Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 7

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Collection: The Cohomology of Groups (Johnson-Freyd/Guo)

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$$1 \to H \to G \to Q \to 1.$$

- homology. M, G-module.

$$E^2_{p,q} = H_p(Q, H_q(H, M)) \Rightarrow H_{p+q}(G, M)$$

- Cohomology.

$$E_2^{p,q} = H^p(Q, H^q(H, M)) \Rightarrow H^{p+q}(G, M)$$

Leray – Lyndon – Hochschild – Serre Spectral Sequence

A homological (graded) spectral sequence

$$\{E^r_{p,q}\}_{r \geq 0,\, r \in \mathbb{Z}}$$

$E^r_{p,q}$ is R-module $\qquad (R = \mathbb{Z})$

(E^r, E^r-page)

w/ differentials. $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$

$(d^r \circ d^r = 0)$

filtration deg $= p \longrightarrow p-r$

total deg $= p+q \longrightarrow p+q-1$

st $E^{r+1} = H(E^r, d^r)$

$$E^{r+1}_{p,q} = \frac{\left(\ker E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}\right)}{\left(\operatorname{Im} E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q}\right)}$$

$$E^r_{p-r,q+r-1} \xleftarrow{d^r} E^r_{p,q} \xleftarrow{d^r} E^r_{p+r,q-r+1}$$

A cohomological S.S. $\{E_r^{p,q}\}_{r\geq 0}$

w/ $d_r: E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$

filtration deg: $p \longrightarrow p+r$

total deg $p+q \longrightarrow p+q+1$

- exact couples
- filtrations.

> two common method to construct S.S.

onstruct S.S.

Exact couples D, E, R-module.

$$\begin{tikzcd}
D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

exact at each vertex

$\ker j = \operatorname{im} i$; $\ker k = \operatorname{im} j$; $\ker i = \operatorname{im} k$.

$D \supset D' = i(D)$ ← image of D under i.

$d = jk : E \to E$. check $d^2 = 0$ $d^2 = j\underbrace{kj}_{0}k = 0$.

$E' = H(E, d)$

Claim.

$$\begin{tikzcd}
D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\
& E' \arrow[ul, "k'"] \arrow[ul, bend left=40, "k(e)"] &
\end{tikzcd}$$

$E' \ni [e]$ $e + de'$

is exact couple

i' restriction of i to D'

$j' : \underset{\rotatebox{90}{$\in$}\, D'}{i(d)} \longmapsto [j(d)] \in H(E, d)$

repeat

$$\begin{tikzcd} D^n \arrow[rr, "i^n"] & & D^n \arrow[dl, "j^n"] \\ & E^n \arrow[ul, "k^n"] & \end{tikzcd}$$

$D_{p,q} \xrightarrow{i} D_{p+1, q-1}$

$D_{p,q} \longrightarrow D_{p,q}$

$\rightsquigarrow E^{n+1} = H(E^n, d^n)$, $d^n = j^n k^n : E^n \longrightarrow E^n$.

Let D, E be <u>bigraded</u> R-module.

$\left(D_{p,q} \text{ R-module}, \quad D_{p,q} \times D_{p',q'} \longrightarrow D_{p+p', q+q'} \right]$

give gradings on morphism

$\deg i = (1, -1)$

$\deg j = (0, 0)$

$\deg k = (-1, 0)$

$\deg d' = \deg jk = (-1, 0)$

$d' : E'_{p,q} \longrightarrow E'_{p-1,q}$

A

✓ c

f

to

– exa

– fi

repeat

$$\begin{tikzcd} D^n \arrow[rr, "i^n"] & & D^n \arrow[dl, "j^n"] \\ & E^n \arrow[ul, "k^n"] & \end{tikzcd}$$

$D_{p,q} \xrightarrow{i} D_{p+1,q-1}$

$D_{p,q} \longrightarrow D_{p,q}$

$\rightsquigarrow E^{n+1} = H(E^n, d^n)$, $d^n = j^n k^n : E^n \longrightarrow E^n$.

