Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 8

Speakers: Meng Guo, Theo Johnson-Freyd

Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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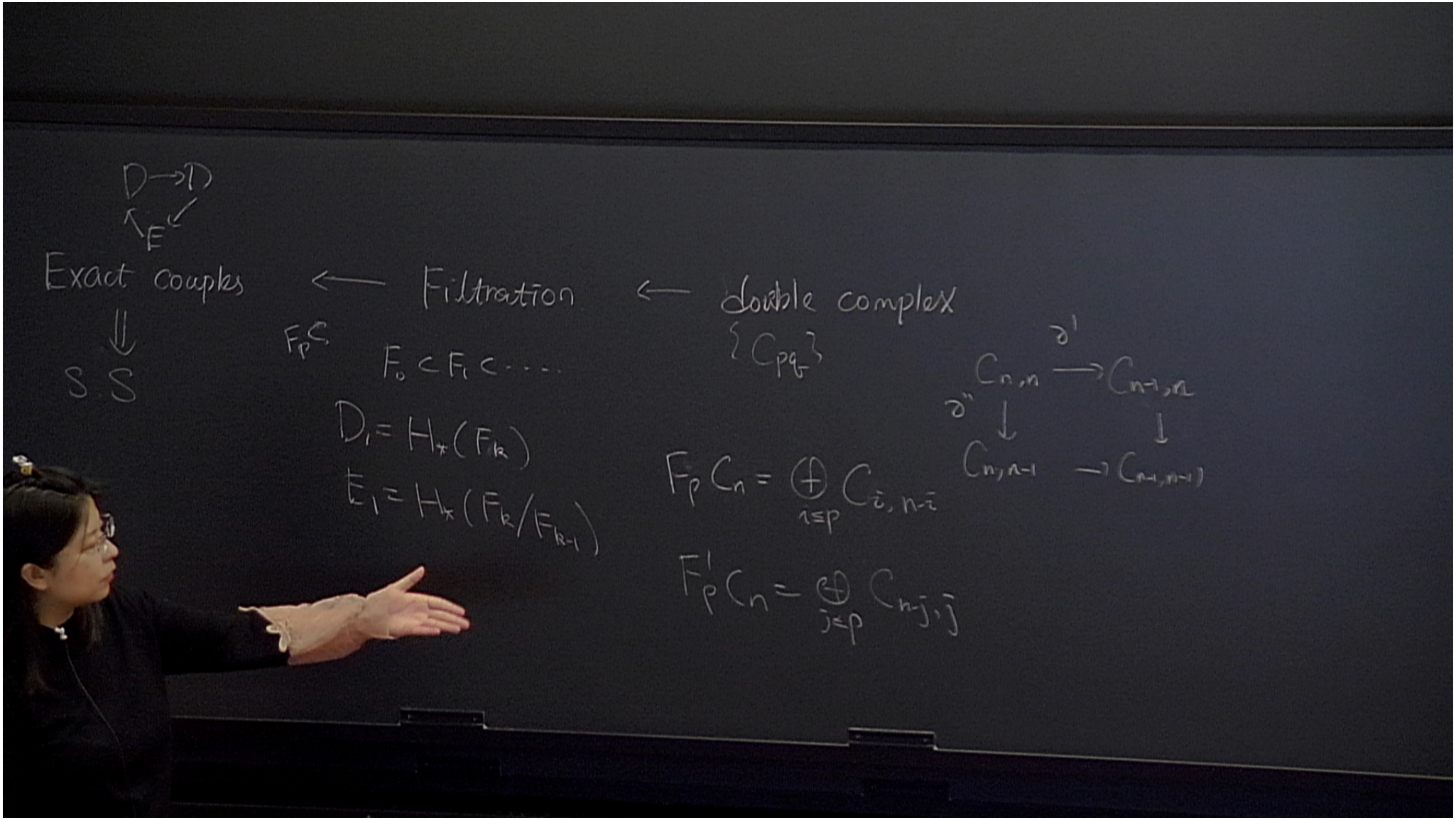
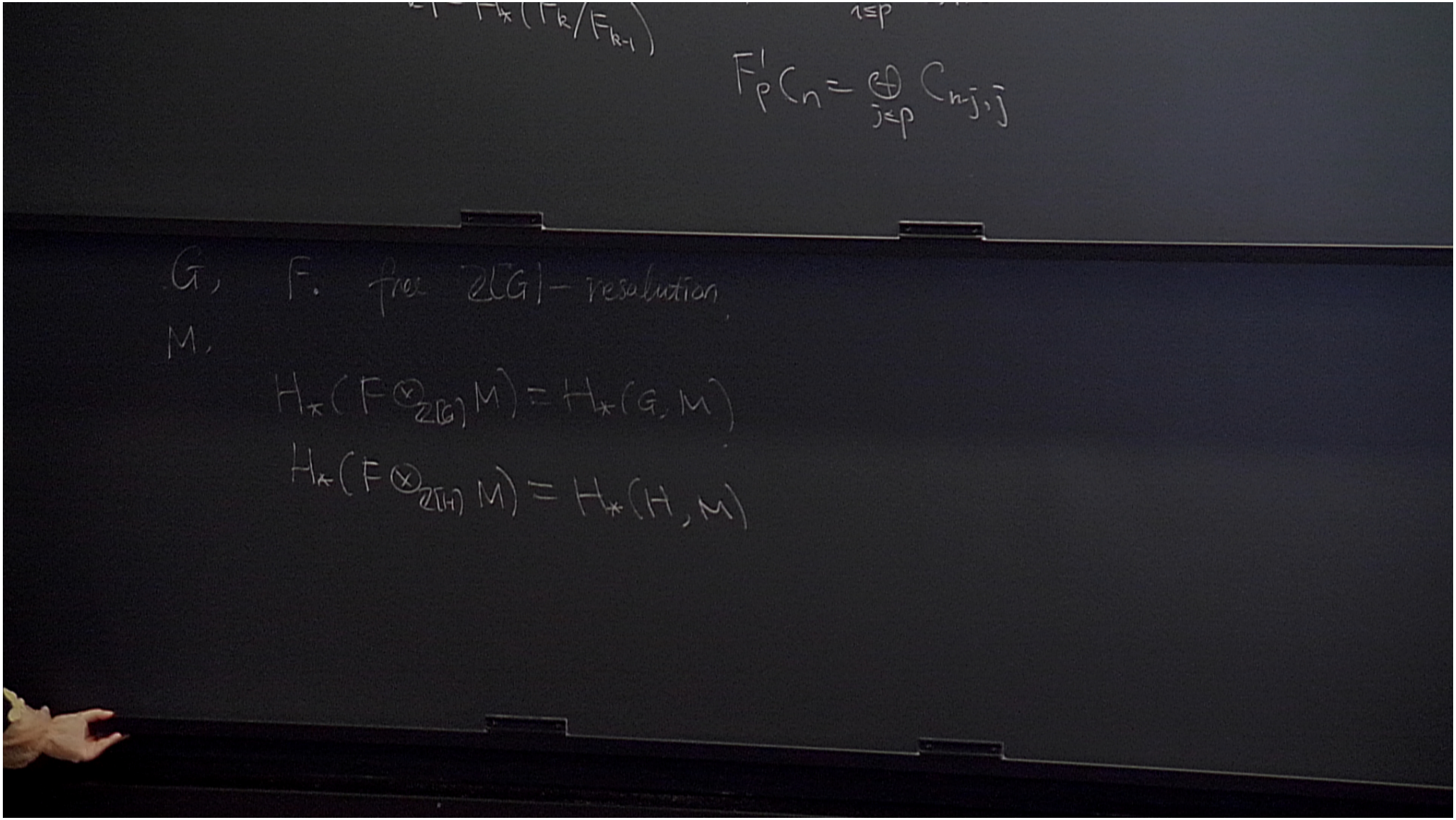
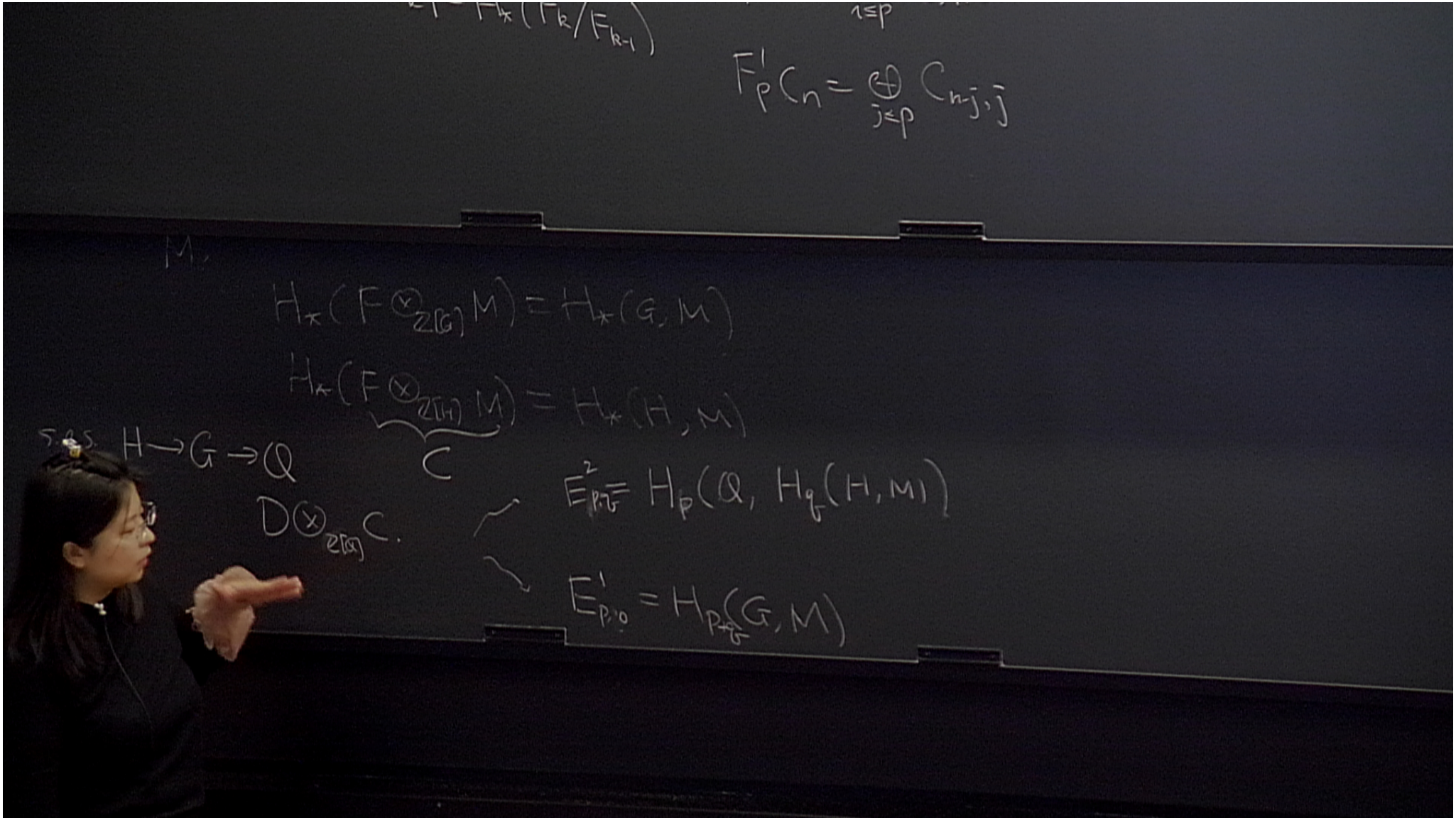
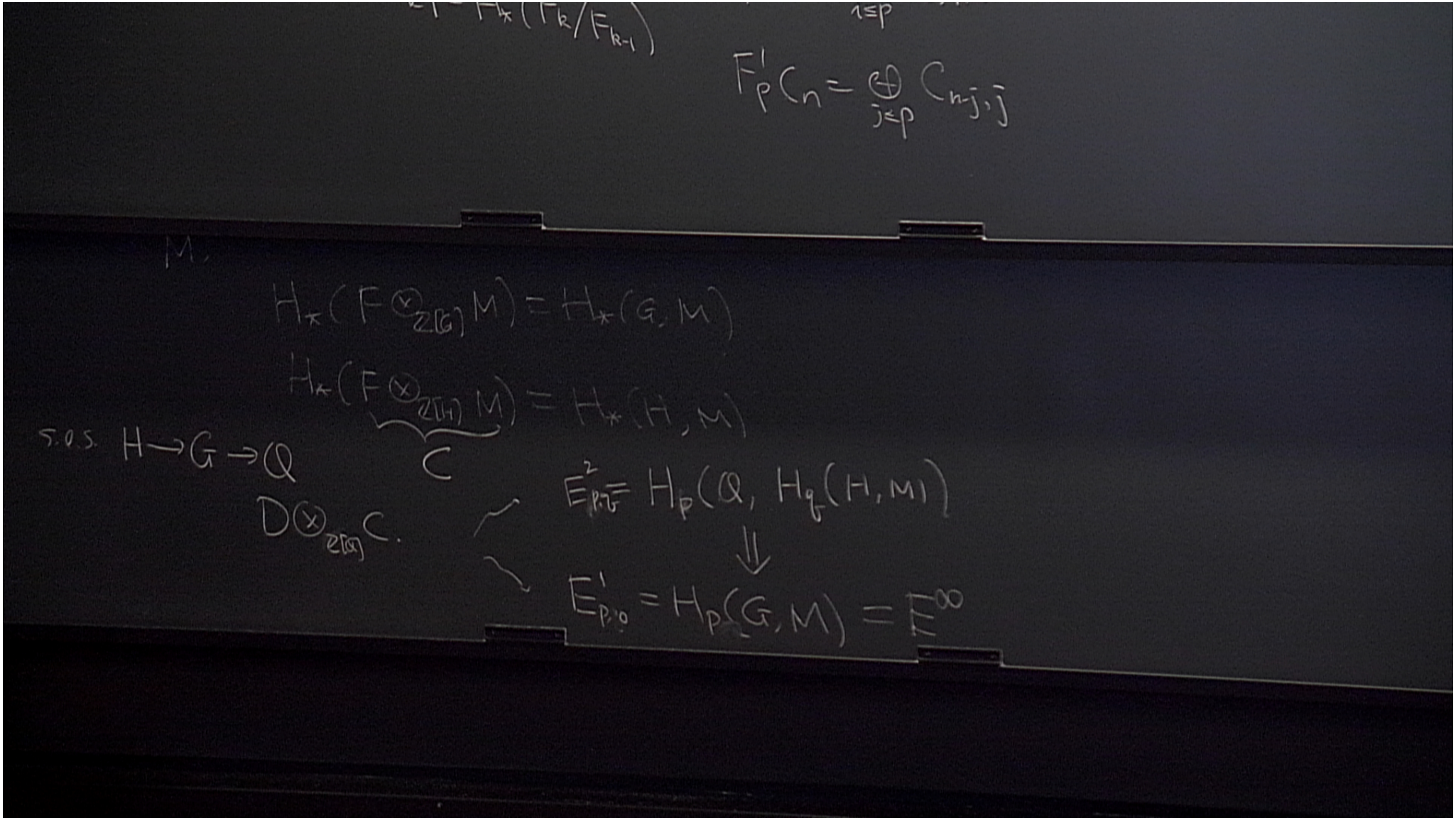
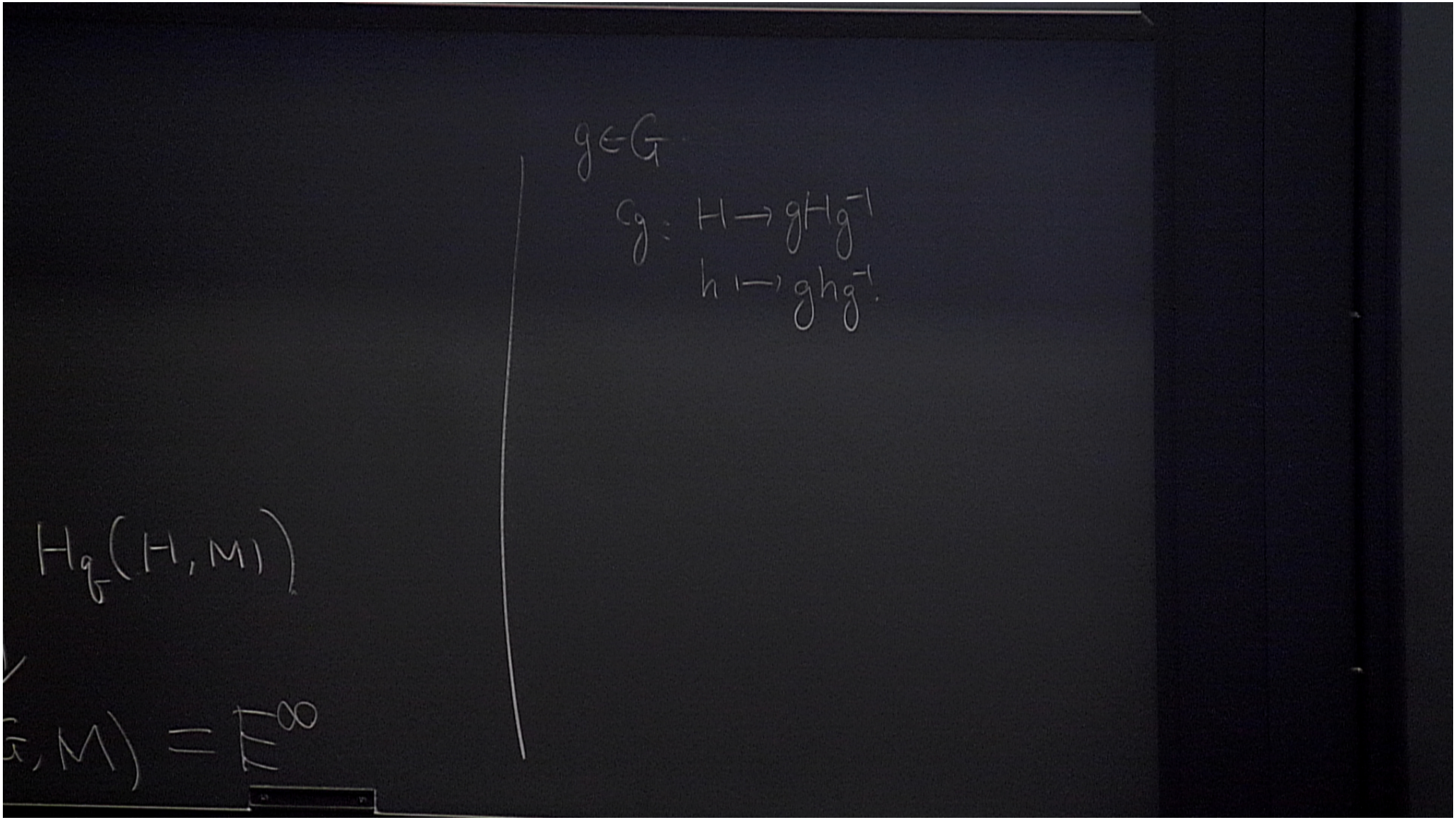
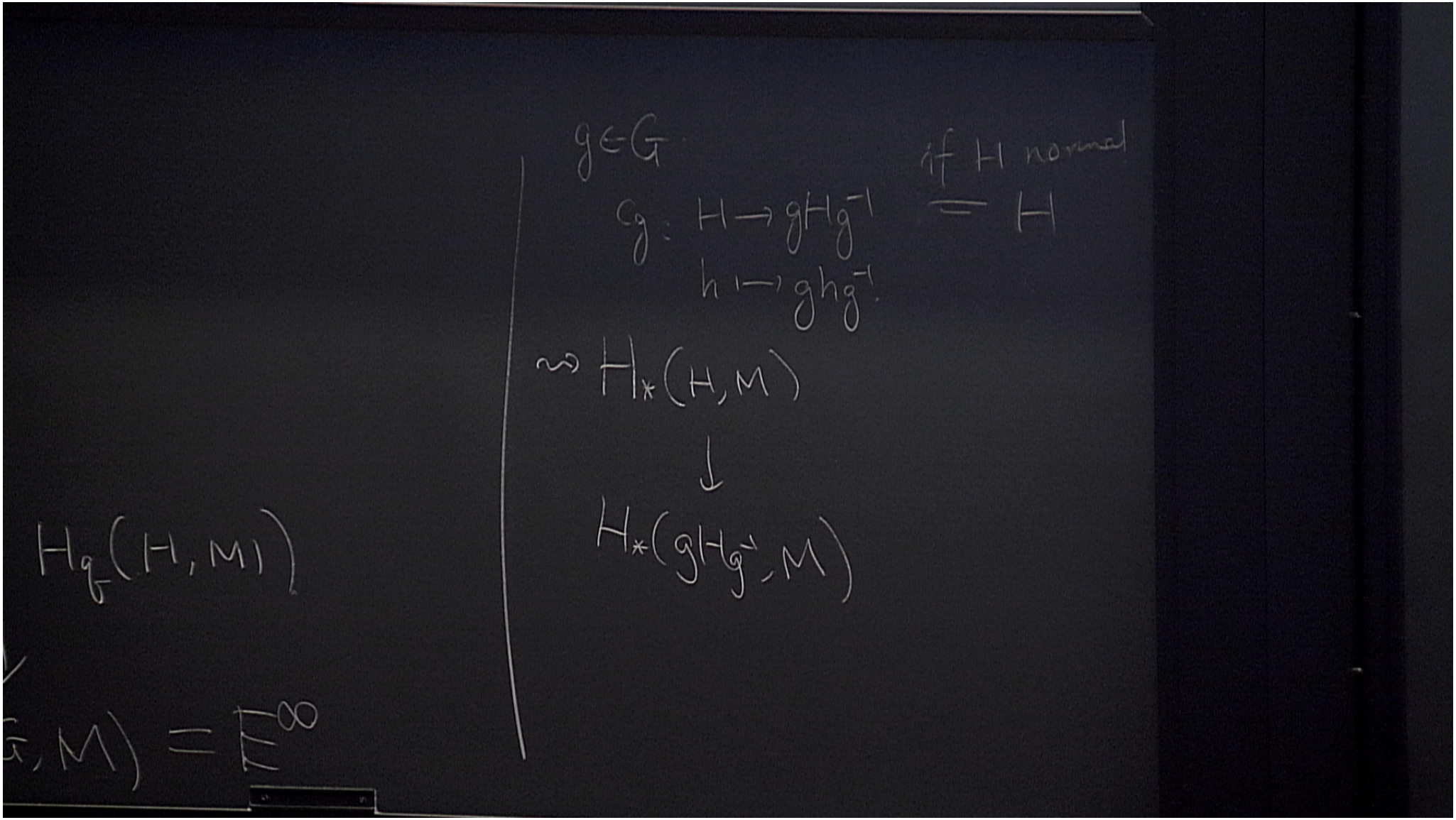

D → D
E
Exact couples ⇐ Filtration ⇐ double complex
S S
$F_p C$
$F_0 \subset F_1 \subset \cdots$
$\{C_{pq}\}$
$D_1 = H_*(F_k)$
$E_1 = H_*(F_k/F_{k-1})$
$F_p C_n = \bigoplus_{i \le p} C_{i,n-i}$
$F'_p C_n = \bigoplus_{j \le p} C_{n-j,j}$
$C_{n,n} \xrightarrow{\partial'} C_{n-1,n}$
∂'' ↓
$C_{n,n-1} \to C_{n-1,n-1}$


$H_*(F_k/F_{k-1})$
$i \leq p$
$F'_p C_n = \bigoplus_{j \leq p} C_{n-j,j}$
G, F_* free $\mathbb{Z}[G]$-resolution.
M.
$H_*(F \otimes_{\mathbb{Z}[G]} M) = H_*(G, M)$
$H_*(F \otimes_{\mathbb{Z}[H]} M) = H_*(H, M)$


$H_*(F_k/F_{k-1})$
$F'_p C_n = \bigoplus_{j \le p} C_{n-j,j}$
M.
$H_*(F \otimes_{\mathbb{Z}[G]} M) = H_*(G, M)$
$H_*(F \otimes_{\mathbb{Z}[H]} M) = H_*(H, M)$
C
SES $H \to G \to Q$
$D \otimes_{\mathbb{Z}[G]} C.$
$E^2_{p,q} = H_p(Q, H_q(H, M))$
$E^1_{p,0} = H_p(G, M)$


$H_*(F_k/F_{k-1})$
$i \leq p$
$F'_p C_n = \bigoplus_{j \leq p} C_{n-j,j}$
M.
$H_*(F \otimes_{\mathbb{Z}[G]} M) = H_*(G, M)$
$H_*(F \otimes_{\mathbb{Z}[H]} M) = H_*(H, M)$
S.E.S. $H \to G \to Q$
C
$D \otimes_{\mathbb{Z}[G]} C.$
$E^2_{p,q} = H_p(Q, H_q(H, M))$
$E'_{p,0} = H_p(G, M) = E^\infty$


$g \in G$
$c_g : H \to gHg^{-1}$
$h \mapsto ghg^{-1}$.
$H_q(H, M)$
$G, M) = E^\infty$


$g \in G$
$c_g : H \to gHg^{-1}$
$h \mapsto ghg^{-1}$
if H normal $= H$
$\curvearrowright H_*(H, M)$
$\downarrow$
$H_*(gHg^{-1}, M)$
$H_q(H, M)$
$G, M) = E^\infty$

$H_q(H, M)$

$G, M) = E^\infty$

$g : H \to gHg^{-1} = H$

$h \mapsto ghg^{-1}$.

