i(G, H^j(\mathbb{Z}_3, U(1))) \Rightarrow H^{i+j}(3G)$
d_2 differential
$H^i(G; \hat{\mathbb{Z}}_3) \to H^{i+2}(G; U(1))$
is $\chi \mapsto \langle \chi \cup K \rangle$
$H^3(\mathbb{Z}_3)$
$H^1(\mathbb{Z}_3, \mathbb{Z}_3)$
$\mathbb{Z}_3$
$U(1)$
$H^2(G, \hat{\mathbb{Z}}_3)$
$H^3(G, U(1))$
$H^4(G, U(1))$
d_2
d_3
d_2


remark: $\in \#SL_{13}(\mathbb{F}_2)$.
done, no H^3.
$\omega^4 \in H^3(BG) = (\text{sub of } H^3(\mathbb{Z}_3)) \cdot (\ker d_2: H^2(G;\hat{\mathbb{Z}}_3) \to H^4(G)) \cdot (\text{coker } d_2: H^1(G;\hat{\mathbb{Z}}_3) \to H^3(G))$
"β"
A cocycle $\omega \in H^3(BG)$ s.t. $\omega|_{\mathbb{Z}_3}$ 2-cocycle
$\omega^4|_{\mathbb{Z}_3} = 0$
α 2-cocycle coeff in $\hat{\mathbb{Z}}_3$.
β solving $d\beta = \langle \alpha \cup \alpha \rangle$
3-cochain
So ω^4 lives here
2 cocycle
$\alpha \in H^2(G;\hat{\mathbb{Z}}_3)$ image of ω^4.
$(d\alpha = 0.)$ also $d_2\alpha = \langle \alpha \cup \alpha \rangle = 0$ in $H^4(G)$
3 cochain β $d\beta = \langle \alpha \cup \alpha \rangle$


Learned: $H^3(M;U(1))_{(3)} \subseteq \mathbb{Z}_3$
Claim: $= \mathbb{Z}_3$.
enough to show $\neq 0$.
$\omega = (\alpha, \beta)$
$\omega = (\kappa, \beta)$
$\omega \in H^3(M;U(1))$
$\alpha \neq 0$.
$K \in H^2(G, \mathbb{Z}_3)$ classifying extension


$\#M = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 13^{3} \cdot 11^{2} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41$
$H^3(M; U(1))_{(3)} = \mathbb{Z}_3$.
done, no H^3.
$\omega^4 \in H^3(BG) = (\text{sub of } H^3(\mathbb{Z}_3)) \cdot (\ker d_2 : H^2(G; \hat{\mathbb{Z}}_3) \to H^4(G))$
A cocycle $\omega^4 \in H^3(BG)$ s.t. $\omega|_{\mathbb{Z}_3}$ 2-cocycle
$\omega^4|_{\mathbb{Z}_3} = 0$
α 2-cocycle coeffs in $\hat{\mathbb{Z}}_3$.
β solving $d\beta = \langle \alpha \cup K \rangle$
3-cochain
So ω^4 lives
2 cocycle
$\alpha \in H^2(G; \hat{\mathbb{Z}}_3)$
$(d\alpha = 0,$ also $d_2^2 \alpha =$
3 cochain
β
$d\beta =$


$H^3(M; U(1))_{(3)} = \mathbb{Z}_3$.
done, no H^3.
$p=2$.
$H^i(2^{1+24})$
$2\text{-str} = S = 2^{1+24} \cdot Co_1$
M
$J=3$ $2^{2300} \cdot 2$ exp=4 $\leq \mathbb{Z}_4$
$J=2$ $\frac{Alt^2(2^{24})}{\text{Lie}}$ 2^{1+275} 0 $\mathbb{Z}_2$
$J=1$ 2^{24} 0 $H^1(Co_1, 2^{24}) = 0$ $H^2(Co_1, 2^{24}) = \mathbb{Z}_2$
$J=0$ $U(1)$ $*$ $H^1(Co_1) = *$ $H^2(Co_1, U(1)) = \mathbb{Z}_2$ $H^3(Co_1)_{(2)} = \mathbb{Z}_4$
$E_2^{i,j} = H^i(Co_1; H^j(2^{1+24}))$
$H^i(Co_1, -)$
$\mathbb{Z}_2$
me. brute force w/ matrices.
Leech Hut
5-pts.
ext to Co_1
Thm w/ DT.


$\omega^4 \in H^3(M, U(1))$ use for lower bound.
ω^4
restricts to a class here which is not 5th by 2.
rest
$M \cong 3^{1+12} : 2Suz \cong 6Suz \cong 2A_7 \cong 2D_5$
$2^n \in Z_{16}$
$S^3 \to S^3/2D_8$
3-manifold
oriented
Meng told you $H^*(D_8)$
$2D_8 \subseteq SU(2) = S^3$
$H^3(2D_8, U(1))$
$D_8 \subseteq O(2) \subseteq SO(3)$
$H^4(\cdot, \mathbb{Z}) \cong \mathbb{Z}/16$.
$H^*(2D_8; H^*(S^3)) \Rightarrow H^*(M^3)$

$\mathbb{M}$ = Fischer-Griess monster is the largest sporadic simple gp.

$$\#\mathbb{M} = 2^{46}\cdot 3^{20}\cdot 5^{9}\cdot 7^{6}\cdot 13^{3}\cdot 11^{2}\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \sim 10^{54}$$

$\times \quad \times \quad \times \qquad\qquad \times$ done, no H^3

$H^3(\mathbb{M};U(1)) = \mathbb{Z}_8 \oplus$ [3] $H^3(\mathbb{M};U(1))_{(3)} = \mathbb{Z}_3$.

$\mathbb{M}$

$E_2^{i,j} = H^i(C_0; H^j(2^{1+24}))$

$j=2$ $\mathbb{Z}/2$ $\quad 0 \quad \mathbb{Z}/2$

$j=1$ 2^{24} $\quad 0 \quad H^1(C_0, 2^{24}) = 0 \quad \neq H^2(C_0, 2^{24}) = \mathbb{Z}_2$

$j=0$ $U(1)$ $\quad \ast \quad H^1(G) = \ast \quad H^2(C_0; U(1)) = \mathbb{Z}_2 \quad H^3(C_0)_{(2)} = \mathbb{Z}_4$

$H^2(C_0, -)$

exp 4

Adj $\frac{(2^{24})}{\text{Leech}}$ 2^{276}

Leech Hit

$\Rightarrow \delta\omega^3 = 0$

$\mathbb{Z}\omega^3$ lifts C_0 is pulled back from C_0; C_0 not D_6 injects H^3

The w/ DT.

1707.08388 + 1810.00463
Z/2 w/ D. Treumann.

Goal: Thm. $H^3(\mathbb{M}, U(1)) \cong \mathbb{Z}_{24} \oplus (\leq 4)$
(prob trivial)

$\mathbb{M}$ = Fischer-Griess monster is the largest sporadic simple gp.

$$\#\mathbb{M} = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 13^3 \cdot \underbrace{11^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71}_{\text{done, no } H^3} \sim 10^{54}$$

$H^3(\mathbb{M}; U(1)) = \mathbb{Z}_8 \oplus$ [3.24] $H^3(\mathbb{M}; U(1))_{(3)} = \mathbb{Z}_3$.