Let D, E be <u>bigraded</u> R-module.

[$D_{p,q}$ R-module, $D_{p,q} \times D_{p',q'} \longrightarrow D_{p+p',q+q'}$]

give gradings on morphism

deg $i = (1,-1)$

deg $j = (0,0)$

deg $k = (-1,0)$

deg $d^1 = $ deg $jk = (-1,0)$

$d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$

$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$

$H_{n+1}(G,\mathbb{Z}/2) \to H_n(G,\mathbb{Z}) \to H_n(G,\mathbb{Z}) \to H_n(G,\mathbb{Z}/2) \to H_{n-1}(G,\mathbb{Z}) \to H_{n-1}(G,\mathbb{Z}) \to \cdots$

A cohomological ... S.S.

w/ $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,}$

filtration deg : $p \longrightarrow p$

total deg $p+q \longrightarrow$

- exact complex
- filtrations.

$>$ two

$$\begin{tikzcd} \text{Exact couples} \arrow[r, leftrightarrow] \arrow[d, squiggly] & \text{filtration} \arrow[dl, squiggly] \\ \text{S.S} & \end{tikzcd}$$

Filtration on chain complex $C_\cdot$

- <u>increasing</u> filtration

$\{F_p C\}_{p\in\mathbb{Z}}$ & $F_p C$ is a subcomplex of $C_\cdot$

$F_p C \subseteq F_{p+1} C \quad \forall p$ $\quad (F_p C)_n$ is a submodule of C_n

$$\begin{tikzcd} F_p C_{n+1} \arrow[r, hook] \arrow[d, "\partial"'] & C_{n+1} \arrow[d, "\partial"] \\ F_p C_n \arrow[r, hook] & C_n \end{tikzcd}$$

$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$

$(C,\mathbb{Z}) \to H_n(C,\mathbb{Z}) \to H_n(C,\mathbb{Z}/2) \to$

$\cdots \subseteq F_{-1}C_\cdot \subseteq F_0C_\cdot \subseteq F_1C \subseteq F_2C \subseteq \cdots F_pC \subseteq F_{p+1}C$

$0 \to D_\cdot \hookrightarrow C_\cdot \to C_\cdot/D_\cdot \to 0$ s.e.s of chain complex

$\rightsquigarrow$ l.e.s $\to H_n(D_\cdot) \to H_n(C_\cdot) \to H_n(C_\cdot/D_\cdot) \to H_{n-1}(D_\cdot) \to \cdots$