$\rightsquigarrow H_*(H, M)$

$\downarrow \quad G^*$

$H_*(gHg^{-1}, M)$

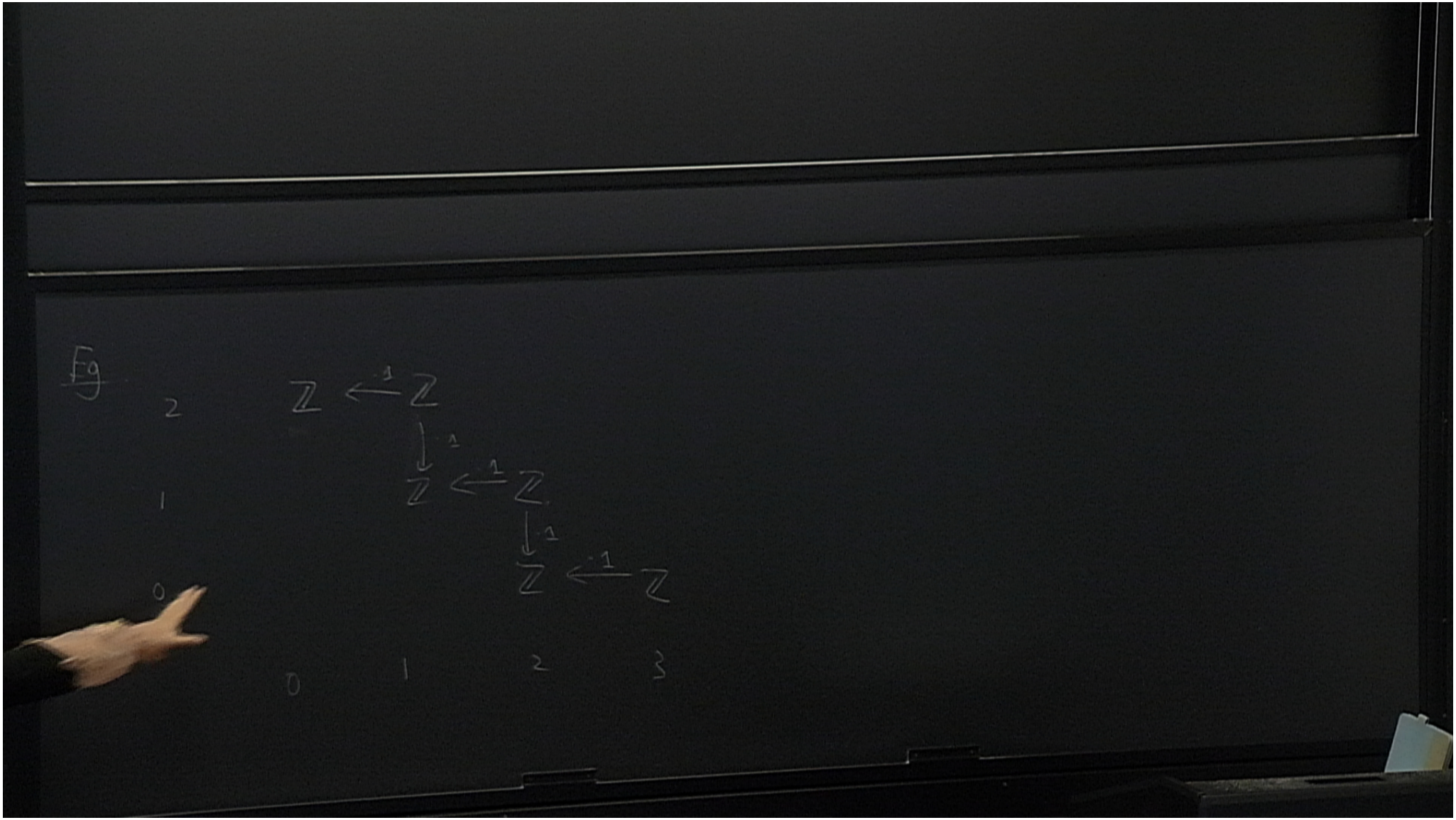
If $g \in H$, $G^* = id$

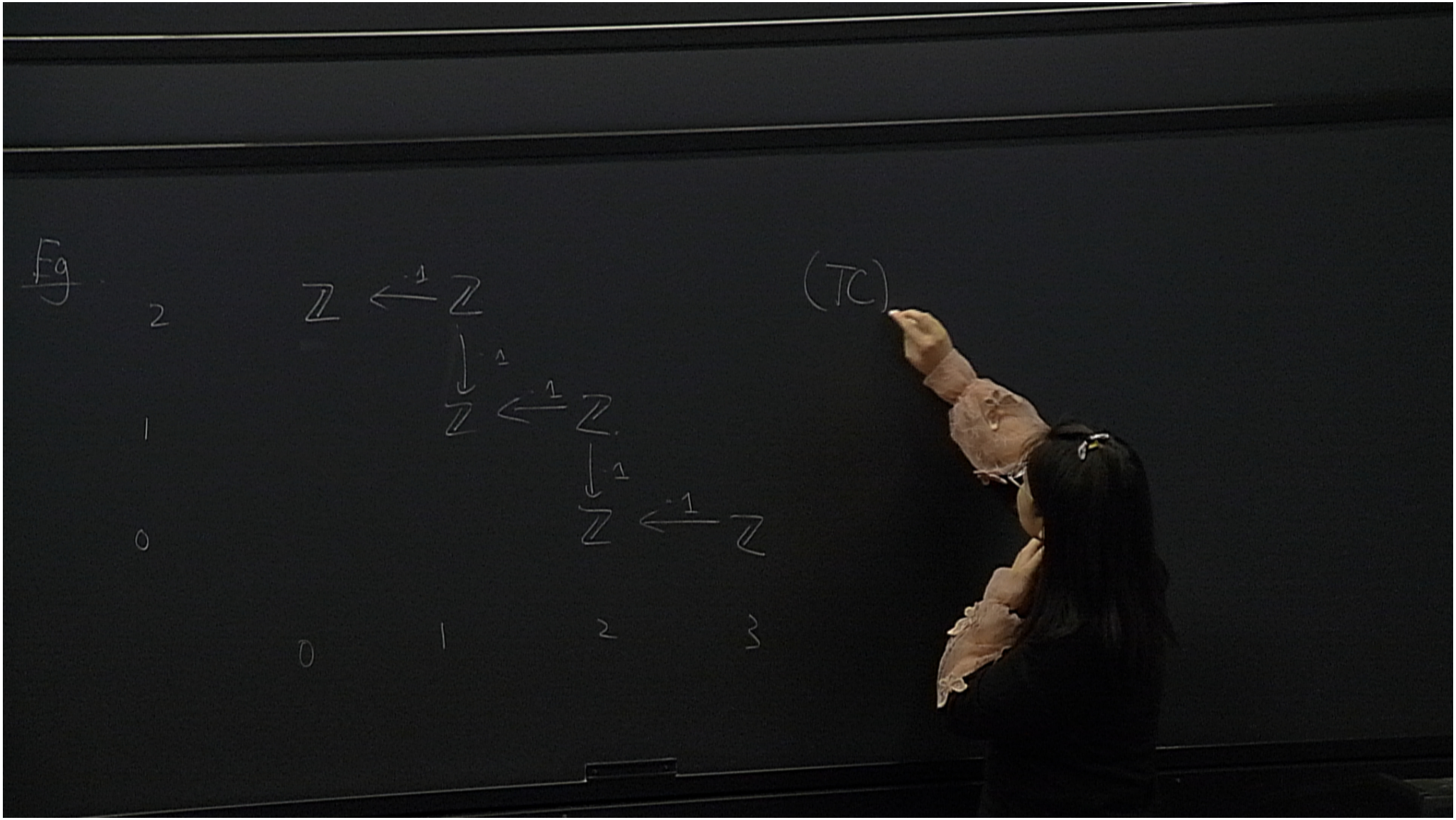
H is normal

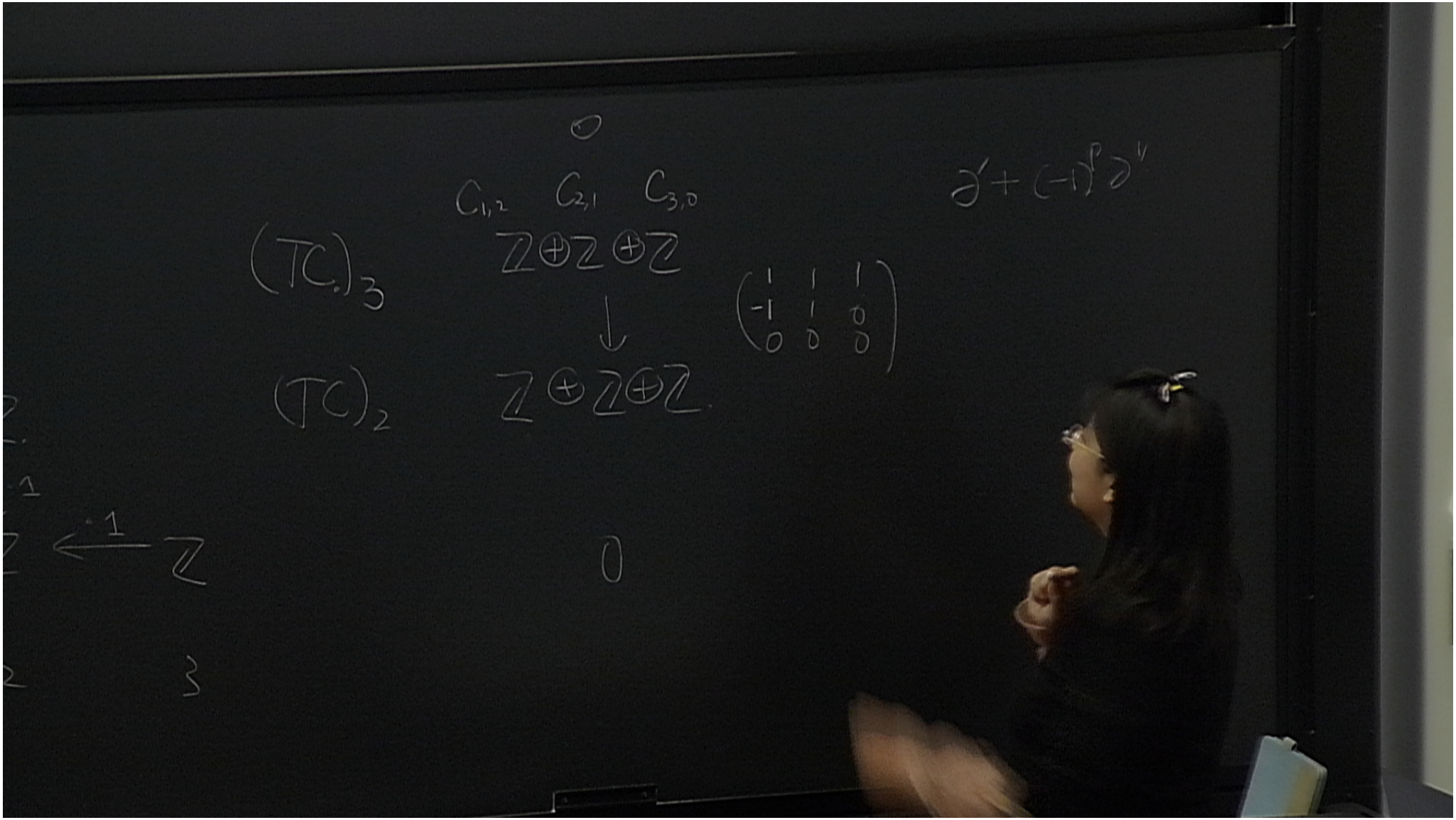
$G \curvearrowright H_*(H, M) \mid \to H_*(H, M)$

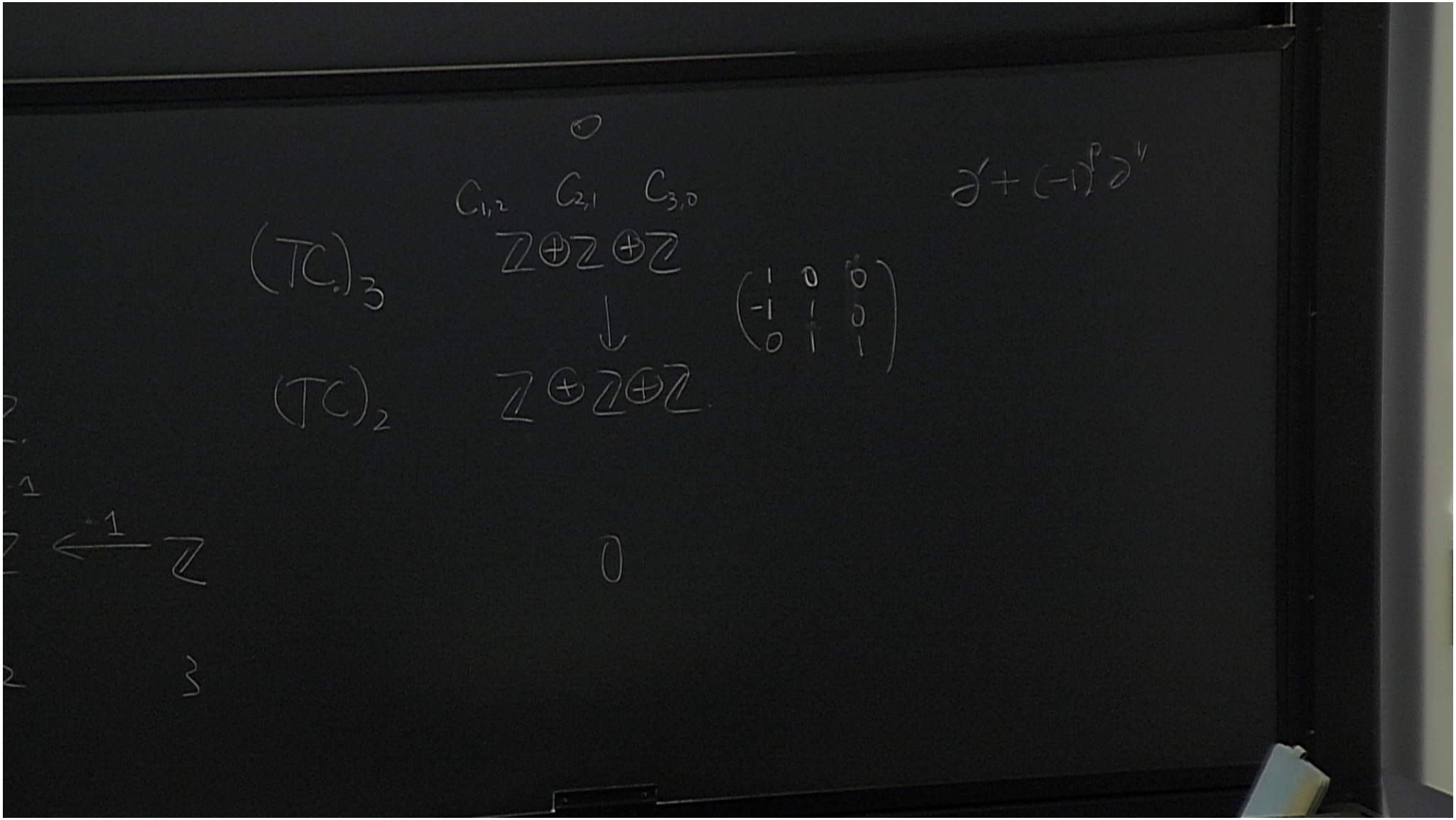
$G/H \curvearrowright$

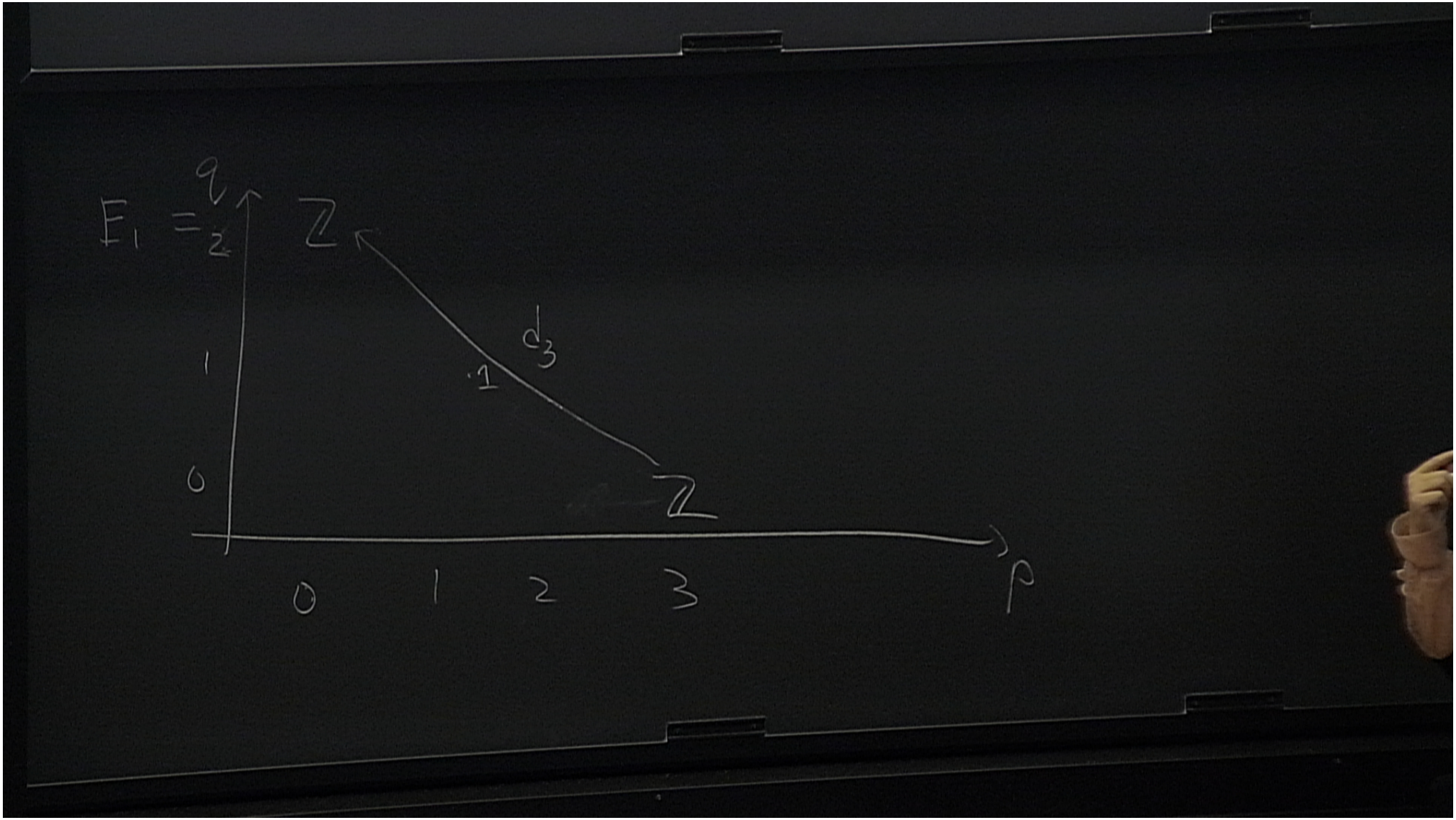
$G \curvearrowright H$

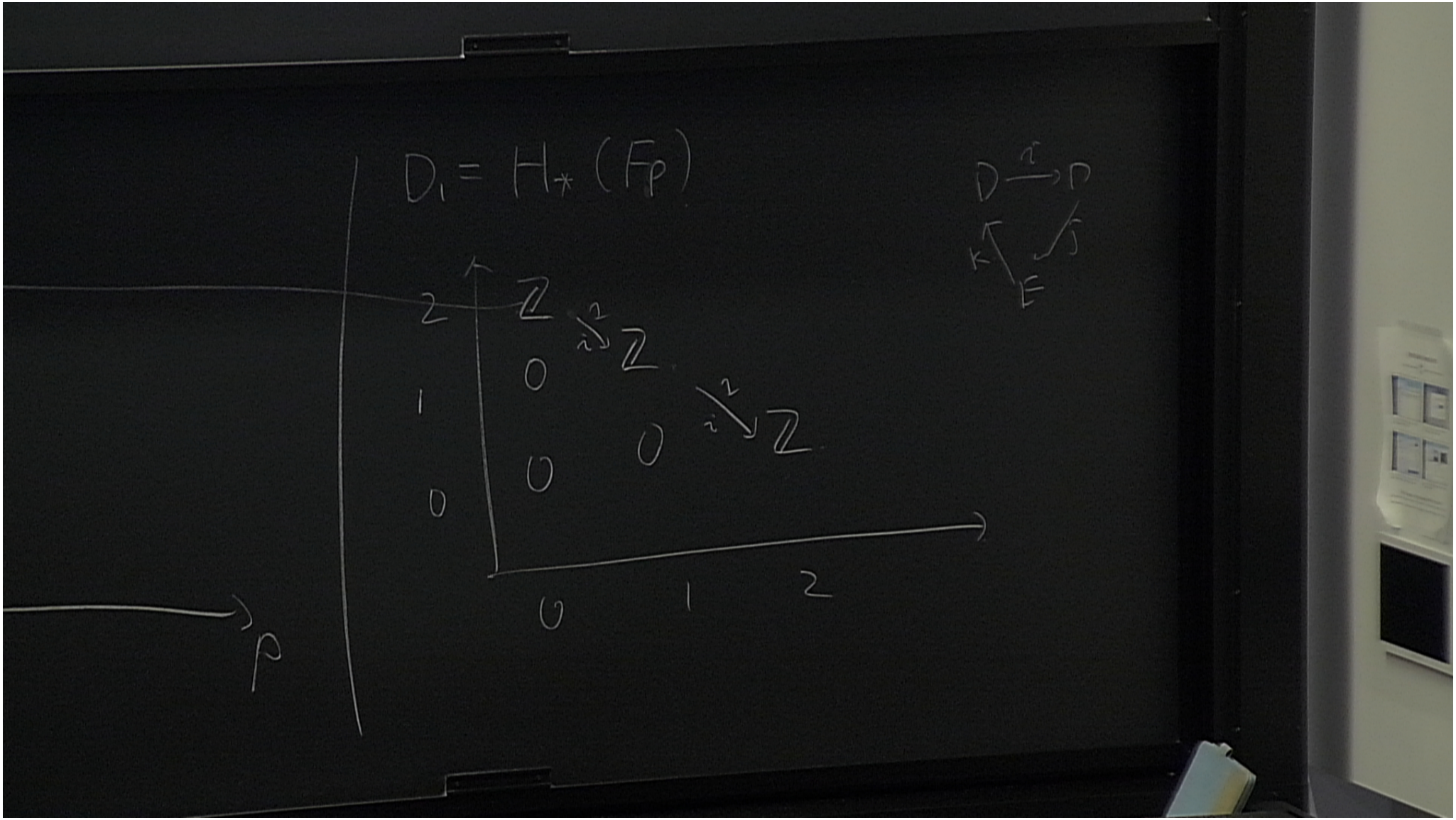

Eg
2
1
0
$\mathbb{Z} \xleftarrow{1} \mathbb{Z}$
$\mathbb{Z} \xleftarrow{1} \mathbb{Z}$
$\mathbb{Z} \xleftarrow{1} \mathbb{Z}$
0 1 2 3