$F_1C \subseteq F_2C \subseteq \underline{F_3C} \subseteq F_4C = C$

$F_3C \to C \to Q_3$

$F_2C \to F_3C \to F_3C/F_2C$

$F_1C \to F_2C \to F_2C/F_1C$

A c

$\checkmark$ dr

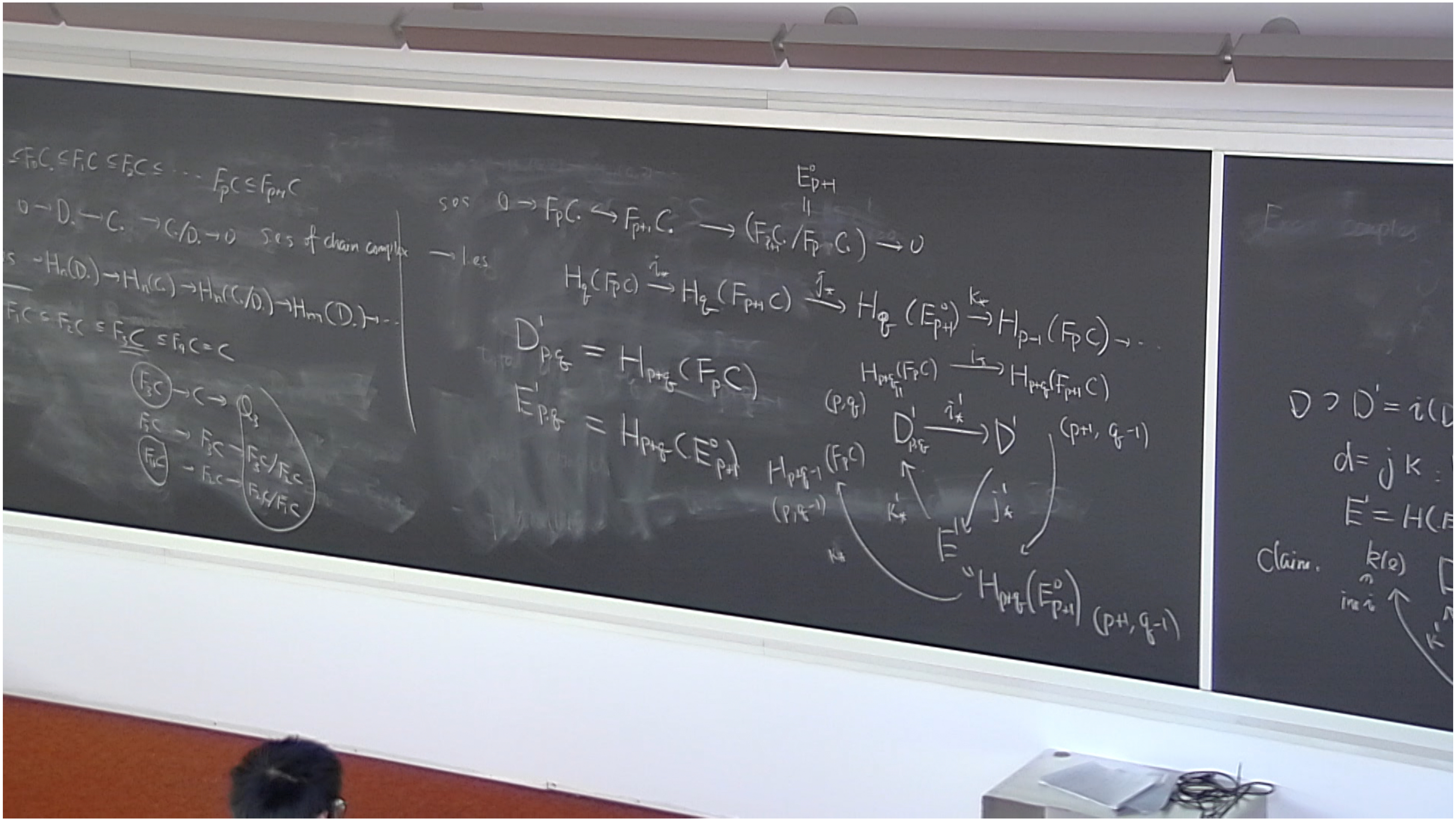
fil

total

– exact

– filt

C_n


F_p C ⊆ F_{p+1} C
0 → D. → C. → C./D. → 0 s.e.s of chain complex → l.e.s
→ H_n(D.) → H_n(C.) → H_n(C./D.) → H_{n-1}(D.) → ...
F_1 C ⊆ F_2 C ⊆ F_3 C ⊆ F_4 C = C
ses 0 → F_p C. ↪ F_{p+1} C. → (F_{p+1} C. / F_p C.) → 0
E^0_{p+1}
H_q(F_p C) → H_q(F_{p+1} C) → H_q(E^0_{p+1}) → H_{q-1}(F_p C) → ...
D^1_{p,q} = H_{p+q}(F_p C)
E^1_{p,q} = H_{p+q}(E^0_{p+1})
(p,q)
(p+1, q-1)
H_{p+q-1}(F_p C)
(p, q-1)
H_{p+q}(E^0_{p+1}) (p+1, q-1)
D ⊃ D' = i(D
d = j k
E' = H(E
Claim. k(e)
im i

Filtration on $C_\bullet$ $\rightsquigarrow$ filtration on $H_*(C_\bullet)$

$$F_p H_* C_\bullet = \mathrm{Im}\left(H_*(F_p C) \to H_*(C) \right)$$
$$\cap\!\!|$$
$$F_{p+1} H_* C = \mathrm{Im}\left(H_*(F_{p+1} C) \to H_*(C) \right).$$

<u>Thm</u> $C = \bigcup_p F_p C$ & $\forall n$, $\exists s(n)$ s.t. $F_{s(n)} C_n = 0$.

$$0 = F_{s(n)} C \subseteq \cdots \subseteq F_{p-1} C \subseteq F_{d(n)} C \subseteq C_n \qquad \forall n$$

$$0 = F_{s(n)} C \cdots \subseteq F_{d(n)} C \subseteq C_n$$

Claim

$$\text{Exact couples} \longleftrightarrow \text{filtration}$$

$$\downarrow$$

$$\text{S.S}$$

- $\{F_p M\}$ $\quad F_p M \subseteq F_{p+1} M$

associated graded module $(\mathrm{Gr}\, M)$

$$(\mathrm{Gr}\, M)_p = F_p M / F_{p-1} M$$

- $\{F_p C_\bullet\}$

$$(\mathrm{Gr}\, C_\bullet)_{p,q} = \left(F_p C / F_{p-1} C\right)_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$$

$H_*(C))$

$\to H_*(C))$

0

$\forall n$

$$E^\infty_{pq} = F_p H_{p+q} C_\cdot \big/ F_{p-1} H_{p+q} C_\cdot$$

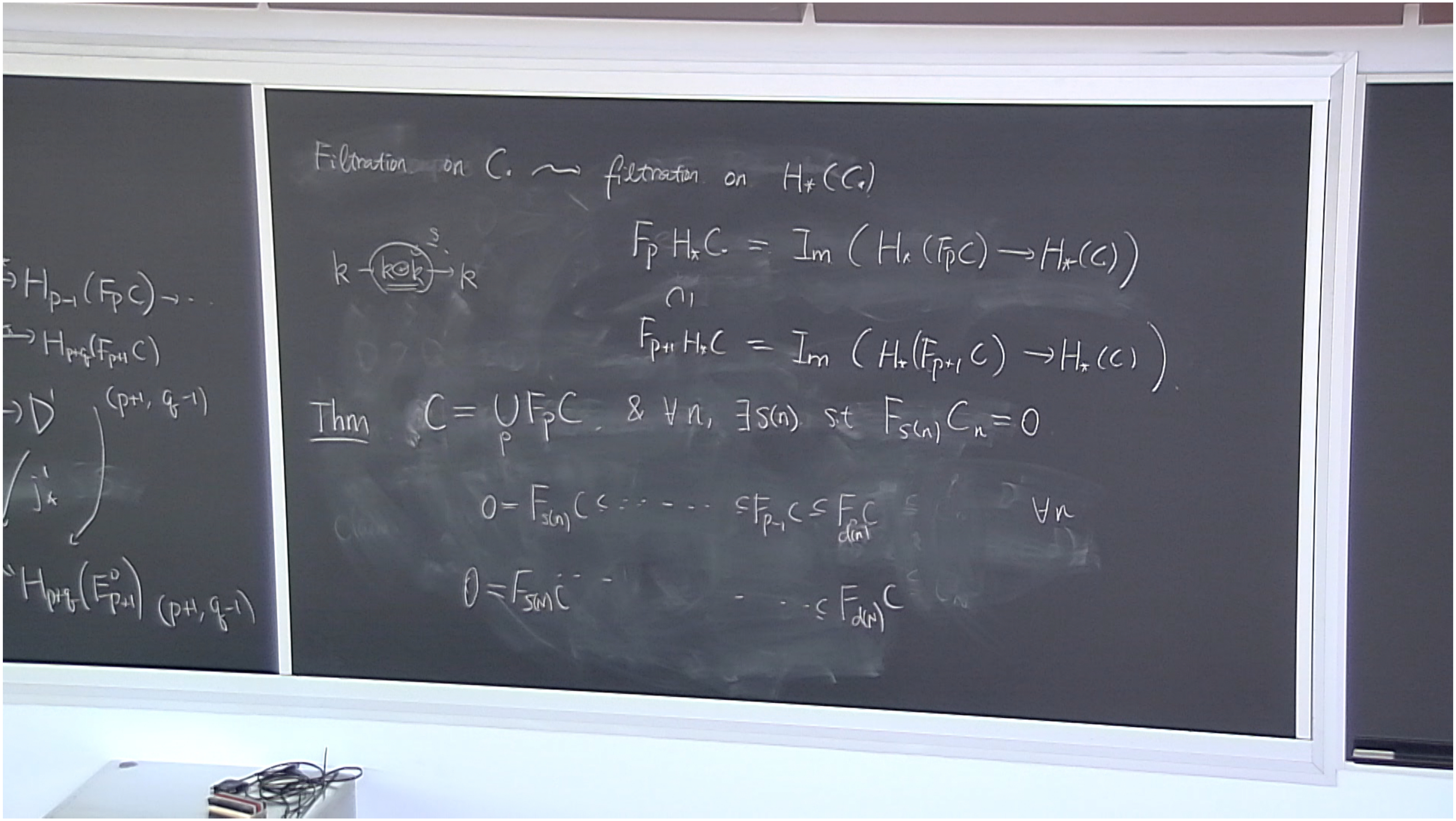
$$\left(E^r_{p,q} \Longrightarrow H_{p+q} C_\cdot \right) \qquad \mathrm{Gr}_\cdot(H_* C)$$