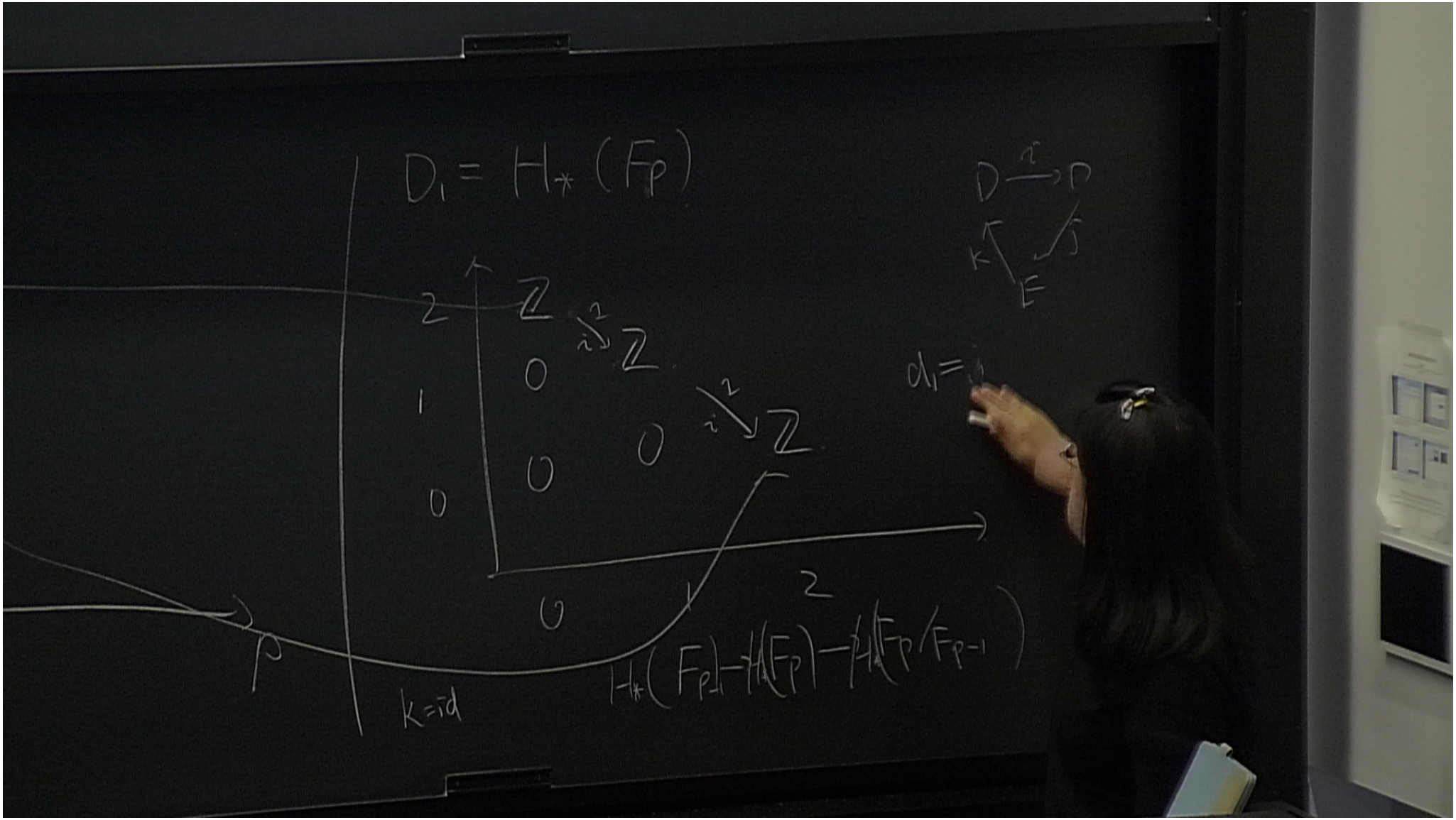

Eg
2
1
0
$\mathbb{Z}$
$\mathbb{Z}$
$\mathbb{Z}$
$\mathbb{Z}$
$\mathbb{Z}$
$\mathbb{Z}$
1
1
1
1
1
0
1
2
3
(TC)

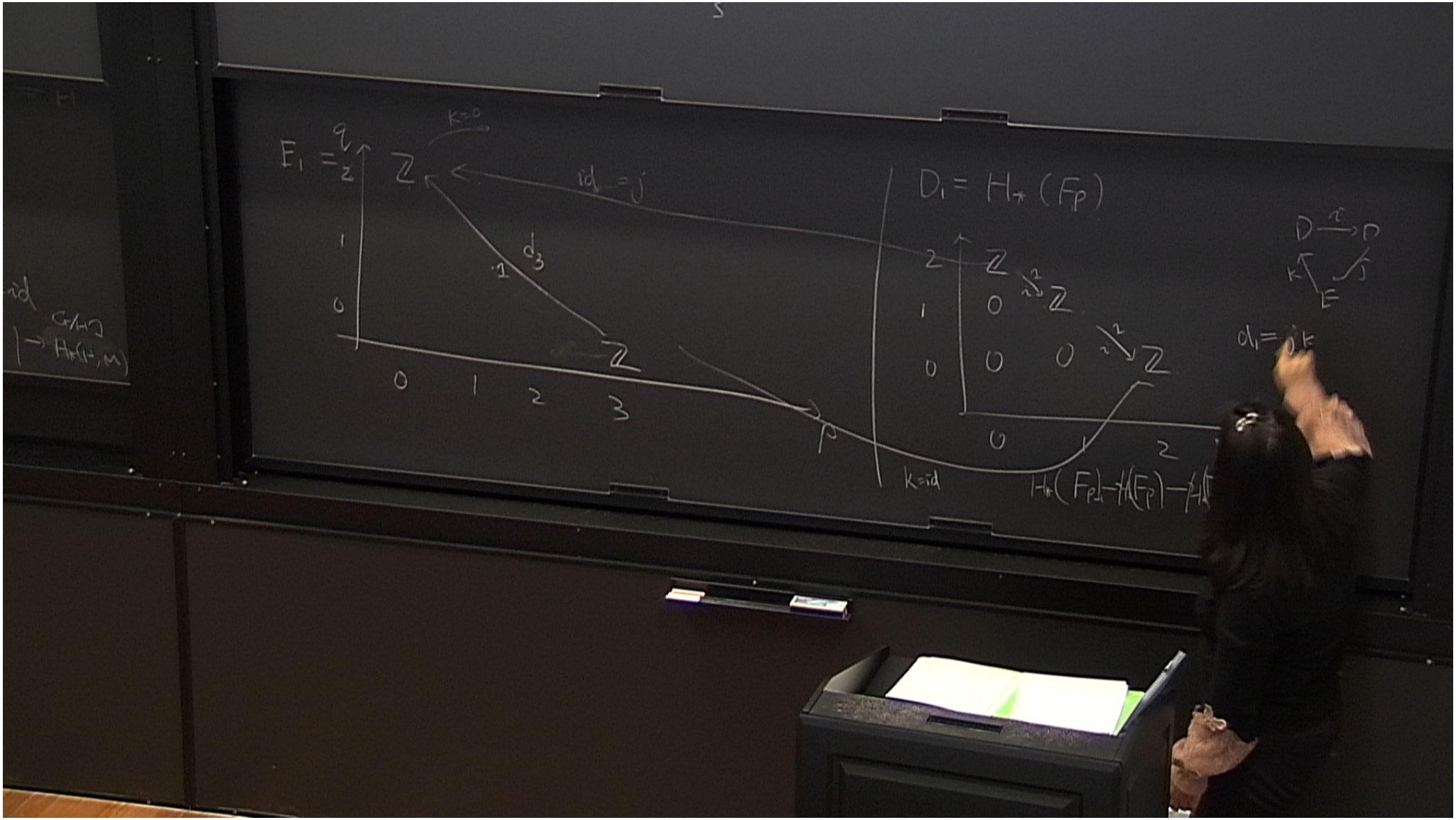

0
$C_{1,2}$ $C_{2,1}$ $C_{3,0}$
$\partial' + (-1)^p \partial''$
$(TC_\cdot)_3$
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(TC)_2$
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
1
1
$\mathbb{Z}$
$\mathbb{Z}$
0
3

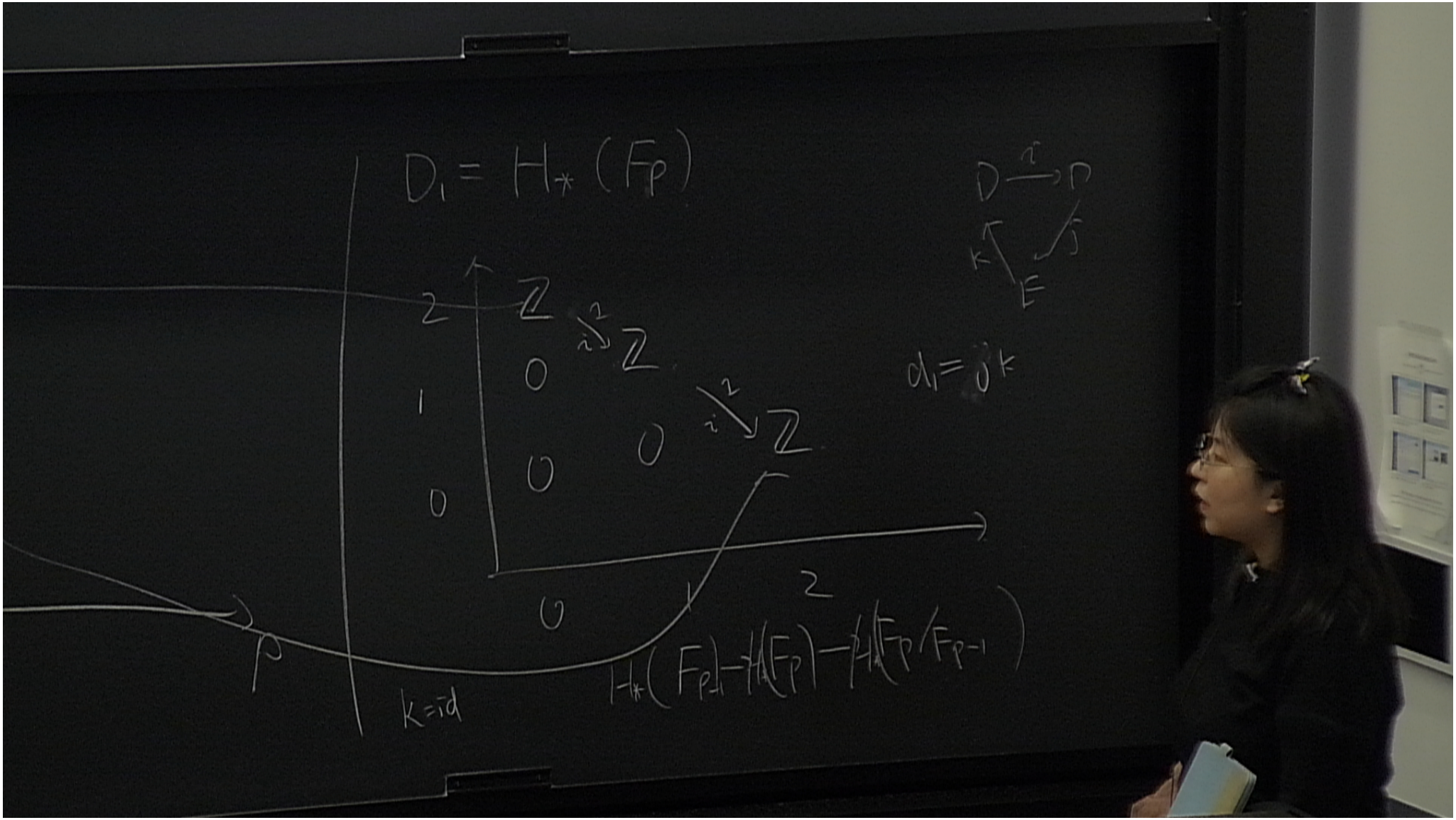

0
$C_{1,2}$ $C_{2,1}$ $C_{3,0}$
$\partial' + (-1)^p \partial''$
$(TC_\cdot)_3$
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
$(TC)_2$
$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
1
1
$\mathbb{Z} \leftarrow \mathbb{Z}$
0
3