$$\begin{array}{ccccccc} & & \mathbb{Z}/2 \; F_1 & & \mathbb{Z}/2 \; F_2 & & \\ & & \| & & \| & & \\ F_0 = 0 & \subset & A & \subset & B & = & D \\ & & \| & & & & \\ & & \mathbb{Z}/2 & & B/A = \mathbb{Z}/2 & & \\ & & \| & & & & \\ & & F_1/F_0 & & & & \end{array}$$

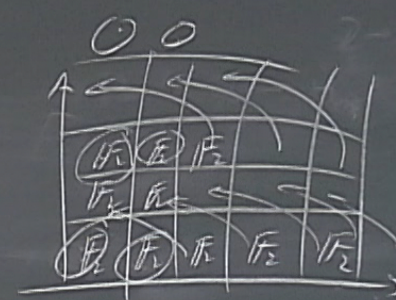
$$\rightsquigarrow \quad \mathbb{Z}/2 \to ? \to \mathbb{Z}/2$$

$\mathbb{Z}/4$

$\mathbb{Z}/2 \oplus \mathbb{Z}/2$


$\to H_{p-1}(F_p C) \to \cdots$
$\to H_{p+q}(F_{p+1} C)$
$\to D^1_{(p+1,\, q-1)}$
j_*^{-1}
$H_{p+q}(E^0_{p+1})$ $(p+1, q-1)$
Filtration on $C_\cdot$ $\rightsquigarrow$ filtration on $H_*(C_\cdot)$
$k \to k \otimes k \to k$
$F_p H_* C_\cdot = \mathrm{Im}\left(H_*(F_p C) \to H_*(C)\right)$
$\cap$
$F_{p+1} H_* C = \mathrm{Im}\left(H_*(F_{p+1} C) \to H_*(C)\right)$
Thm $C = \bigcup_p F_p C$ & $\forall n, \exists s(n)$ s.t. $F_{s(n)} C_n = 0$
$0 = F_{s(n)} C \subseteq \cdots \subseteq F_{p-1} C \subseteq F_{d(n)} C \subseteq \cdots \quad \forall n$
$0 = F_{s(n)} C \cdots \subseteq F_{d(n)} C$