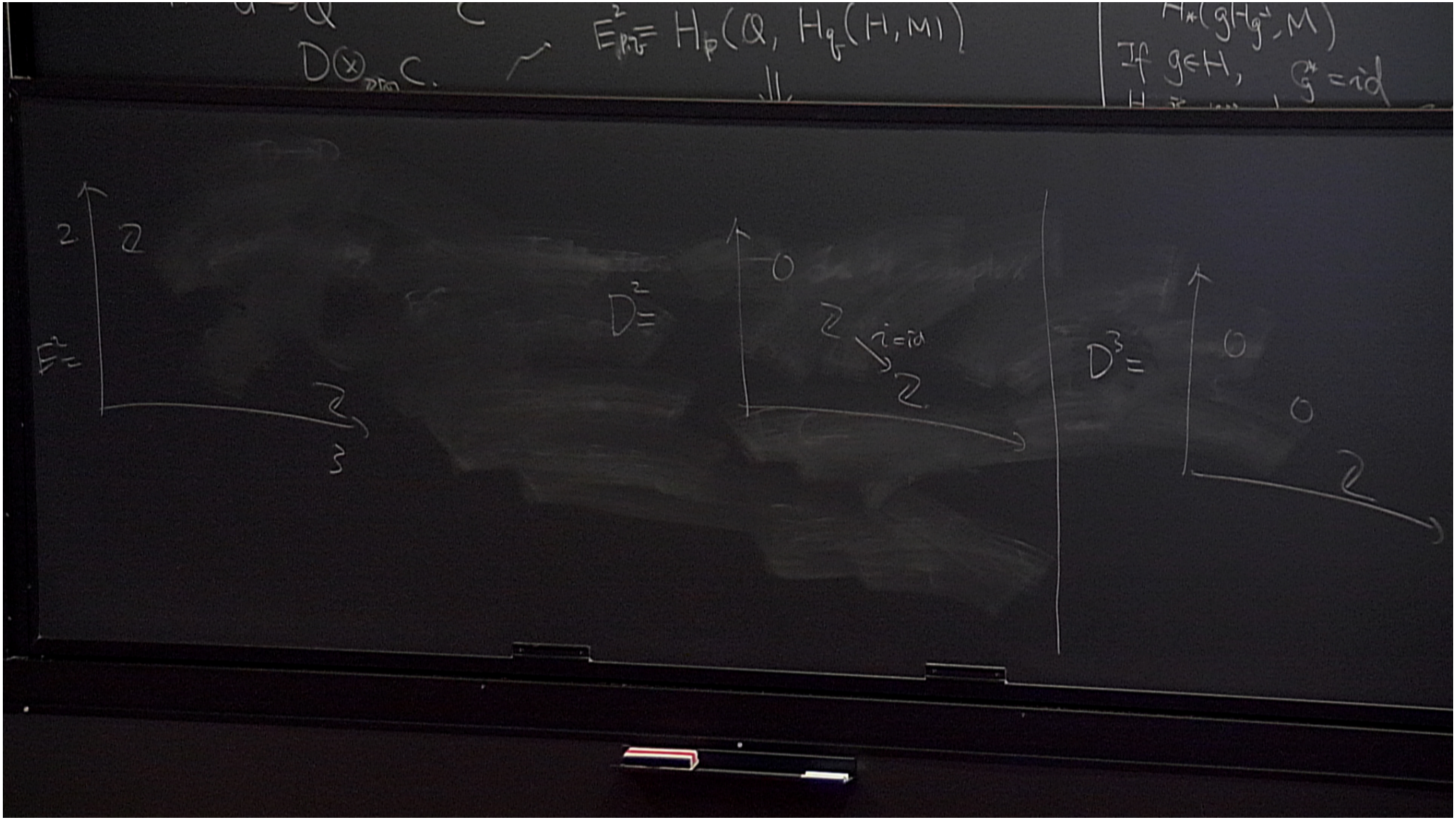

$E_1 = \frac{q}{2}$
Z
d_3
1
1
0
Z
0
1
2
3
p

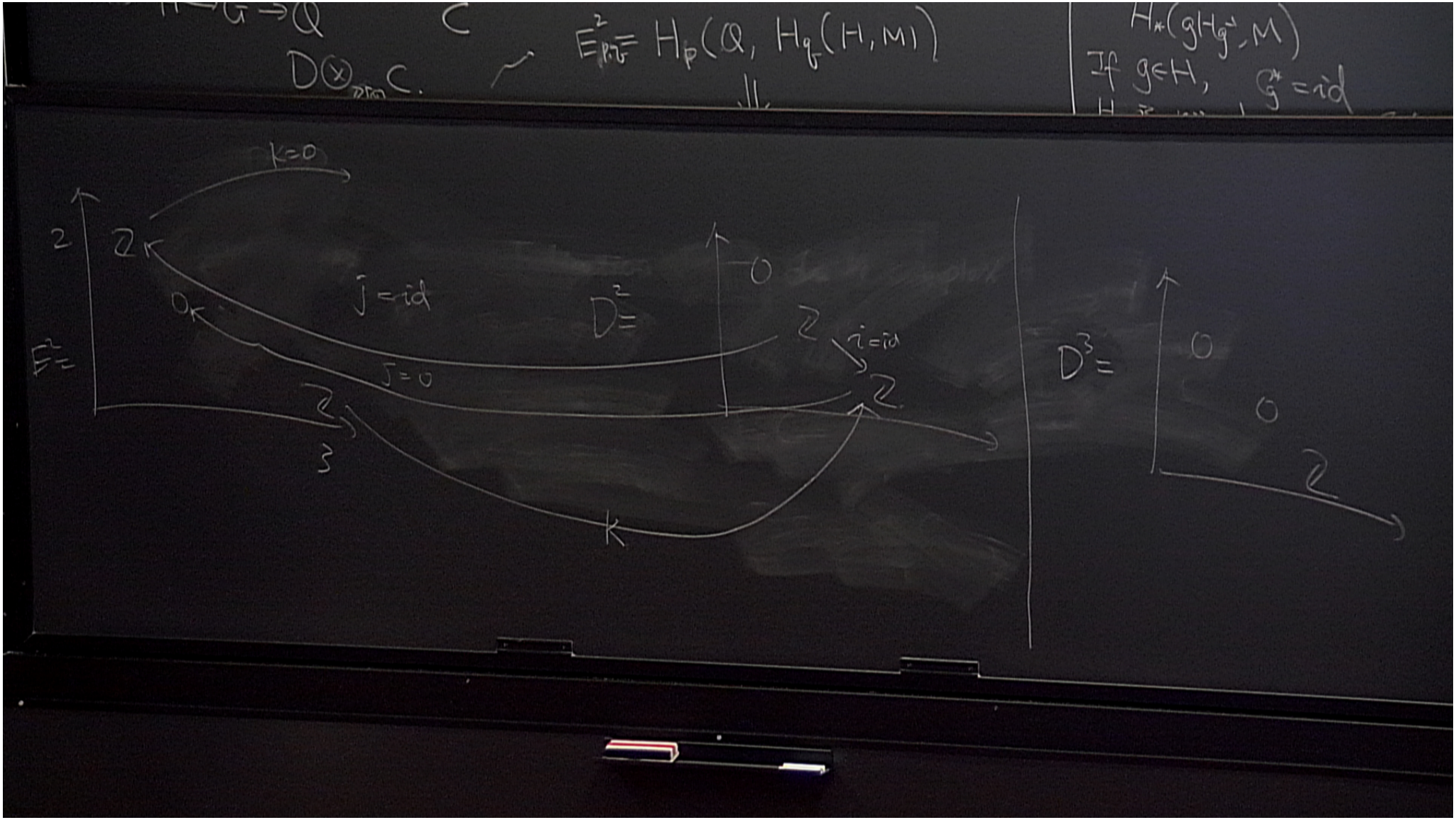

D1 = H* (Fp)
D
D
E
2
1
0
0
1
2
p

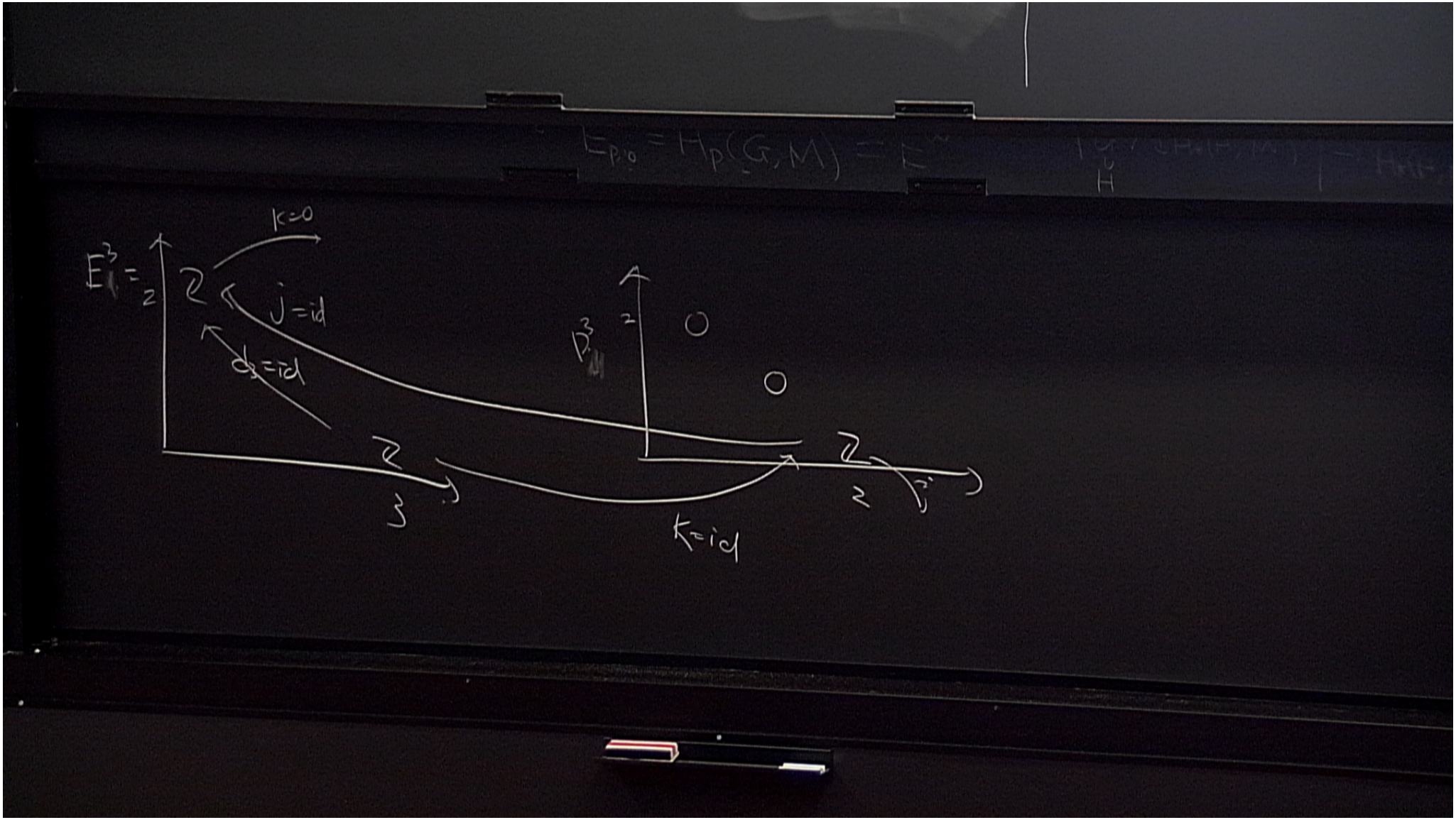

D1 = H* (Fp)
2
1
0
Z
0
Z
0
0
Z
0
1
2
k=id
d1=

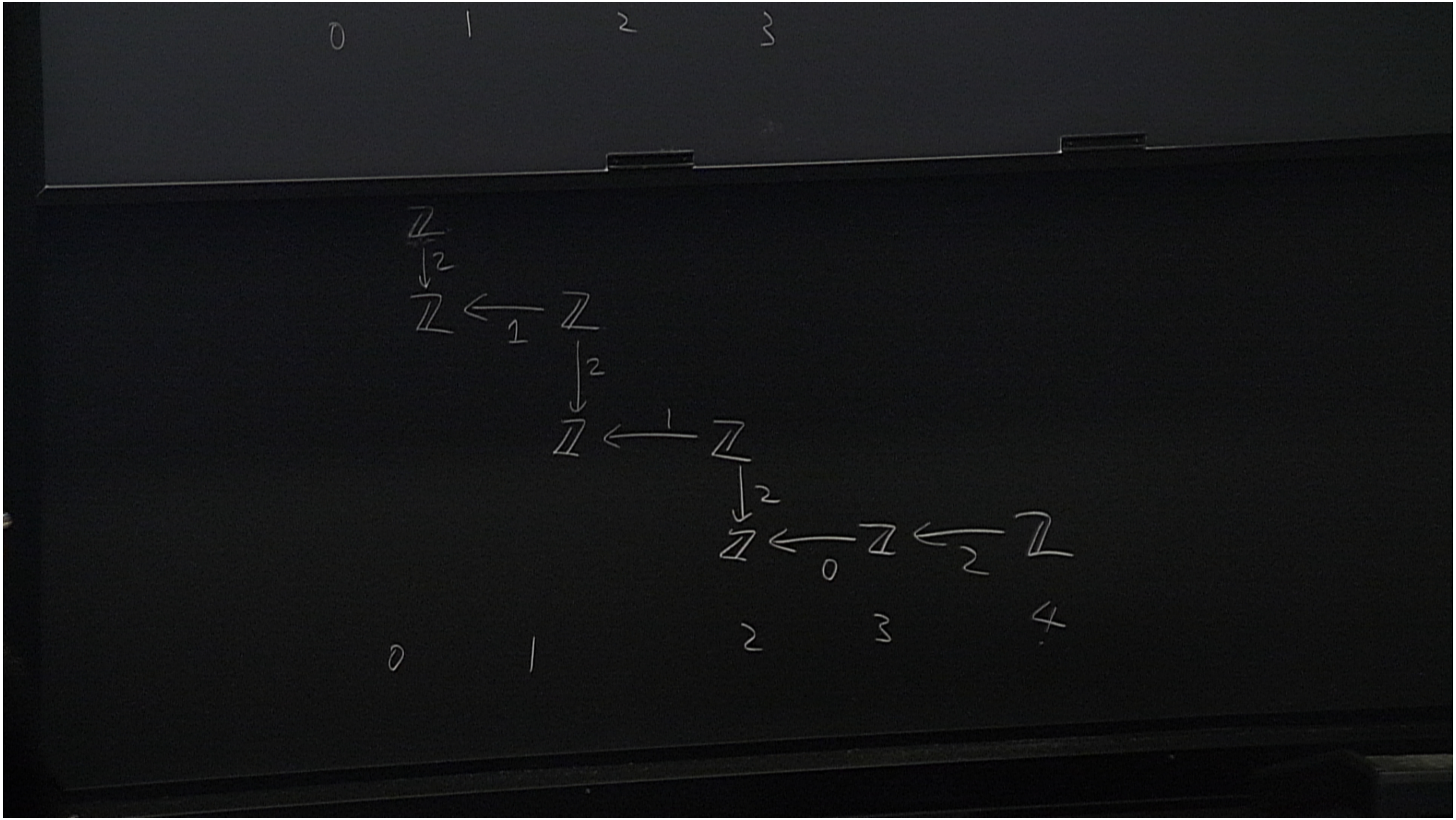

E1 = q
Z
k=0
id = j
d3
Z
0 1 2 3
D1 = H*(Fp)
Z
Z
Z
k=id
d1 = jk

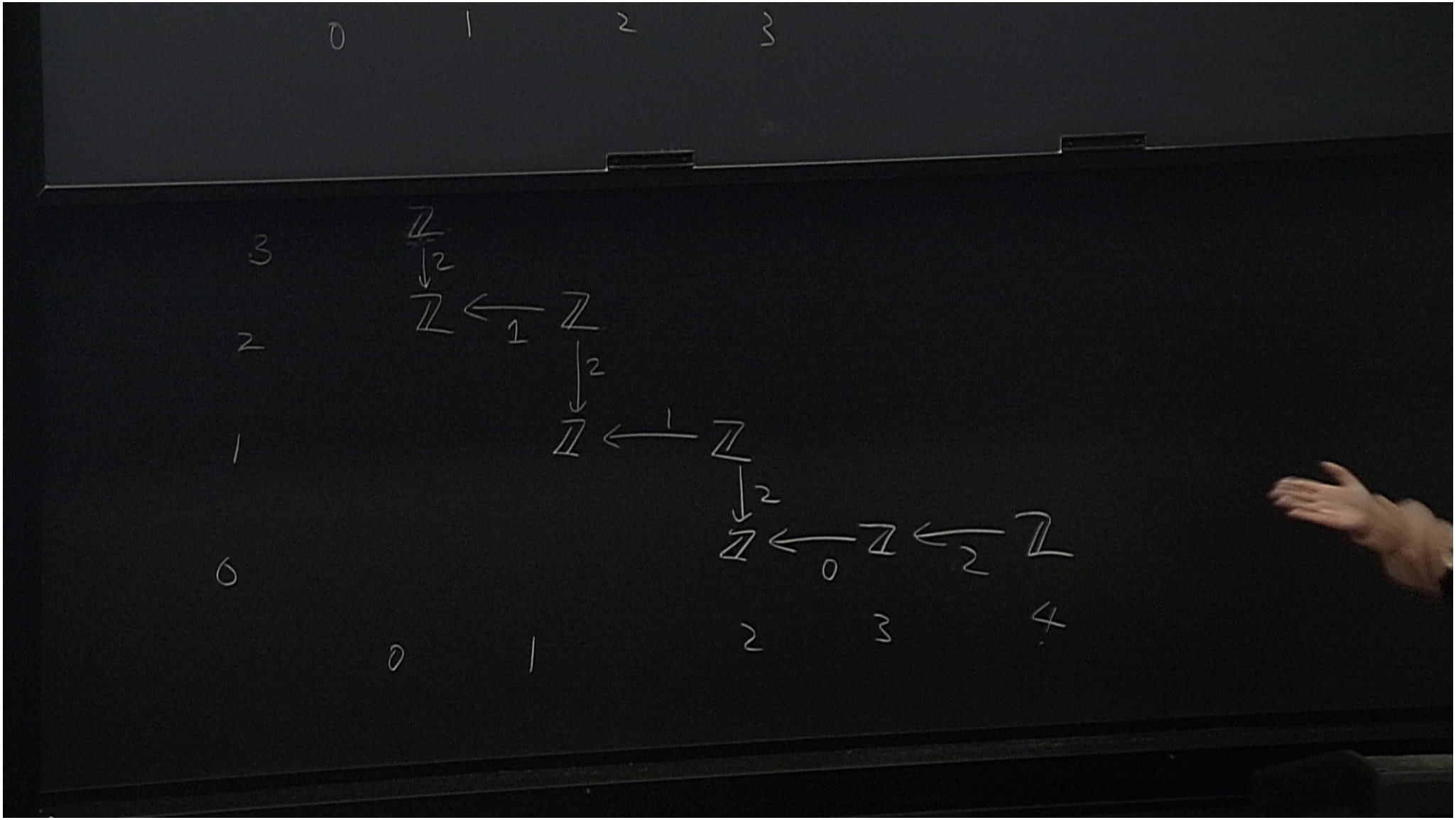

D1 = H* (Fp)
d1 = jk
k = id

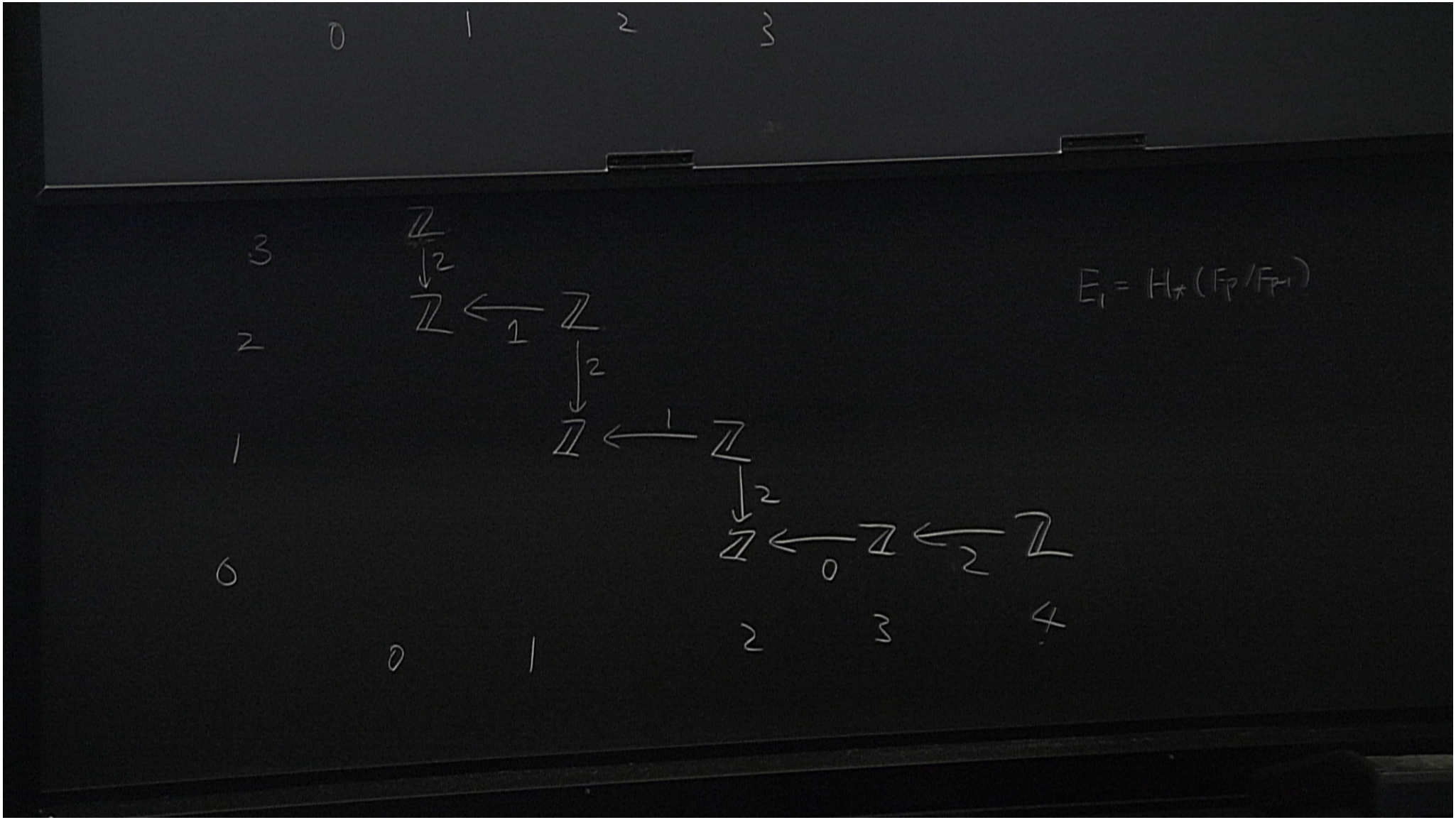

$D \otimes_{R[G]} C$.
$E^2_{p,q} = H_p(Q, H_q(H, M))$
If $g \in H$, $\tilde{g}^* = id$
$E^2 =$
$D^2 =$
$i = id$
$D^3 =$

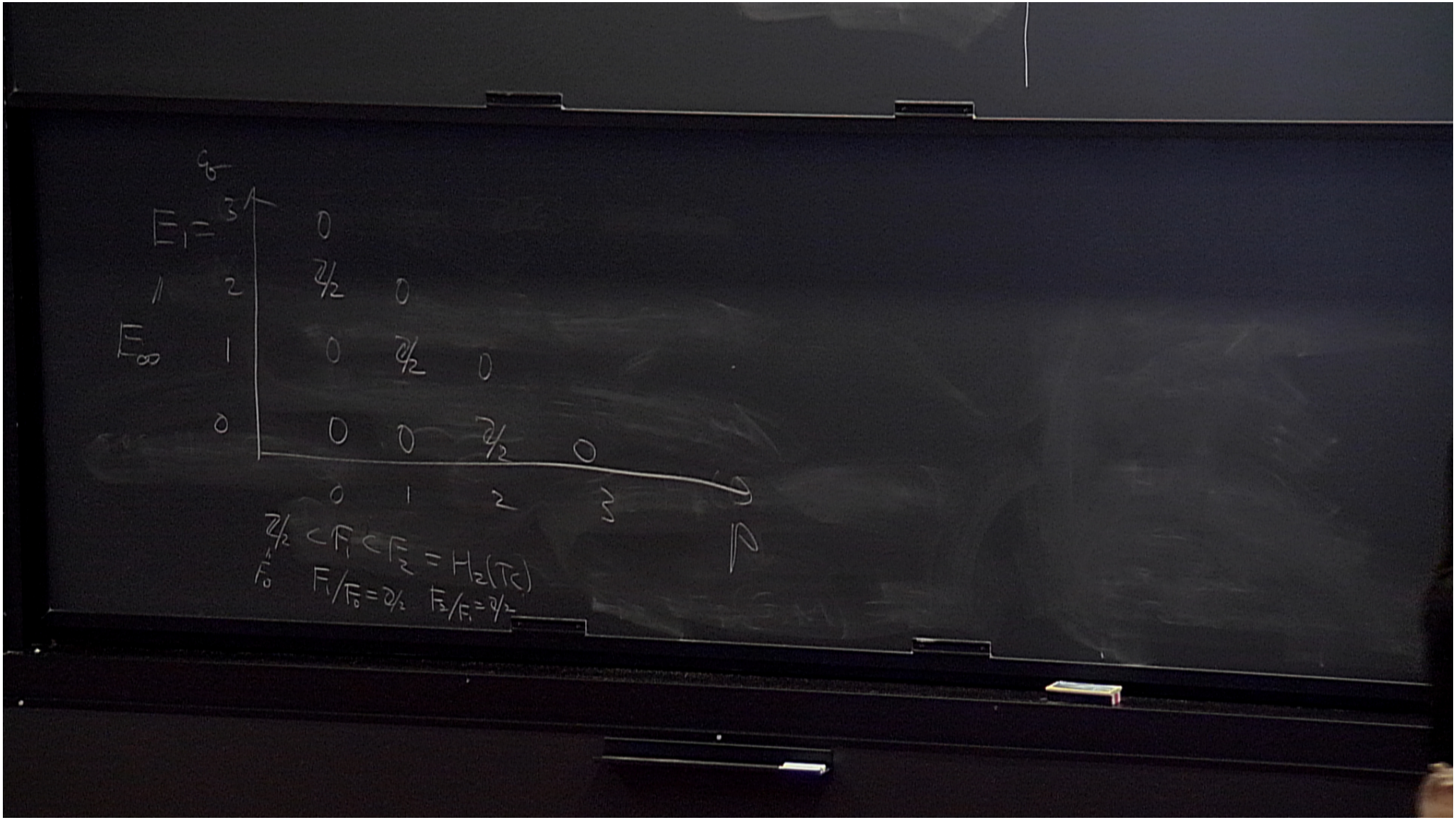

$E^2_{p,q} = H_p(Q, H_q(H,M))$
$H_*(gHg^{-1}, M)$
If $g \in H$, $\tilde{g}^* = id$
$k=0$
$j=id$
$j=0$
$j=id$
$E^2 =$
$D^2 =$
$D^3 =$

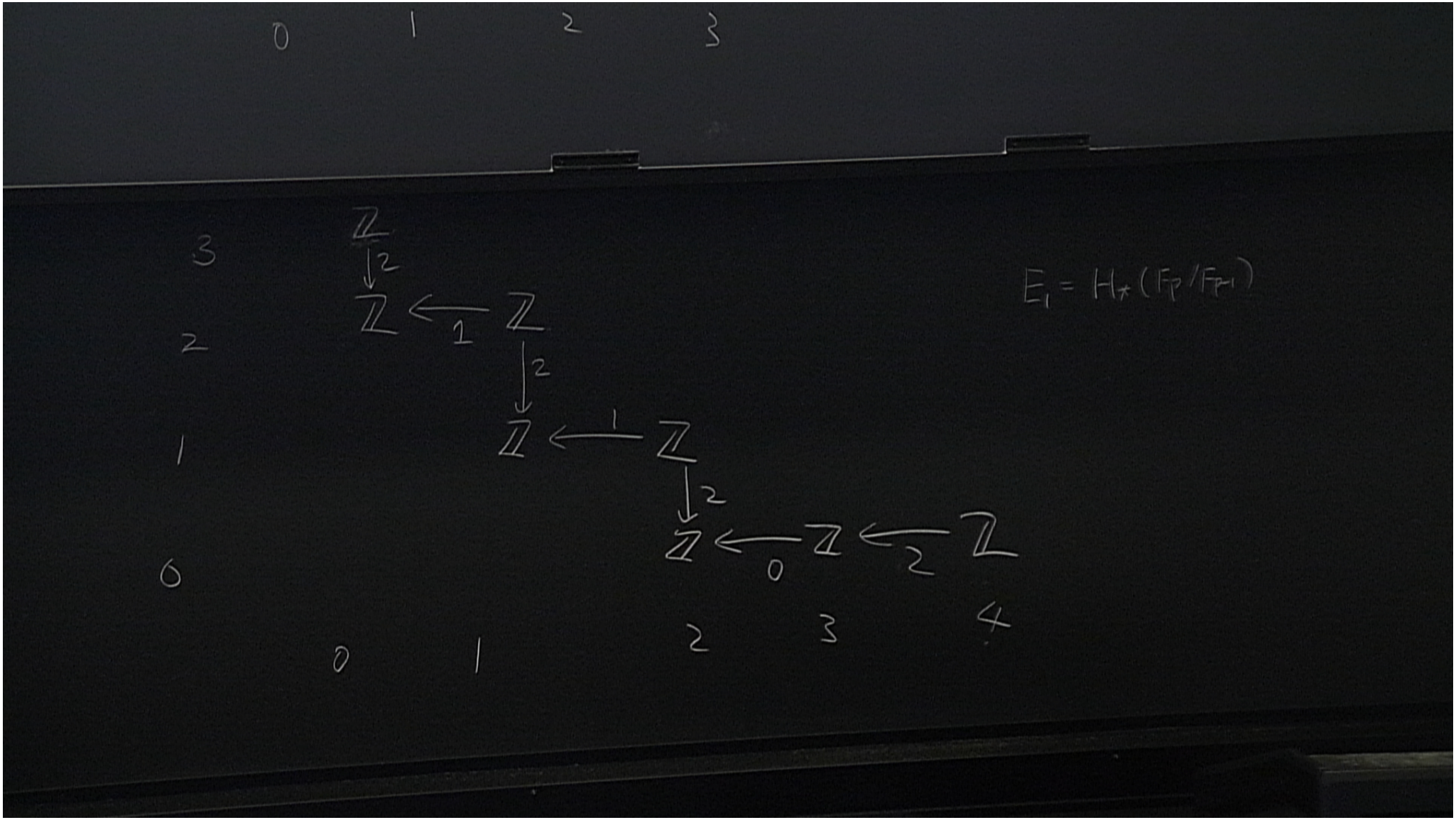

k=0
J=id
K=id

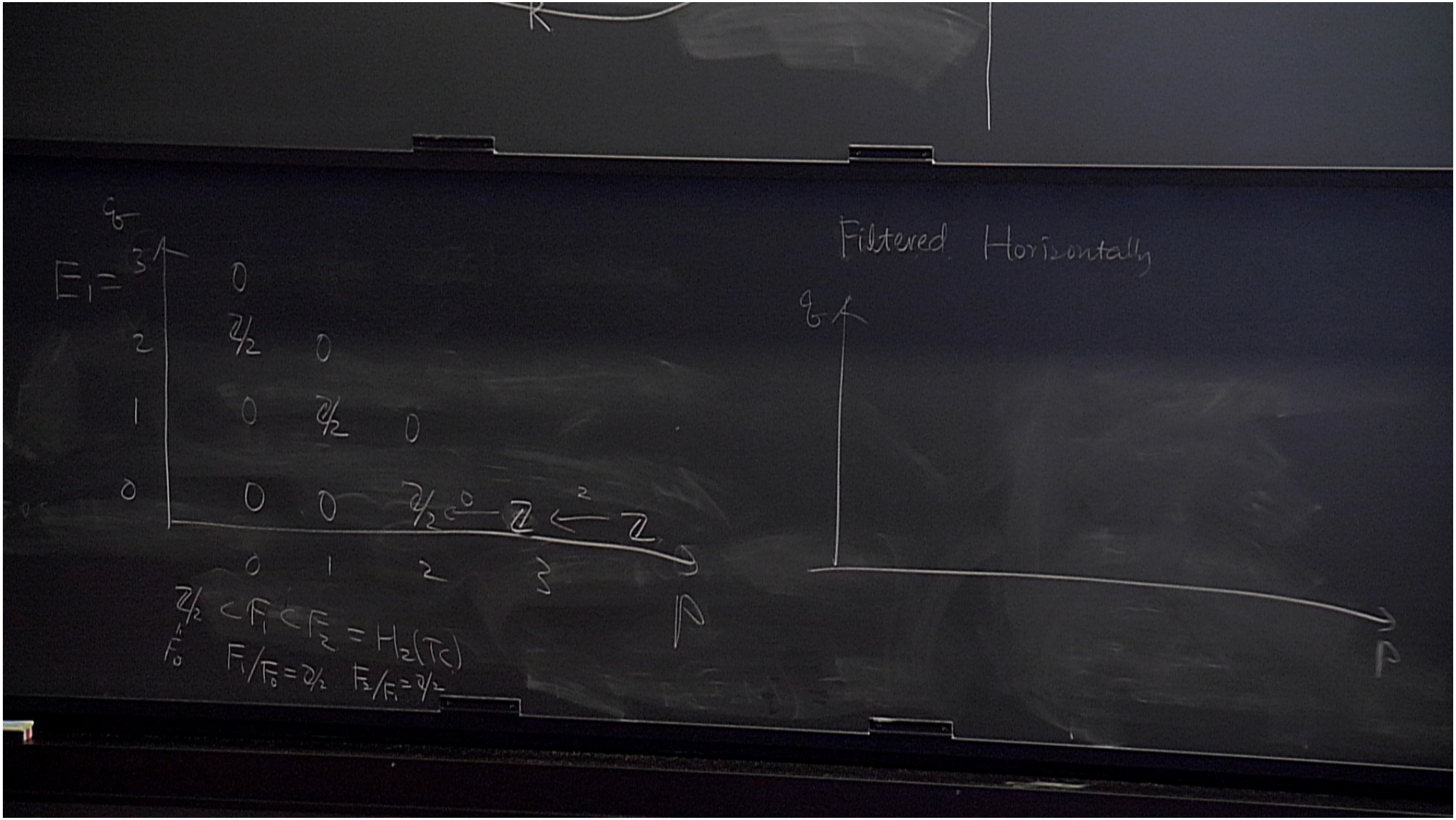

0 1 2 3
Z
2
Z ← Z
1
2
Z ← Z
1
2
Z ← Z ← Z
0 2
0 1 2 3 4

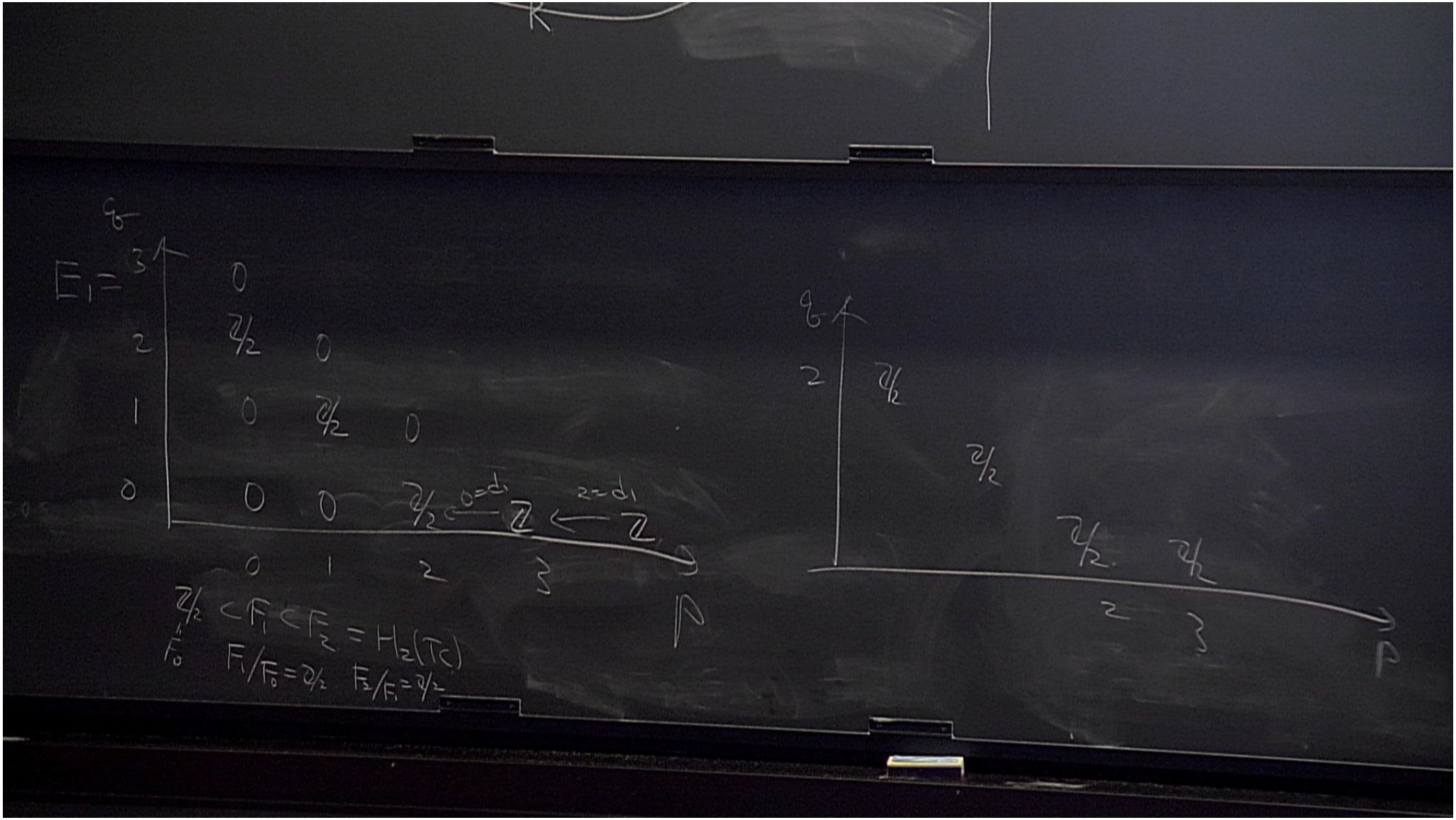

0 1 2 3
3
2
1
0
Z
2
Z
1
Z
2
Z
1
Z
2
Z
0
Z
2
Z
0 1 2 3 4


0 1 2 3
3
2
1
0
0 1 2 3 4
$E_1 = H_*(F_p/F_{p-1})$


$E_1 =$
E_∞
3
2
1
0
$\mathbb{Z}$
$\mathbb{Z}/2$
0 1 2 3
$\mathbb{Z}/2 \subset F_1 \subset F_2 = H_2(TC)$
F_0 $F_1/F_0 = \mathbb{Z}/2$ $F_2/F_1 = \mathbb{Z}/2$


0 1 2 3
3
2
1
0
0 1 2 3 4
$E_1 = H_*(F_p/F_{p-1})$

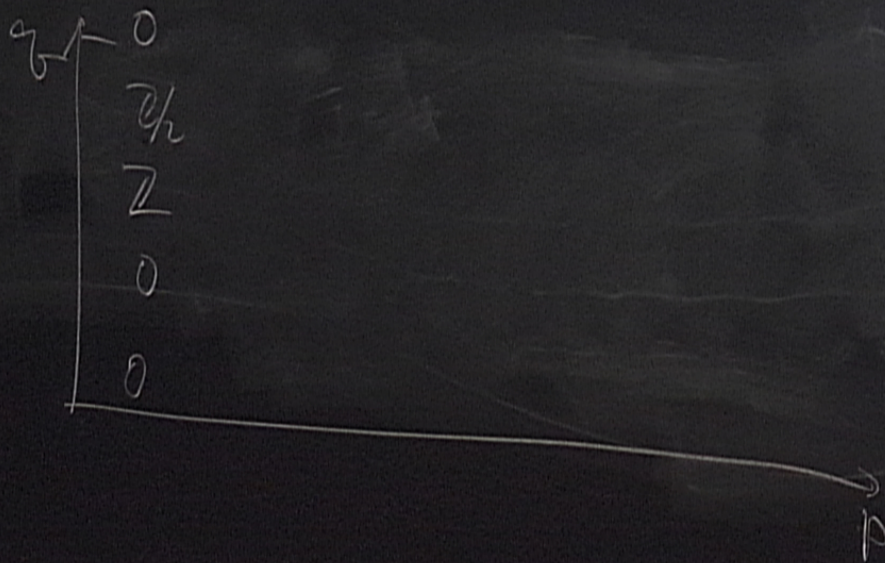

$E_1 =$
3
2
1
0
0
$\mathbb{Z}_2$ 0
0 $\mathbb{Z}_2$ 0
0 0 $\mathbb{Z}_2 \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z}$
0 1 2 3
p
$\mathbb{Z}_2 < F_1 < F_2 = H_2(TC)$
F_0
$F_1/F_0 = \mathbb{Z}_2$ $F_2/F_1 = \mathbb{Z}_2$
Filtered Horizontally
q
p


$E_1 =$
q
3
2
1
0
$\mathbb{Z}/2$
$\mathbb{Z}$
p
$\mathbb{Z}/2 \subset F_1 \subset F_2 = H_2(T^2)$
F_0
$F_1/F_0 = \mathbb{Z}/2$
$F_2/F_1 = \mathbb{Z}/2$

$\mathbb{Z}/2 \subset F_1 \subset F_2 = H_2(TC)$

F_0 $F_1/F_0 = \mathbb{Z}/2$ $F_2/F_1 = \mathbb{Z}/2$

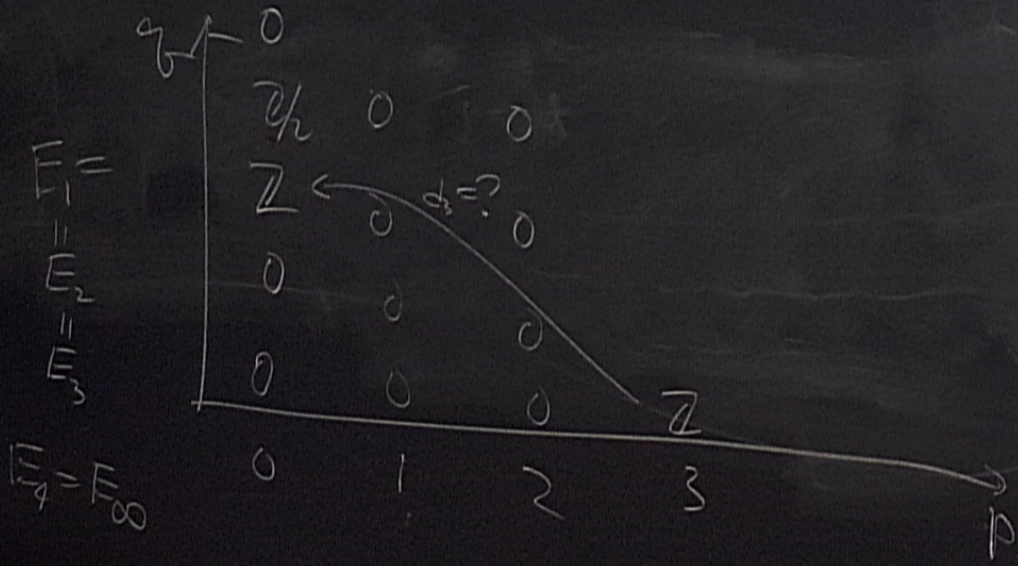
Another filtration



$\mathbb{Z}/2 \subset F_1 \subset F_2 = H_2(TC)$

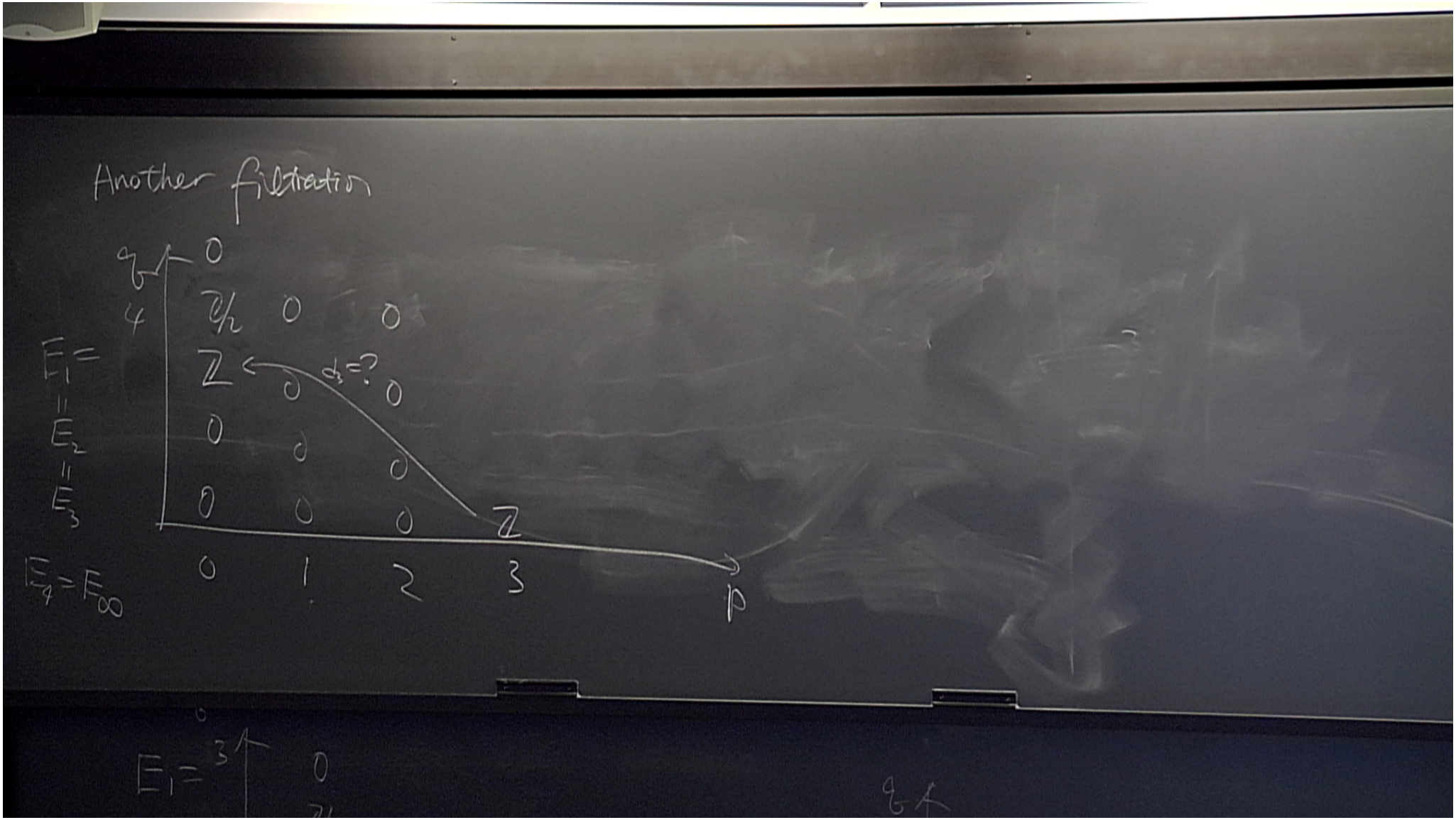
F_0 $\quad F_1/F_0 = \mathbb{Z}/2 \quad F_2/F_1 = \mathbb{Z}/2$

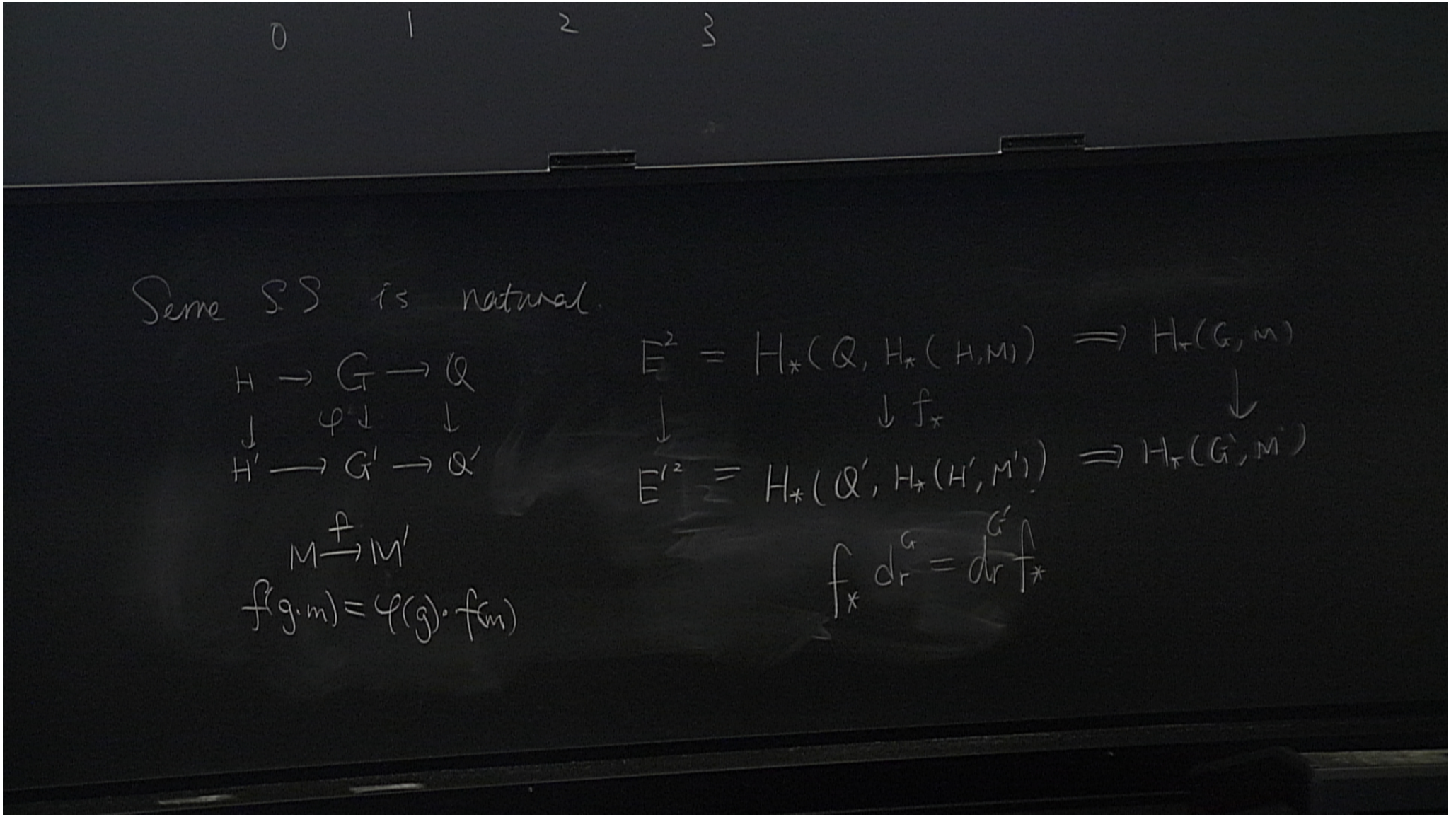
Another filtration

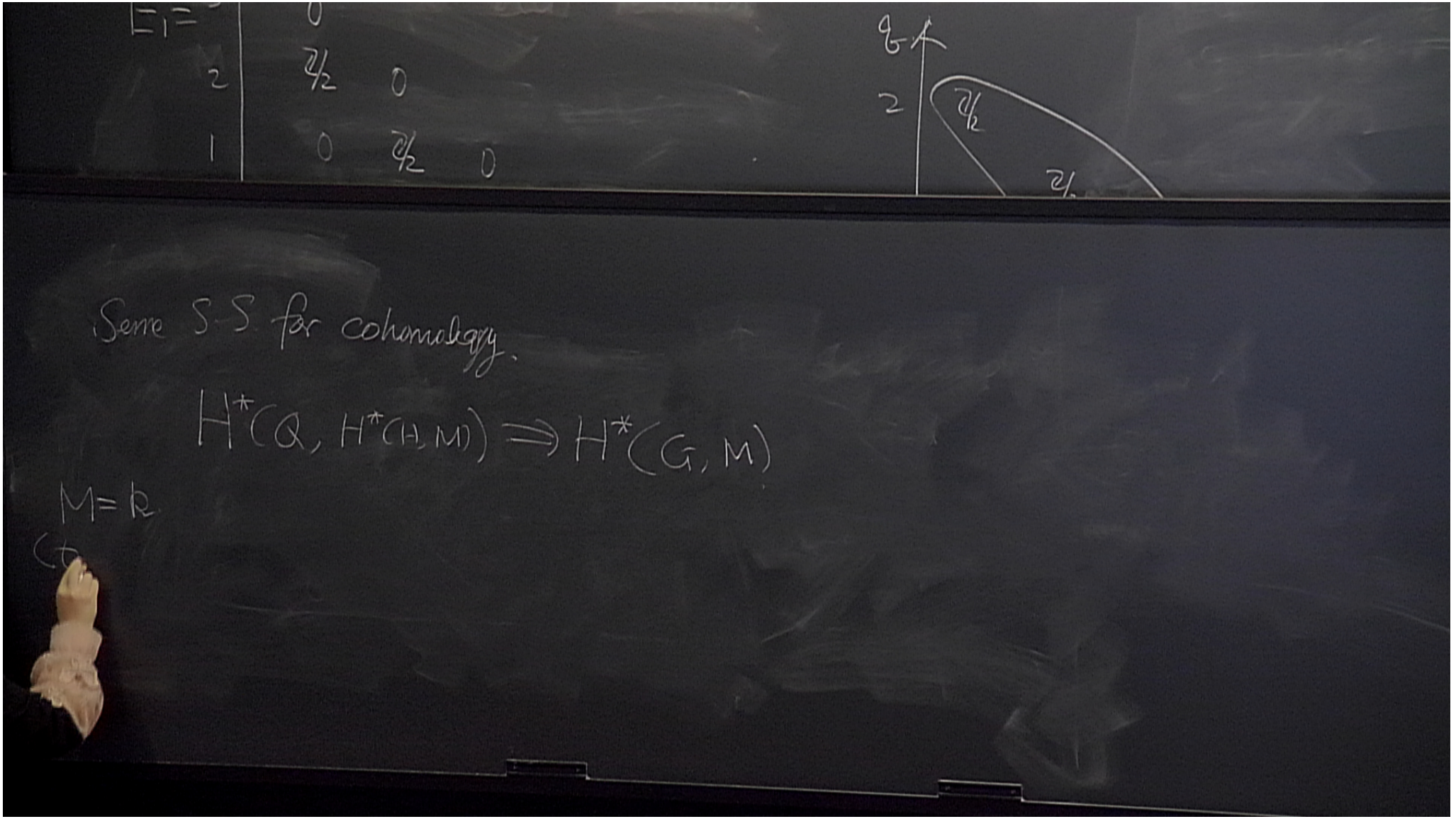


$E_1 = E_2 = E_3$

$E_4 = E_\infty$


Another filtration
q
4
0
Z/2 0 0
Z 0 0
0 0 0
0 0 0 Z
0 1 2 3
p
d3 = ?
E1 =
E2
E3
E4 = E∞
E1 =
3
0