$$\frac{F_p C_{p+q}}{F_{p-1} C_{p+q}}$$

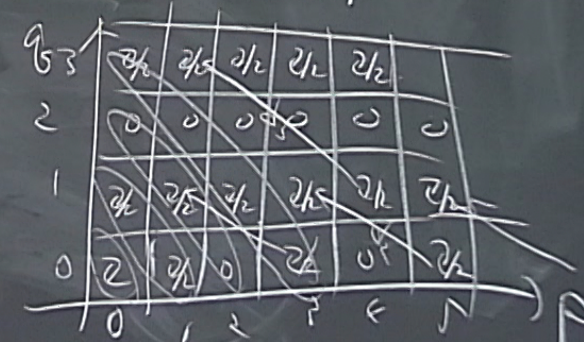


$$C_2 \to C_4 \to C_2$$

$$\mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2$$

$$E_2 = H_p(\mathbb{Z}_2, H_q(\mathbb{Z}_2, \overset{\mathbb{F}_2}{\mathbb{Z}})) \Rightarrow H_{p+q}(C_4, \overset{\mathbb{F}_2}{\mathbb{Z}})$$

$$H_n(\mathbb{Z}_2, \overset{\mathbb{F}_2}{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$



double complex

$C_{p,q}$

$$\begin{array}{ccc} C_{p,q} & \xrightarrow{\partial'} & C_{p-1,q} \\ {\scriptstyle \partial''}\downarrow & & \downarrow{\scriptstyle \partial'} \\ C_{p,q-1} & \xrightarrow[\partial']{} & C_{p-1,q-1} \end{array}$$

$\partial'\partial'' = \partial''\partial'$

$\partial'\partial' = 0$

$\partial''\partial'' = 0$

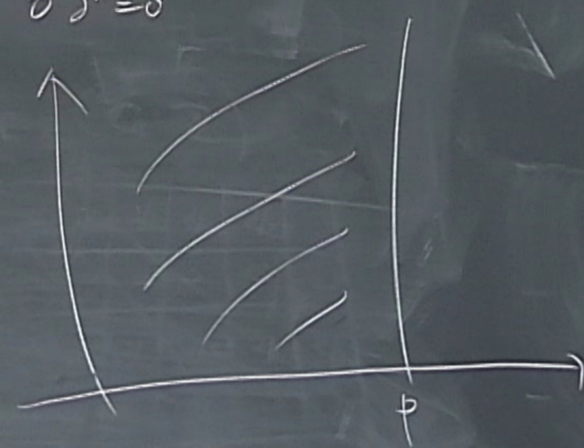
total complex $(TC)_n = \bigoplus_{p+q=n} C_{p,q}$