0 1 2 3
Serre SS is natural.
$H \to G \to Q$
$H' \to G' \to Q'$
$M \xrightarrow{f} M'$
$f(g \cdot m) = \varphi(g) \cdot f(m)$
$E^2 = H_*(Q, H_*(H, M)) \Rightarrow H_*(G, M)$
$E'^2 = H_*(Q', H_*(H', M')) \Rightarrow H_*(G', M')$
$f_* d_r^G = d_r^{G'} f_*$



Serre S.S. for cohomology.

$$H^*(Q, H^*(H, M)) \Rightarrow H^*(G, M)$$

$M = \mathbb{R}$

$E_1 = \begin{array}{c|c} 3 & 0 \\ & \geq 1 \end{array}$

q

Serre S.S for cohomology.

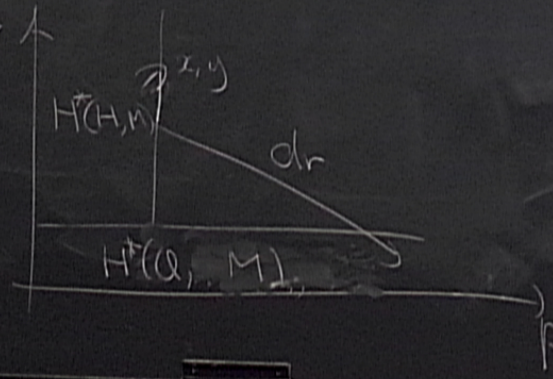
$$H^*(Q, H^*(H, M)) \Rightarrow H^*(G, M)$$

$M = k$

(trivial G-mod).

Assumption:

Central extension



$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{|x|} x \, d_r(y)$$

3

$k = id$

$M = \mathbb{F}_2 = R$

$\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$

$H^*(\mathbb{Z}/2, \mathbb{F}_2) = \mathbb{F}_2[x], \quad |x| = 1$

$H^p(\mathbb{Z}/2, H^q(\mathbb{Z}/2, \mathbb{F}_2)) \Longrightarrow H^{p+q}(\mathbb{Z}/4, \mathbb{F}_2)$

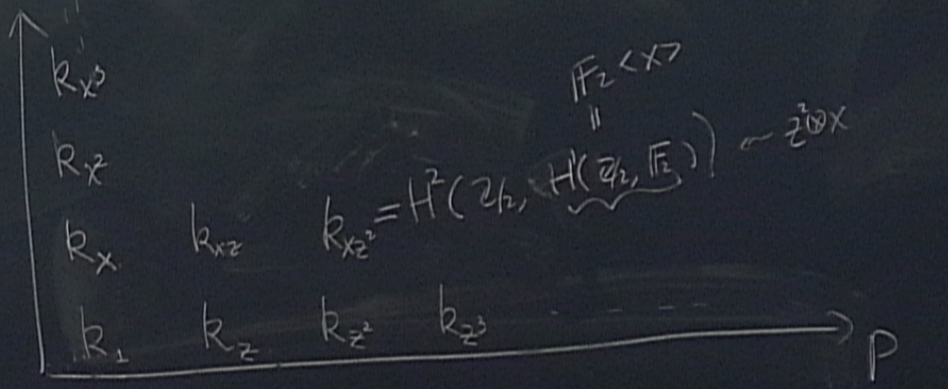
$H^p(\mathbb{Z}/2, \mathbb{F}_2[x])$

$=$

$H^p(\mathbb{Z}/2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$

$=$

$\mathbb{F}_2[z] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$



$M \to M'$

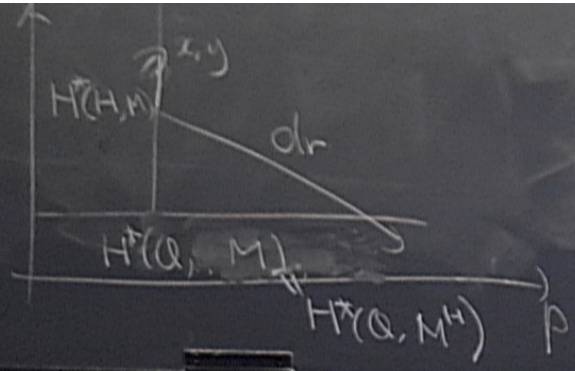
$f(g \cdot m) = \varphi(g) \cdot f(m)$

$f_* d_r^G = d_r^{G'} f_*$

(trivial G-module).

Assumption:

Central extension



$$d_r(x\cdot y) = d_r(x)\,y + (-1)^{|x|} x\, d_r(y)$$

Aside: for cohomology Serre S.S

$$E_r^{s,t} \times E_r^{s',t'} \longrightarrow E_r^{s+s',\,t+t'}$$

Compatible w/ differential d_r

$$d_r(x\cdot y) = d_r(x)\cdot y + (-1)^{s+t} x\, d_r(y)$$

$M = \mathbb{F}_2 = R$

$\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$

$H^*(\mathbb{Z}/2, \mathbb{F}_2) = \mathbb{F}_2[x], \quad |x| = 1$

$H^p(\mathbb{Z}/2, H^q(\mathbb{Z}/2, \mathbb{F}_2)) \Rightarrow H^{p+q}(\mathbb{Z}/4, \mathbb{F}_2)$

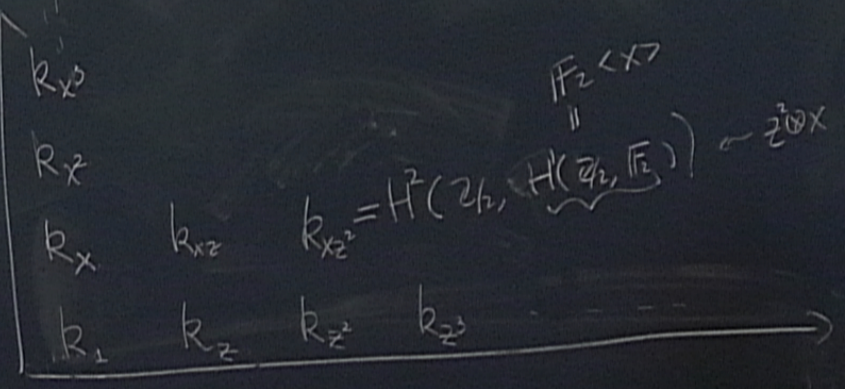
$H^p(\mathbb{Z}/2, \mathbb{F}_2[x])$

$=$

$H^p(\mathbb{Z}/2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$

$=$

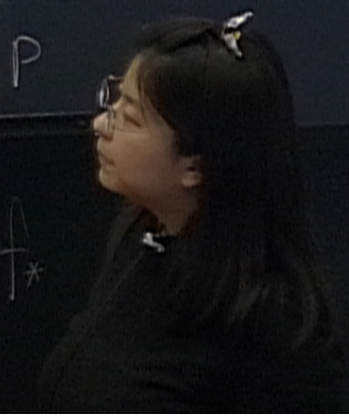
$\mathbb{F}_2[z] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$



$M \to M'$

$f(g \cdot m) = \varphi(g) \cdot f(m)$

$f_* d_r^G = d_r^G f_*$



$M = \mathbb{F}_2 = k$ $\quad \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$ ↷ classified by $z^2 \in H^2(\mathbb{Z}/2, \mathbb{F}_2)$ $\quad H^*(\mathbb{Z}/2, \mathbb{F}_2) = \mathbb{F}_2[x]$, $\quad |x| = 1$

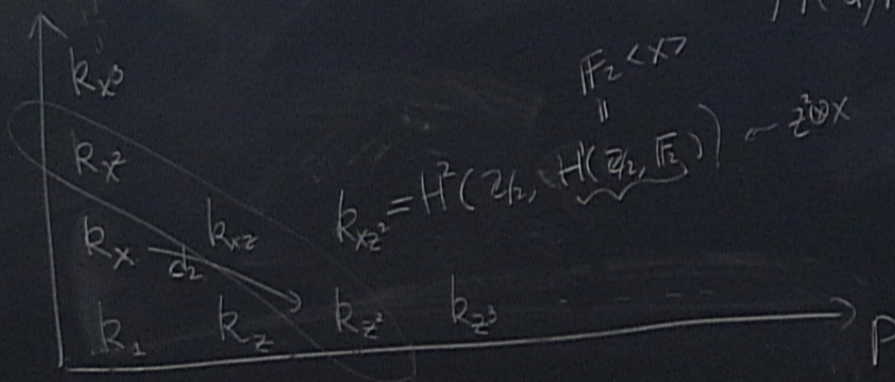
$H^p(\mathbb{Z}/2, H^q(\mathbb{Z}/2, \mathbb{F}_2)) \Rightarrow H^{p+q}(\mathbb{Z}/4, \mathbb{F}_2)$ $\qquad E_{2,0}^{\infty}$

$H^2(G, M) = F_0 \supset F_1 \supset F_2 = E_{2,0}^{\infty}$

$H^p(\mathbb{Z}/2, \mathbb{F}_2[x])$

$H^p(\mathbb{Z}/2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$

$\mathbb{F}_2[z] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$



$R_{xz^2} = H^2(\mathbb{Z}/2, H(\mathbb{Z}/2, \mathbb{F}_2))$ $\quad \mathbb{F}_2\langle x \rangle$ $\quad z^2 \otimes x$

$E_{2,0}^2 \twoheadrightarrow E_{2,0}^{\infty} \hookrightarrow H^2(G, M)$

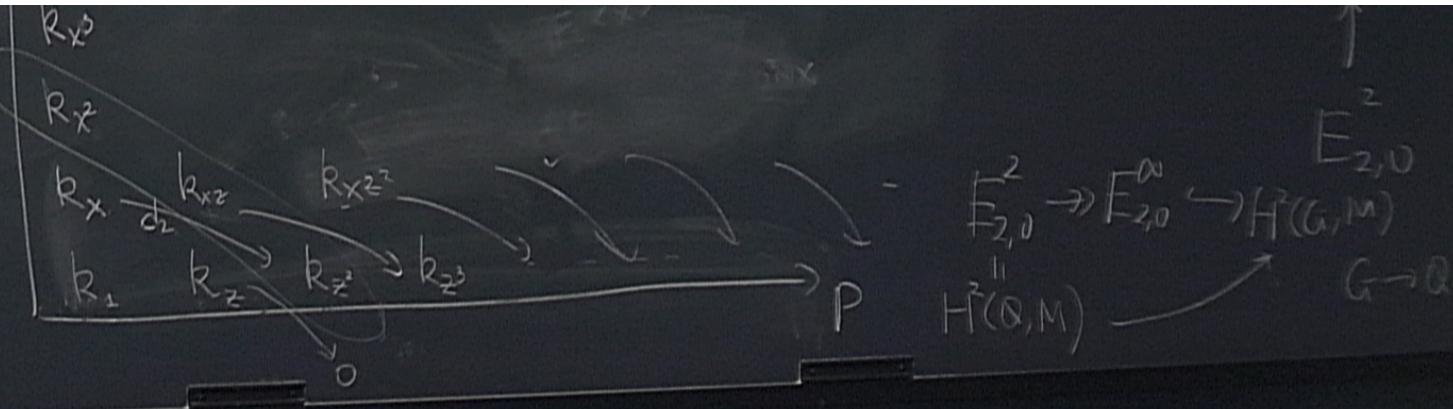
$H^p(\mathbb{Z}/2, \mathbb{F}_2[x])$

$=$

$H^p(\mathbb{Z}/2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$

$=$

$\mathbb{F}_2[z] \otimes_{\mathbb{F}_2} \mathbb{F}_2[x]$



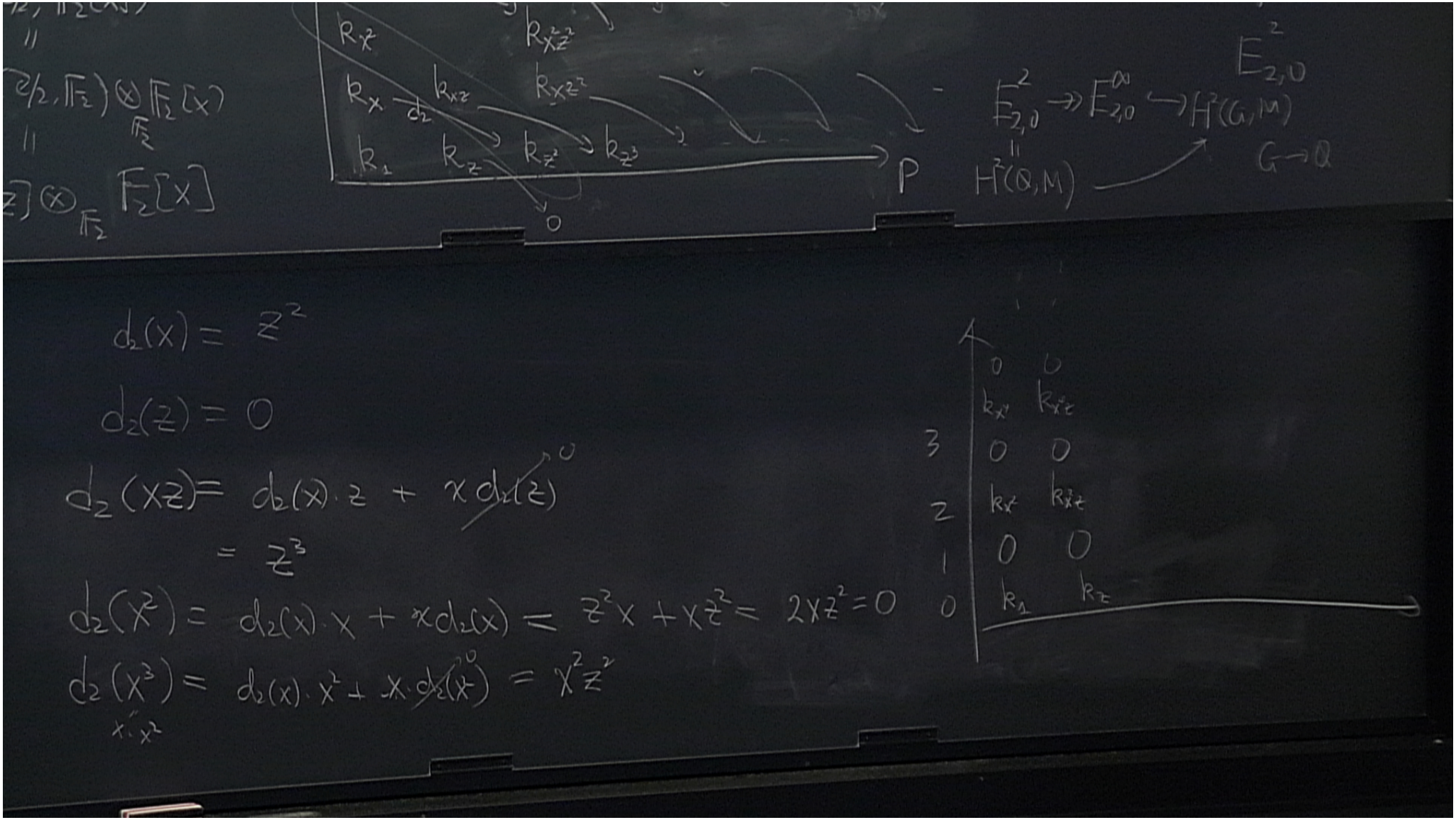
$d_2(x) = z^2$

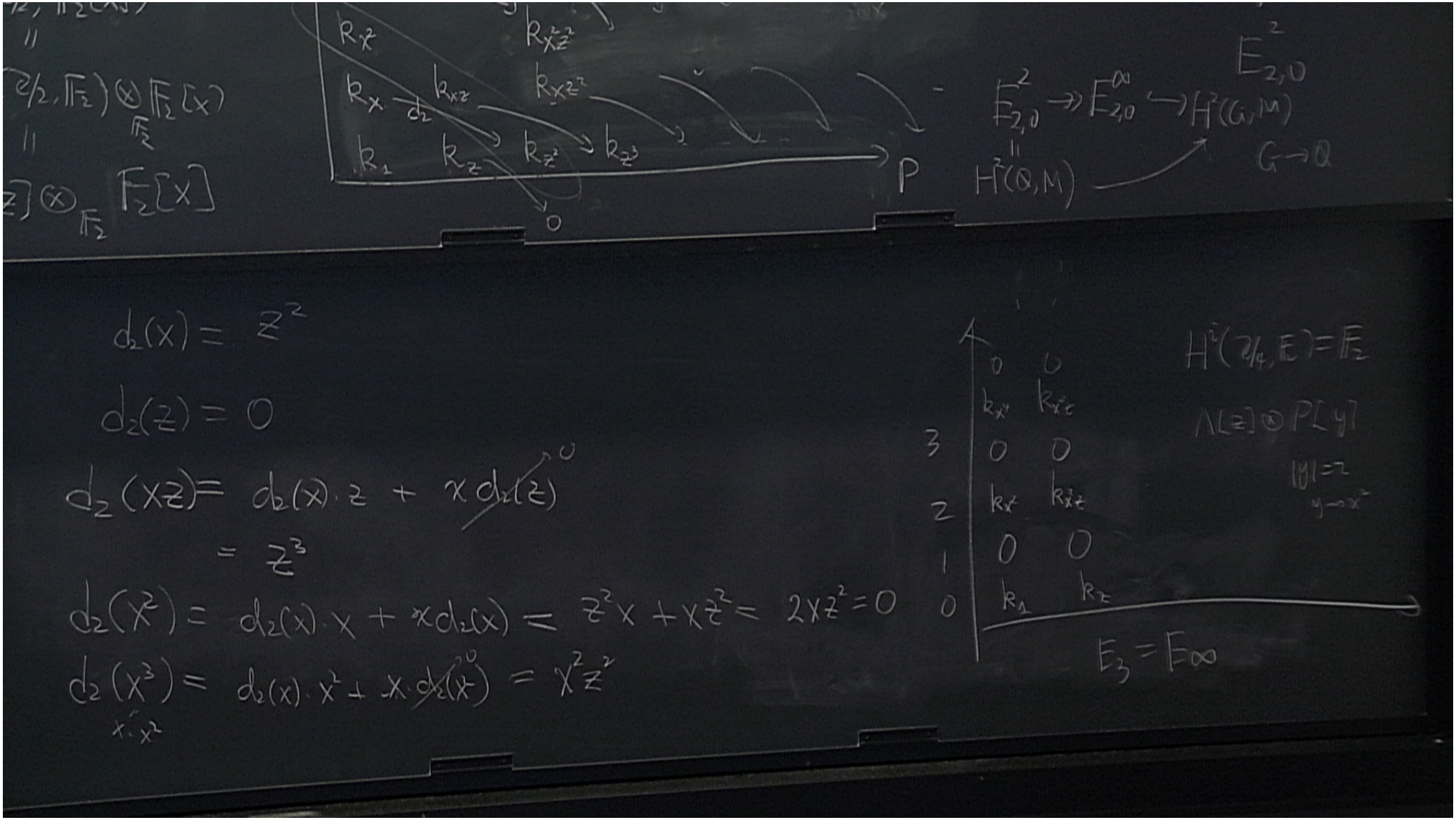
$d_2(z) = 0$

$d_2(xz) = d_2(x) \cdot z + x\, d_2(z)$

$= z^3$

$d_2(x^2) = d_2(x) \cdot x + x\, d_2(x) = z^2 x + x z^2 = 2xz^2 = 0$

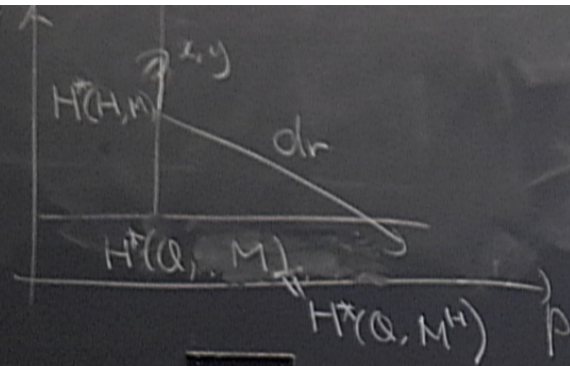

R_{x^2}
$R_{x^2z^2}$
R_x
R_{xz}
R_{xz^2}
d_2
R_1
R_z
R_{z^2}
R_{z^3}
0
P
$E_{2,0}^2$
$E_{2,0}^\infty$
$H^2(G,M)$
$E_{2,0}^2$
$H^2(Q,M)$
$G \to Q$
$d_2(x) = z^2$
$d_2(z) = 0$
$d_2(xz) = d_2(x)\cdot z + x\,d_2(z)$
$= z^3$
$d_2(x^2) = d_2(x)\cdot x + x\,d_2(x) = z^2x + xz^2 = 2xz^2 = 0$
$d_2(x^3) = d_2(x)\cdot x^2 + x\cdot d_2(x^2) = x^2z^2$
$x\cdot x^2$
3
2
1
0
k_{x^2}
R_{x^2z}
R_{x^3}
R_{x^3z}
R_1
R_z


$d_2(x) = z^2$
$d_2(z) = 0$
$d_2(xz) = d_2(x)\cdot z + x\,d_2(z) = z^3$
$d_2(x^2) = d_2(x)\cdot x + x\,d_2(x) = z^2x + xz^2 = 2xz^2 = 0$
$d_2(x^3) = d_2(x)\cdot x^2 + x\cdot d_2(x^2) = x^2z^2$
$x\cdot x^2$
$H^i(\mathbb{Z}/4, \mathbb{E}) = \mathbb{F}_2$
$\Lambda[z] \otimes P[y]$
$|y| = 2$
$y \to x^2$
$E_3 = E_\infty$
$H^2(G, M)$
$G \to Q$
$H^2(Q, M)$
$E_{2,0}^2$
$E_{2,0}^\infty$
$E_{2,0}$
R_x
R_{xz}
R_{xz^2}
R_{z^3}
k_{xz}
k_{xz^2}
k_{z^3}
d_2
k_1
k_z
k_{z^2}
P
0

(trivial G-module).

Assumption:

Central extension



$$d_r(x\cdot y) = d_r(x)\, y + (-1)^{|x|} x\, d_r(y)$$

D_8.

$$\mathbb{Z}/2 \to D_8 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \rightsquigarrow H^2(\mathbb{Z}/2\oplus\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{F}_2[x,y]$$

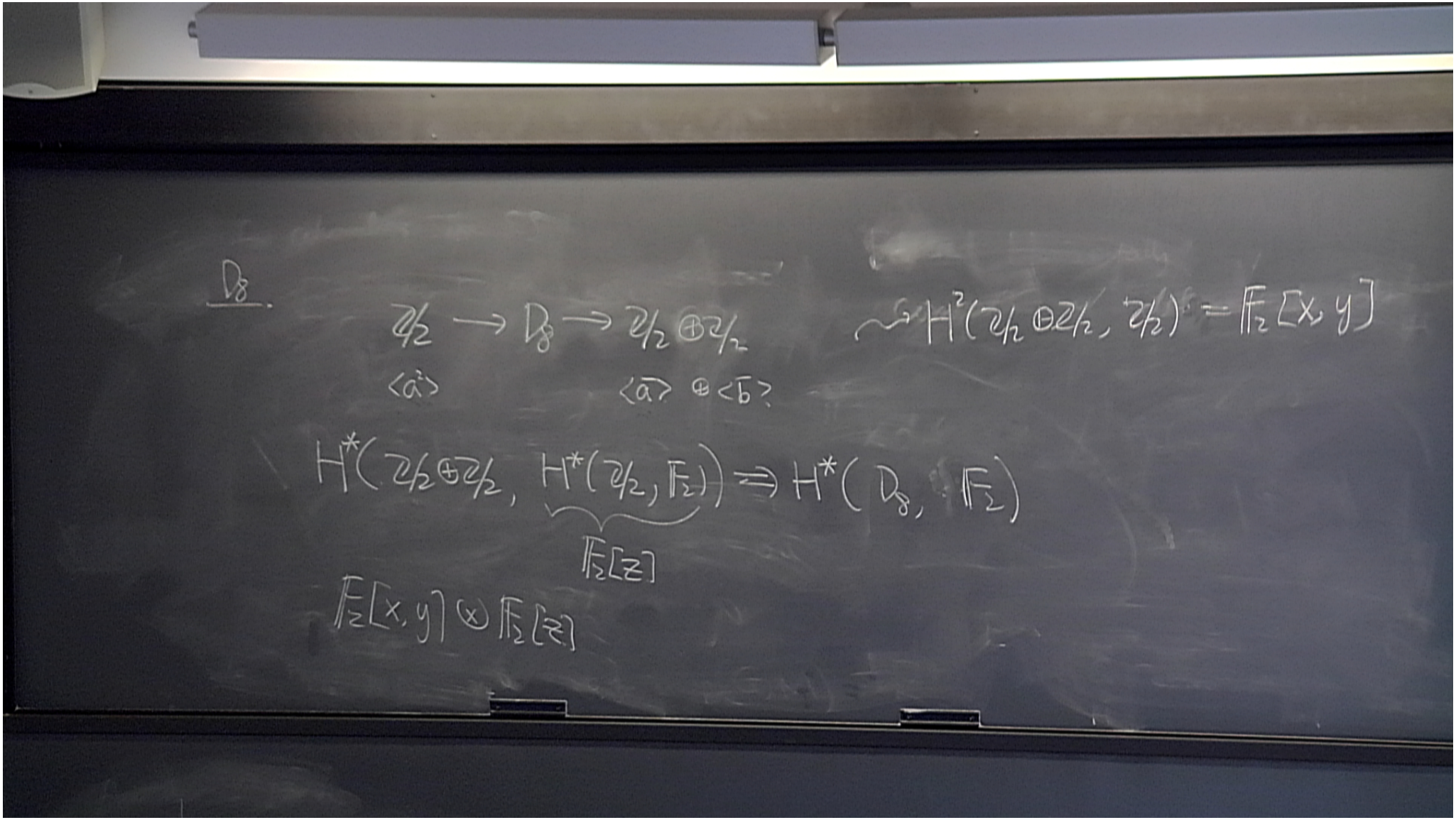
$\langle a^2 \rangle$ $\qquad$ $\langle \bar{a} \rangle \oplus \langle \bar{b} \rangle$

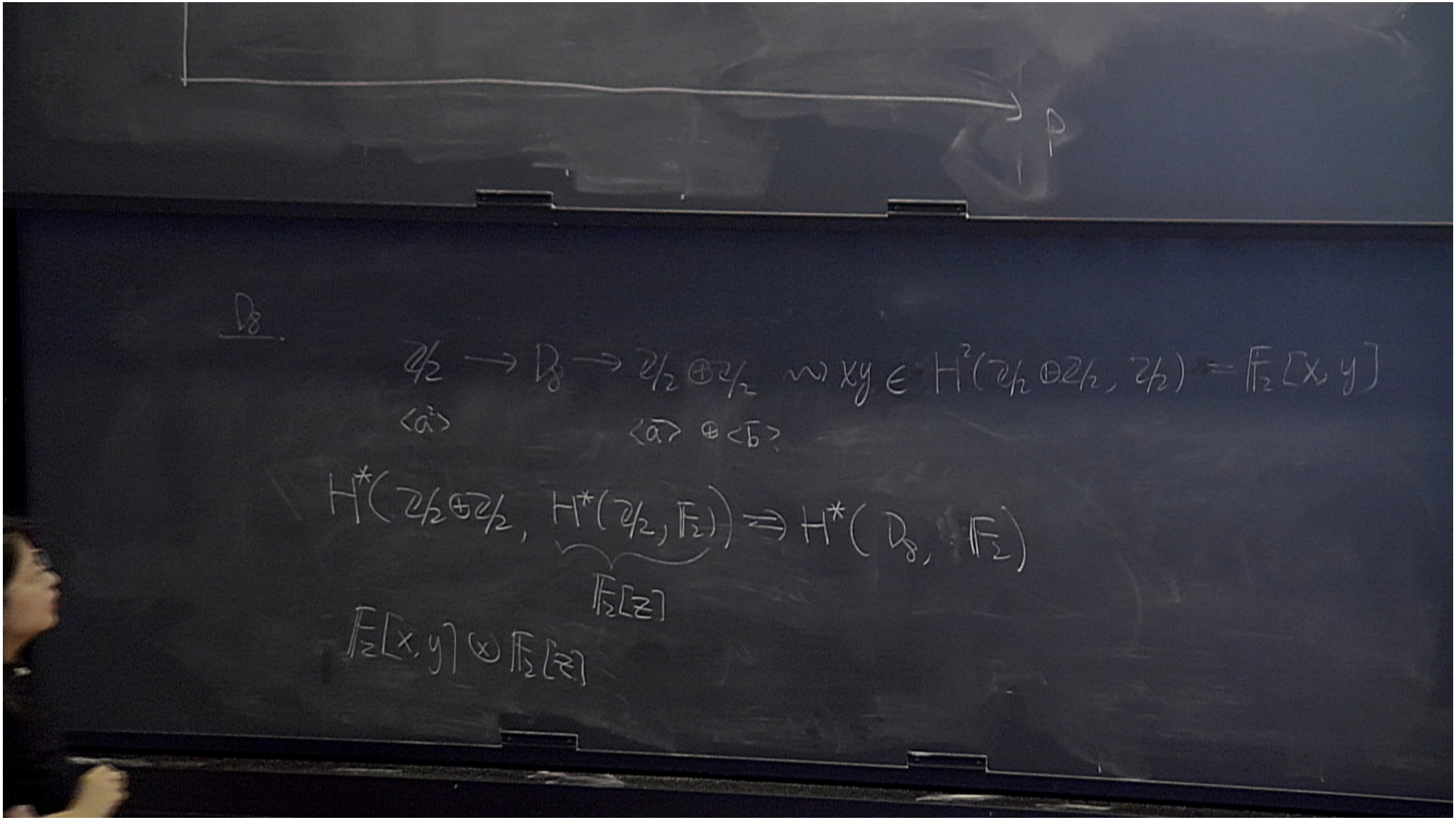
$$D_8 = \langle a, b \mid a^4 = b^2 = 1,\ aba = b \rangle$$

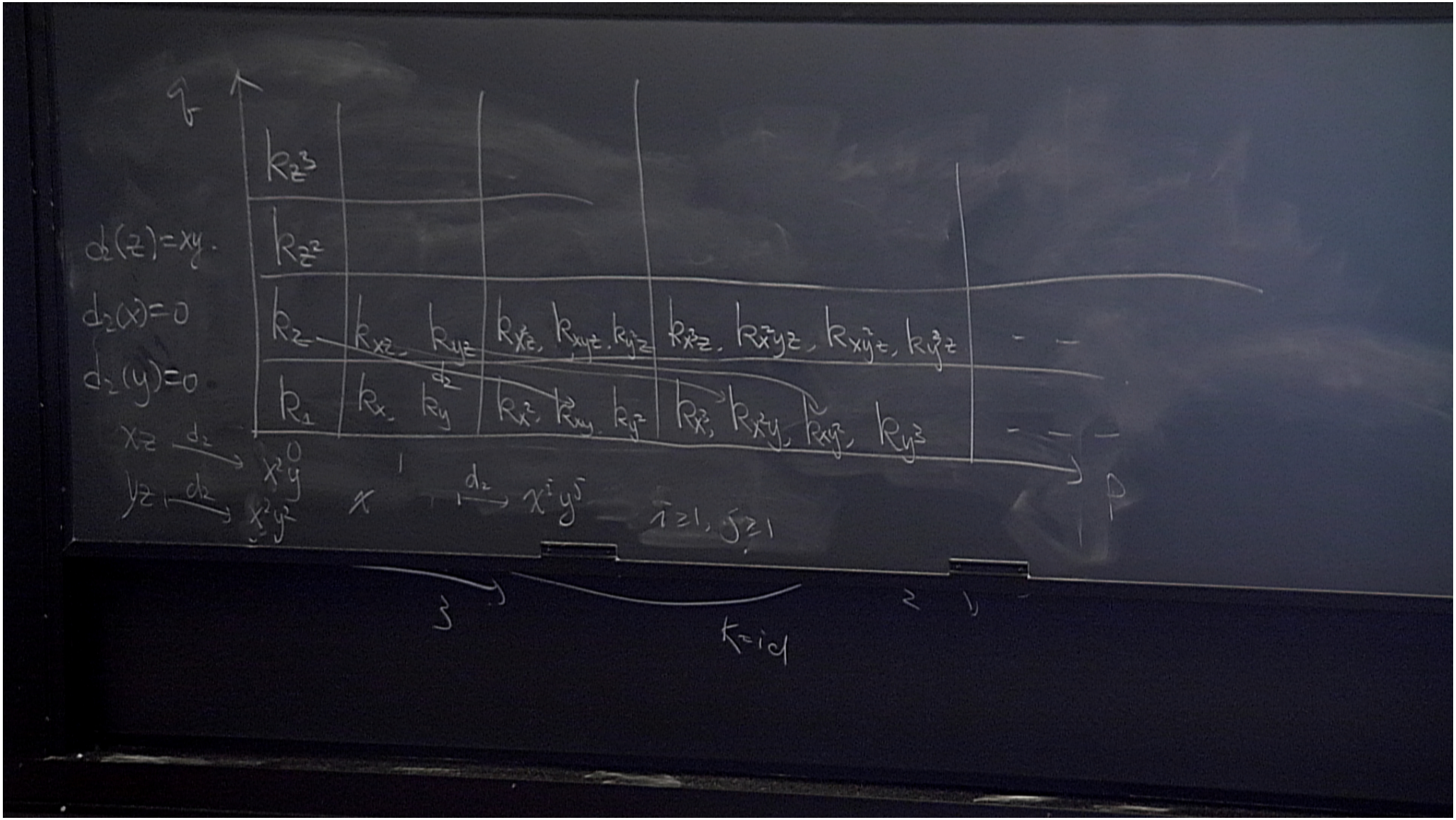
a^2 $\qquad$ $(\bar{a}\bar{b})^2 = 1$, $\bar{a}\bar{b} = \bar{b}\bar{a}$

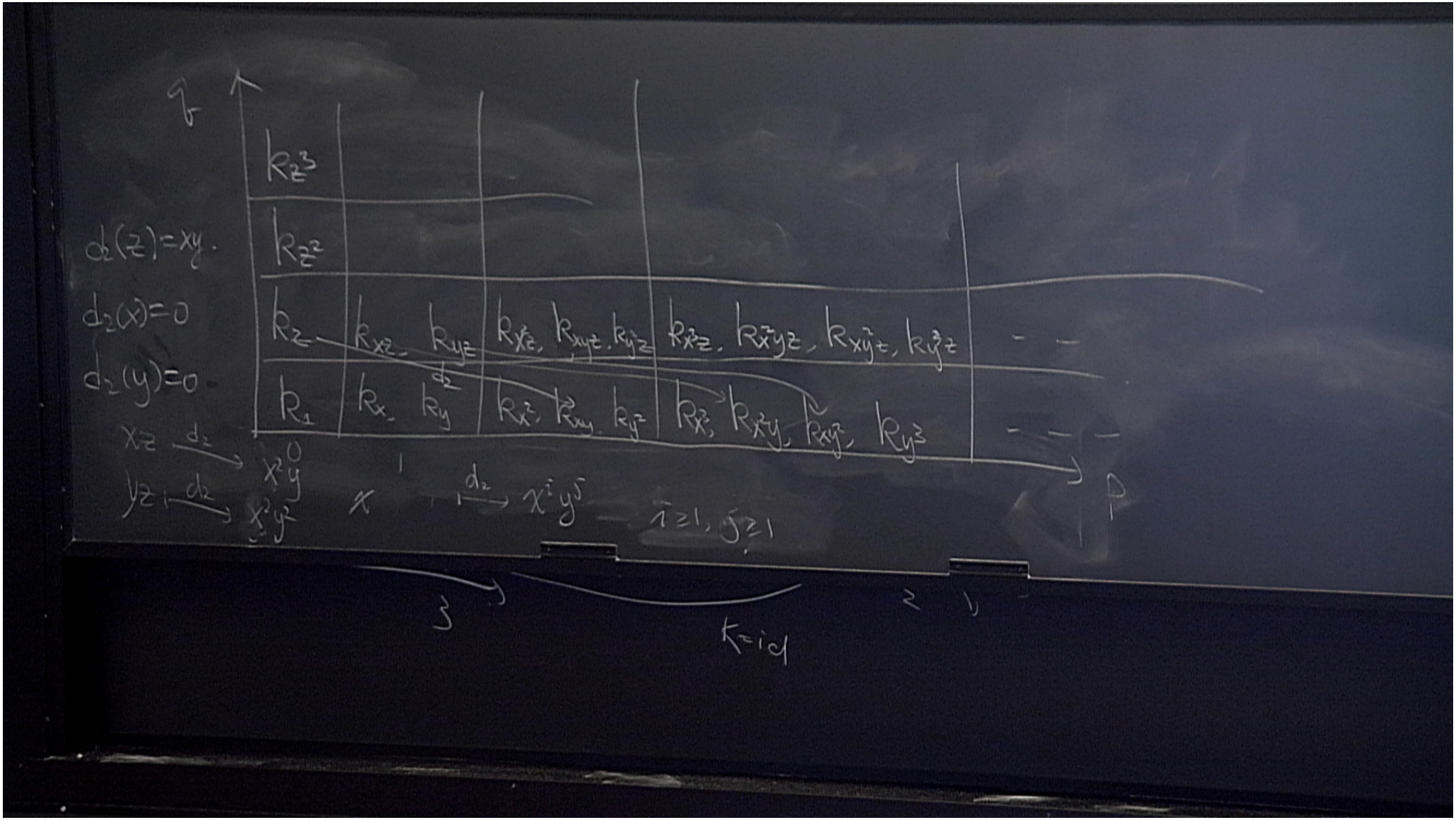
$$\mathbb{Z}/4 \to D_8 \to \mathbb{Z}/2$$

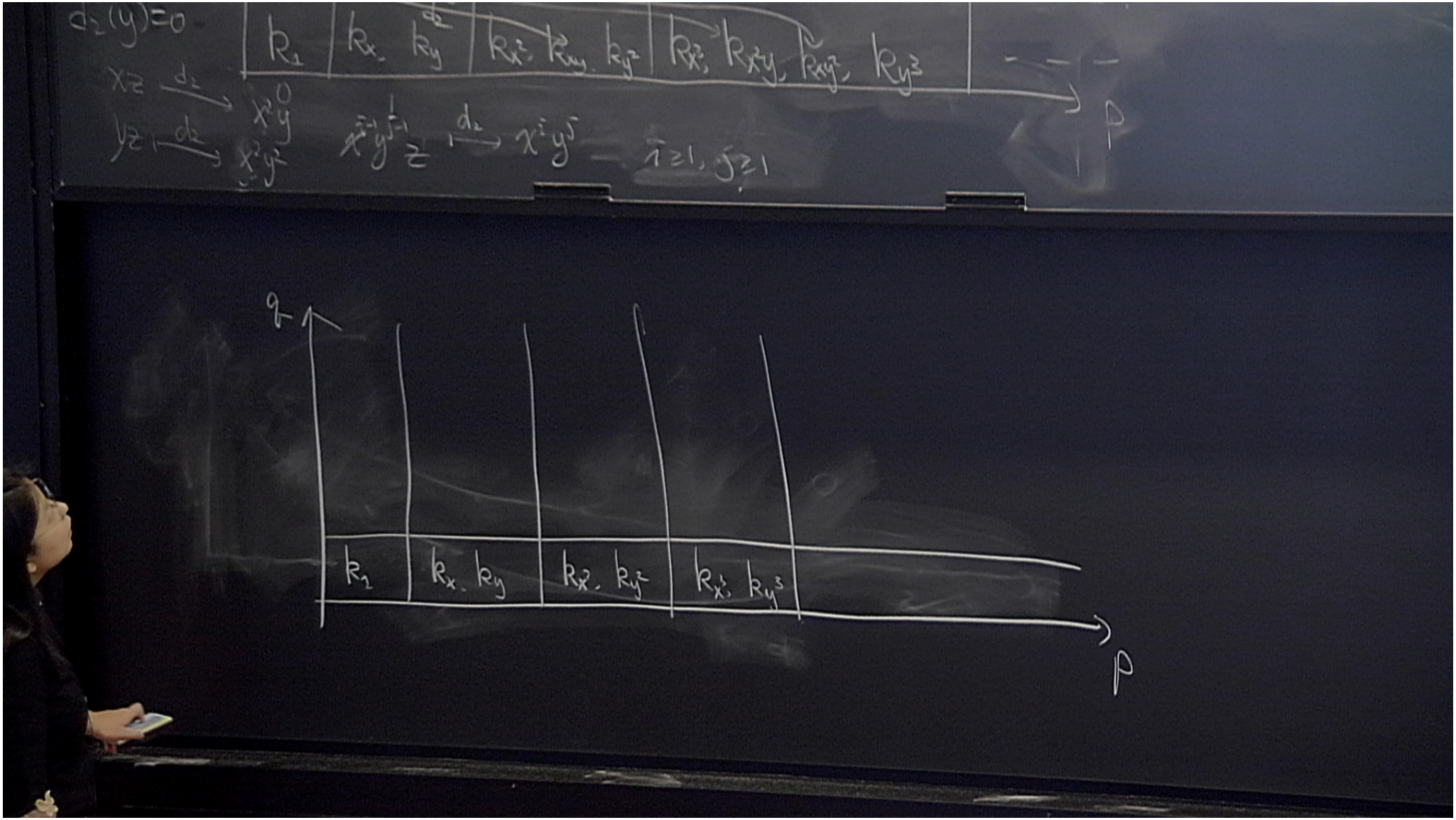
$\langle a \rangle \hookrightarrow D_8$ $\qquad$ $\langle \bar{b} \rangle$

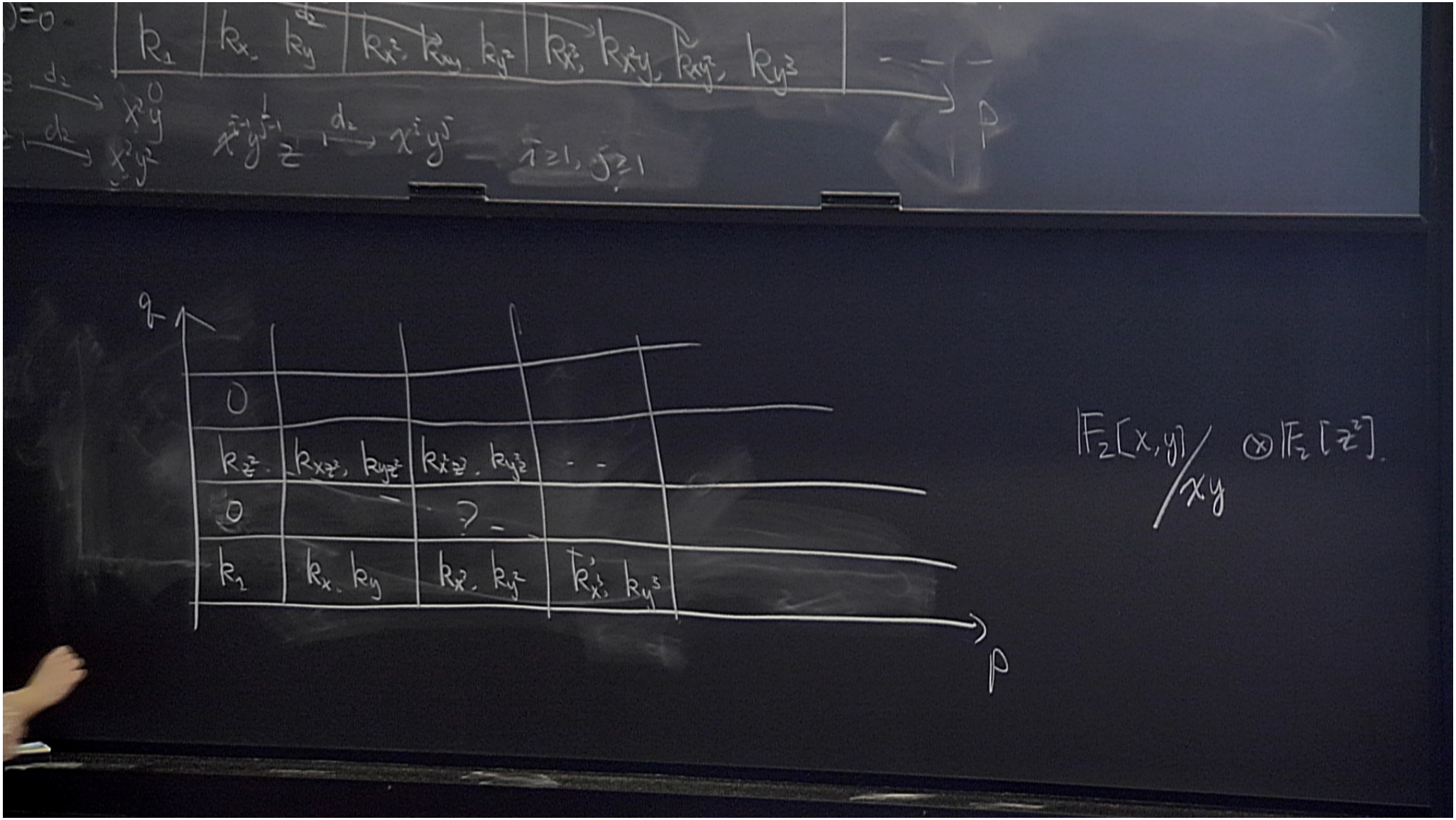

D_8.
$\mathbb{Z}/2 \to D_8 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\langle a^2 \rangle$
$\langle a \rangle \oplus \langle b \rangle$
$\rightsquigarrow H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{F}_2[x, y]$
$H^*(\mathbb{Z}/2 \oplus \mathbb{Z}/2, H^*(\mathbb{Z}/2, \mathbb{F}_2)) \Rightarrow H^*(D_8, \mathbb{F}_2)$
$\mathbb{F}_2[z]$
$\mathbb{F}_2[x, y] \otimes \mathbb{F}_2[z]$

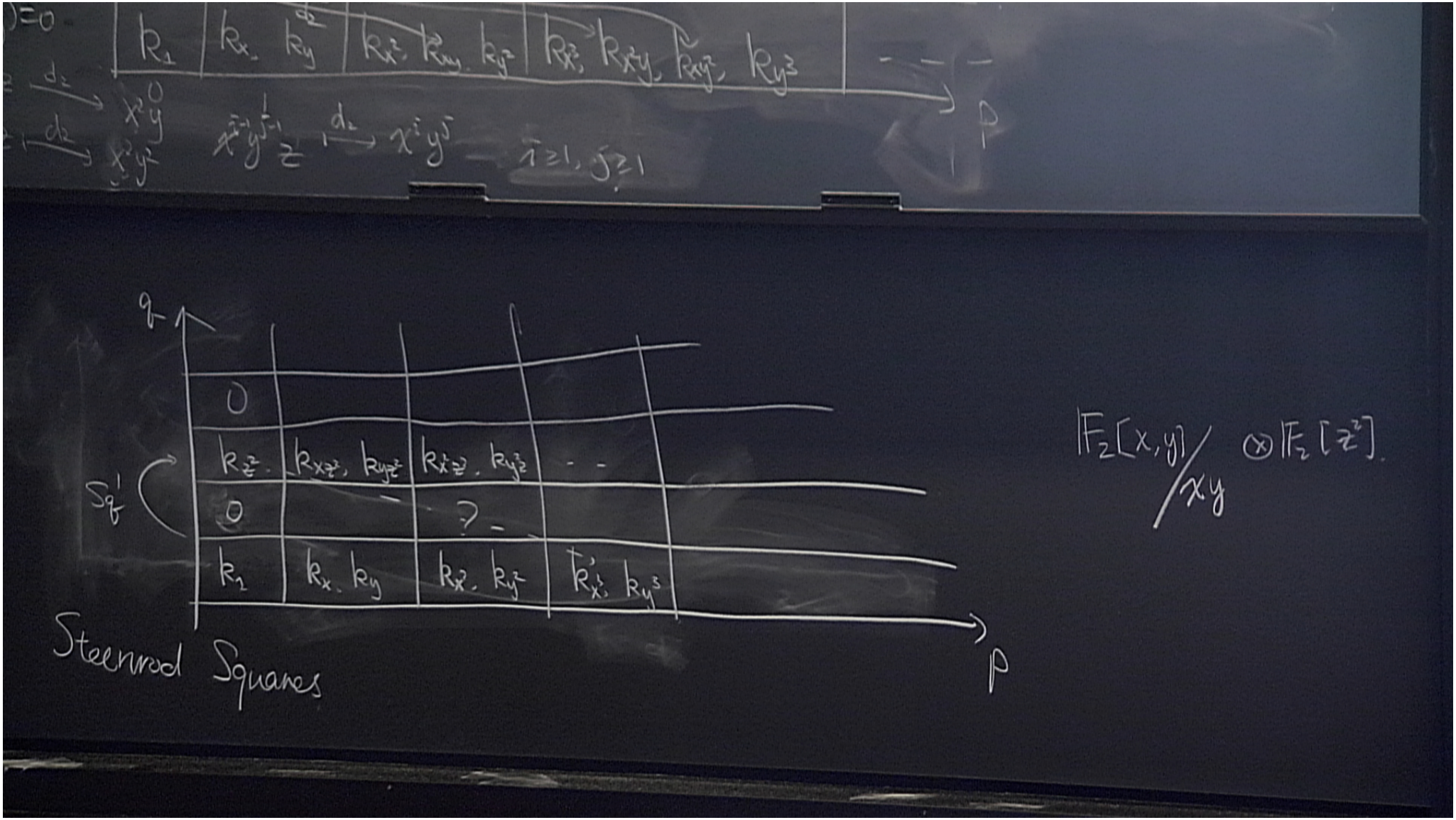

D8
$\mathbb{Z}_2 \to D_8 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightsquigarrow xy \in H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{F}_2[x,y]$
$\langle a \rangle$
$\langle a \rangle \oplus \langle b \rangle$
$H^*(\mathbb{Z}_2 \oplus \mathbb{Z}_2, H^*(\mathbb{Z}_2, \mathbb{F}_2)) \Rightarrow H^*(D_8, \mathbb{F}_2)$
$\mathbb{F}_2[z]$
$\mathbb{F}_2[x,y] \otimes \mathbb{F}_2[z]$


R_z^3
R_z^2
R_z
R_{xz} R_{yz}
R_{x^2z} R_{xyz} R_{y^2z}
R_{x^3z} R_{x^2yz} R_{xy^2z} R_{y^3z}
R_1
R_x R_y
R_{x^2} R_{xy} R_{y^2}
R_{x^3} R_{x^2y} R_{xy^2} R_{y^3}
$d_2(z)=xy$
$d_2(x)=0$
$d_2(y)=0$
$xz \xrightarrow{d_2} x^2y$
$yz \xrightarrow{d_2} xy^2$
$x^i \xrightarrow{d_2} x^{i-1}y$
$i=1, j=1$
d_2
d_2
$d_2


q
k_{z^3}
k_{z^2}
$d_2(z)=xy$
$d_2(x)=0$
$d_2(y)=0$
k_z k_{xz} k_{yz} k_{x^2z} k_{xyz} k_{y^2z} k_{x^3z} k_{x^2yz} k_{xy^2z} k_{y^3z}
k_1 k_x k_y k_{x^2} k_{xy} k_{y^2} k_{x^3} k_{x^2y} k_{xy^2} k_{y^3}
d_2
$xz \xrightarrow{d_2} x^2y$
$yz \xrightarrow{d_2} x^2y^2$
$x \xrightarrow{d_2} x^i y^j$
$i \geq 1, j \geq 1$
p
3
2
1
$k=id$


$d_2(y)=0$
k_1
k_x k_y
k_x^2 k_{xy} k_y^2
k_x^3 k_x^2y k_{xy^2} k_y^3
$x^2 \xrightarrow{d_2} x^2y^0$
$y^2 \xrightarrow{d_2} x^0y^2$
$x^{i-1}y^{j-1} z \xrightarrow{d_2} x^iy^j$
$i \geq 1, j \geq 1$
p
q
k_1
k_x k_y
k_x^2 k_y^2
k_x^3 k_y^3
p


k_1 | k_x k_y | k_x^2 k_{xy} k_y^2 | k_x^3 k_x^2y k_{xy^2} k_y^3
d_2
x^2y^0
d_2
x^2y^2
$x^{i-1}y^{j-1}$ z $\xrightarrow{d_2}$ x^iy^j $i \geq 1, j \geq 1$
p
q
0
k_{z^2} k_{xz^2} k_{yz^2} $k_x^2z^2$ $k_y^2z^2$
0 ?
k_1 k_x k_y k_x^2 k_y^2 k_x^3 k_y^3
p
$\mathbb{F}_2[x,y]/xy \otimes \mathbb{F}_2[z^2]$.


k_1 | k_x, k_y | k_x^2, k_{xy}, k_y^2 | k_x^3, k_x^2y, k_{xy^2}, k_y^3
x^0y^0
x^1y^0, x^0y^1
$x^{i-1}y^{j-1} \xrightarrow{d_2} x^iy^j$ $i \geq 1$, $j \geq 1$
p
q
0
k_{z^2}, k_{xz^2}, k_{yz^2}, $k_{x^2z^2}$, $k_{y^2z^2}$
0
?
k_1, k_x, k_y, k_{x^2}, k_{y^2}, k_{x^3}', k_{y^3}
S_q^1
Steenrod Squares
p
$\mathbb{F}_2[x,y]/xy \otimes \mathbb{F}_2[z^2]$.

Steenrod Squares / Power Operations

$(p=2)$

$Sq^0, Sq^1, Sq^2, Sq^3, \cdots$

$Sq^i : H^n(X, \mathbb{F}_2) \longrightarrow H^{n+i}(X, \mathbb{F}_2)$ $\quad i \geq 0$, $\forall n$.

↑ Cohomology operations (↑ spaces)

satisfying ① $Sq^0 = id : H^n(X, \mathbb{F}_2) \longrightarrow H^n(X, \mathbb{F}_2)$

② $Sq^1 = \beta$ ← Bockstein operations associated to $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$

$\mathbb{Z}/2$

Cohomology operations

Satisfying ① $Sq^0 = id : H^n(X, \mathbb{F}_2) \to H^n(X, \mathbb{F}_2)$

② $Sq^1 = \beta$ ← Bockstein operation associated to $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$

L.e.s

$$\cdots \to H^n(X, \mathbb{Z}/2) \to H^n(X, \mathbb{Z}/4) \to H^n(X, \mathbb{Z}/2) \xrightarrow{\beta} H^{n+1}(X, \mathbb{Z}/2) \to \cdots$$

③ $|x| = n$, $Sq^n x = x^2 (= x \cup x)$

④ If $i > |x|$, $Sq^i x = 0$

⑤ Cartan's formula $Sq^n(x \cdot y) = \sum_{i=0}^{n} Sq^i(x)\, Sq^{n-i}(y)$ ← $x \cup y$

} ⇒ existence & uniqueness of Sq^i

Steenrod Squares / Power Operations

$(p=2)$

$Sq^0, Sq^1, Sq^2, Sq^3, \cdots$

$Sq^i : H^n(X, \mathbb{F}_2) \to H^{n+i}(X, \mathbb{F}_2)$ $\quad i \geq 0$, $\forall n$.

spaces

Cohomology operations

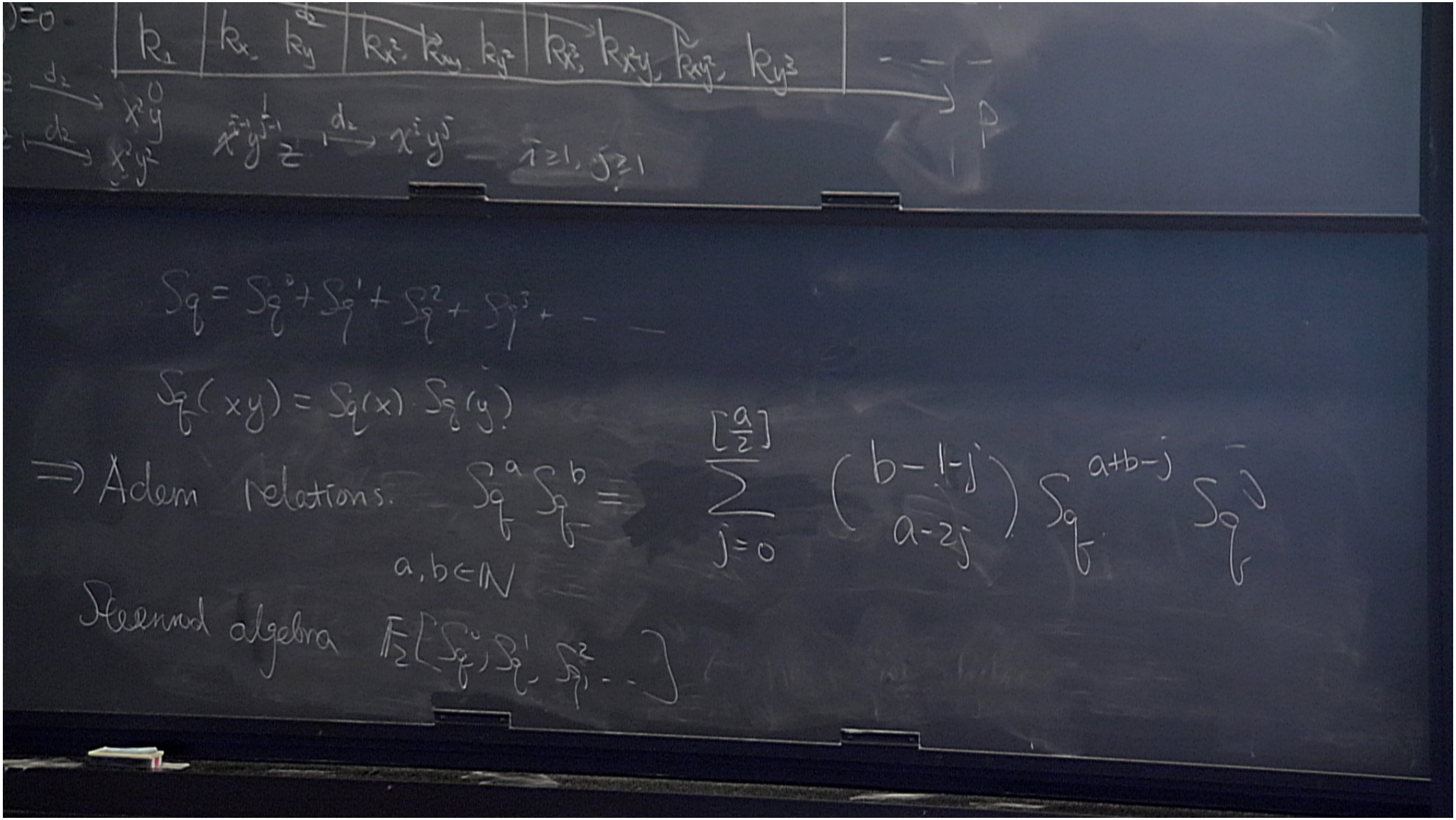
natural : $X \xrightarrow{f} Y$

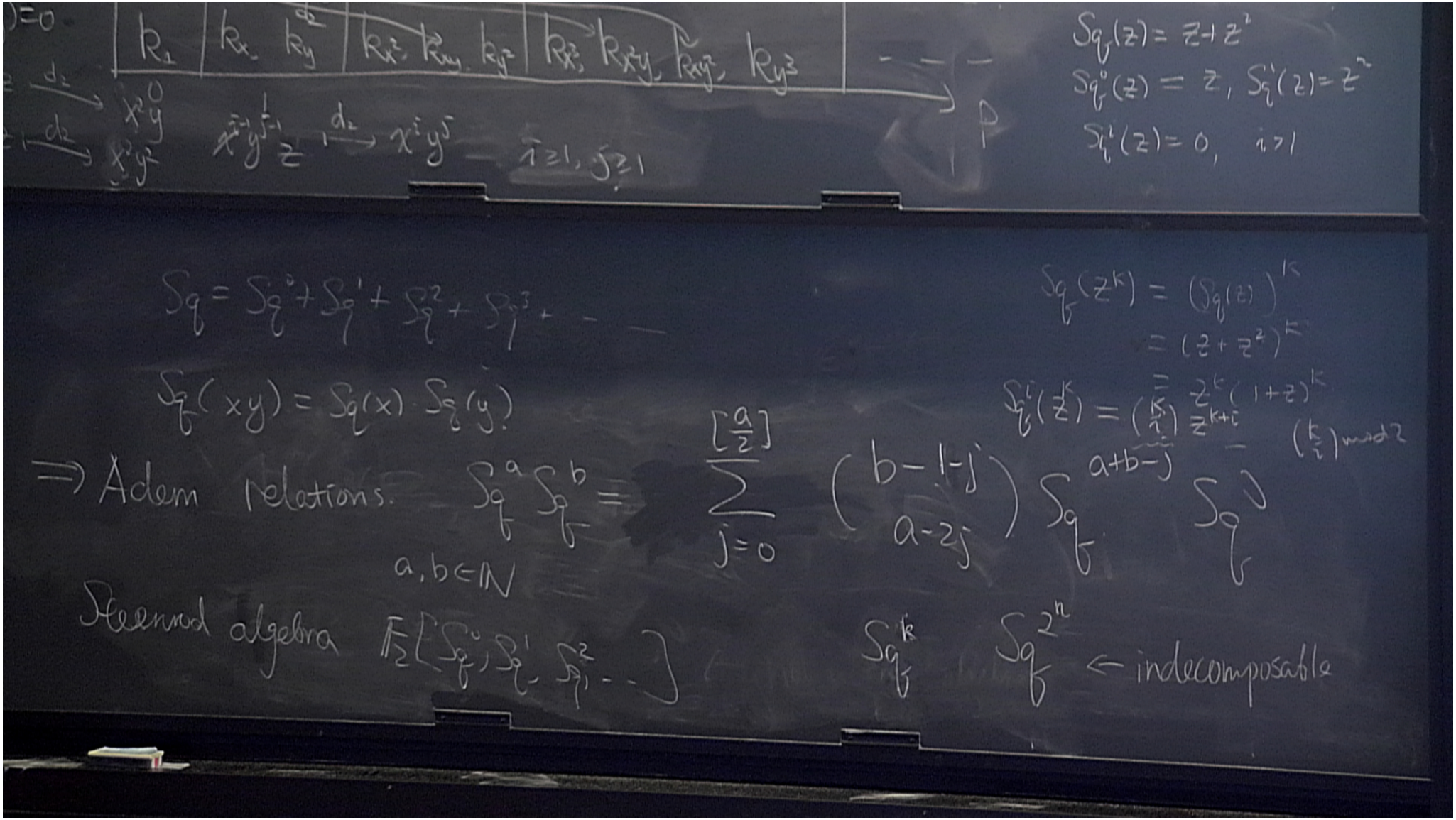
$$\begin{array}{ccc} H^n(X,\mathbb{F}_2) & \xrightarrow{Sq^i} & H^{n+i}(X,\mathbb{F}_2) \\ f^* \uparrow & & \uparrow f^* \\ H^n(Y,\mathbb{F}_2) & \xrightarrow{Sq^i} & H^{n+i}(Y,\mathbb{F}_2) \end{array}$$

satisfying ① $Sq^0 = id : H^n(X, \mathbb{F}_2) \to H^n(X, \mathbb{F}_2)$

② $Sq^1 = \beta$ ← Bockstein operation associated to $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2$

③ $|x| = n$, $Sq^n x = x^2 (= x \cup x)$ $\Rightarrow$ existence & uniqueness


k_1 | k_x k_y | k_{x^2} k_{xy} k_{y^2} | k_{x^3} k_{x^2y} k_{xy^2} k_{y^3} | - - -
d_2
x^0y^0
d_2
x^0y^1
$x^{i-1}y^{j-1}$ $\xrightarrow{d_2}$ x^iy^j $i \geq 1, j \geq 1$
P
$Sq = Sq^0 + Sq^1 + Sq^2 + Sq^3 + \cdots$
$Sq(xy) = Sq(x) \cdot Sq(y)$
$\Rightarrow$ Adem relations. $Sq^a Sq^b = \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$
$a, b \in \mathbb{N}$
Steenrod algebra $\mathbb{F}_2[Sq^0, Sq^1, Sq^2, \ldots]$


k_1 | k_x k_y | k_{x^2} k_{xy} k_{y^2} | k_{x^3} k_{x^2y} k_{xy^2} k_{y^3}
d_2
d_2
x^0y^0
x^0y^1
$x^{i-1}y^{j-1}$
d_2
x^iy^j
$i \geq 1, j \geq 1$
$Sq(z) = z + z^2$
$Sq^0(z) = z, \; Sq^1(z) = z^2$
$Sq^i(z) = 0, \quad i > 1$
$Sq = Sq^0 + Sq^1 + Sq^2 + Sq^3 + \cdots$
$Sq(xy) = Sq(x) \cdot Sq(y)$
$Sq(z^k) = (Sq(z))^k$
$= (z + z^2)^k$
$= z^k(1+z)^k$
$Sq^i(z^k) = \binom{k}{i} z^{k+i}$
$\binom{k}{i} \bmod 2$
$\Rightarrow$ Adem relations. $Sq^a Sq^b = \sum_{j=0}^{[\frac{a}{2}]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$
$a, b \in \mathbb{N}$
Steenrod algebra $\mathbb{F}_2[Sq^0, Sq^1, Sq^2, \ldots]$
Sq^k
Sq^{2^n} $\leftarrow$ indecomposable