$$\begin{array}{c} C_{\cdot,q} \\ \downarrow \\ C_{\cdot,q-1} \end{array}$$

e.g. $C_{p,q} = C_p \otimes C'_q$

$(TC)_n = (C \otimes C')_n$

$C_{p,\cdot} \to C_{p-1,\cdot}$



H_q

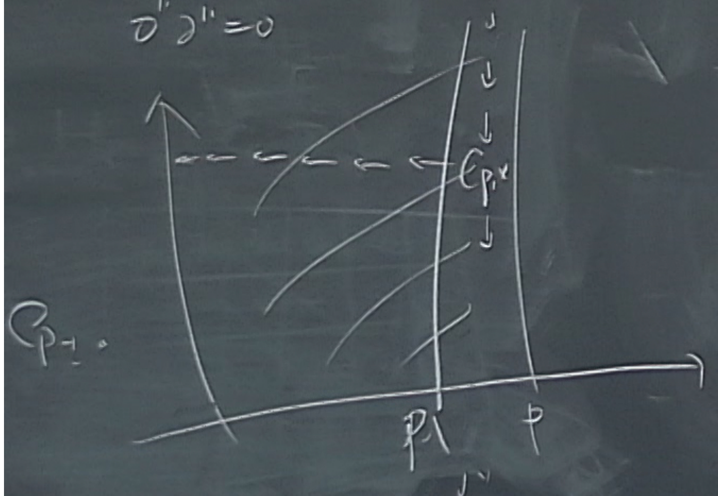
$D'_{p,q} =$

$E'_{p,q} =$

$\partial'\partial'' = \partial''\partial'$

$\partial'\partial' = 0$

$\partial''\partial'' = 0$



$$F_p C = \bigoplus_{i \le p} C_{i,*}$$

$$F_p C / F_{p-1} C = C_{p,*}$$

$$E^0_{p,q} = \left(F_p C / F_{p-1} C\right)_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q} = C_{p,q}$$

$$E^1_{p,q} = H_\cdot(C_{p,\cdot}, \partial'')$$

$$\Downarrow$$

$$H_*(TC)$$

$$E^1_{p,q} = H_\cdot(C_{\cdot,q}; \partial') \Rightarrow H_*(TC)$$

$C_{p+q} = C_{p,q}$

$$1 \to H \to G \to Q \to 1$$

$F_\bullet$, free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$,

$$H_*(G,M) = H_*(F_\bullet \otimes_{\mathbb{Z}[G]} M)$$

each F_n is also a module over $\mathbb{Z}[H]$

$\mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G] \curvearrowright M$

$\Big\} \Rightarrow$ $F_\bullet$ is free resol of $\mathbb{Z}$ over $\mathbb{Z}[H]$

$$C_\bullet = F_\bullet \otimes_{\mathbb{Z}[H]} M \Rightarrow H_*(C_\bullet) = H_*(H,M)$$

$$F_\bullet \otimes_{\mathbb{Z}[G]} M \cong \mathbb{Z} \otimes_{\mathbb{Z}[Q]} C_\bullet = \mathbb{Z} \otimes_{\mathbb{Z}[Q]} (\underbrace{F_\bullet \otimes_{\mathbb{Z}[H]} M}_{\mathbb{Z}[G]\ \ \mathrm{Ind}_H^G M})$$

$$\partial'\partial'' = \partial''\partial'$$

$$\partial'\partial' = 0$$

$$\partial''\partial'' = 0$$

$$M_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$$

$$(M_H)_Q = M_G = M/\{gm - m \mid g \in G\} = H_0(G, M)$$

$$\mathbb{Z} \otimes_{\mathbb{Z}[Q]} (\mathbb{Z} \otimes_{\mathbb{Z}[H]} M) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$$

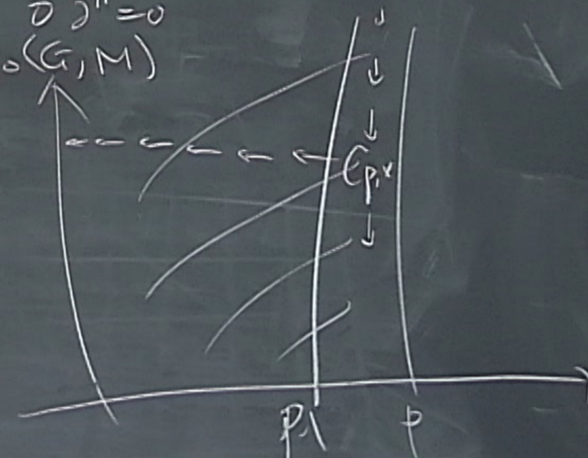
$g \cdot (f \otimes m)$

$(f g^{-1} \otimes_{\mathbb{Z}[H]} g m)$

$g H$

q

C_{p+q}



Now take a free solution $D_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}[Q]$.

$$\underline{\underline{D_p}} \otimes_{\mathbb{Z}[Q]} C_q . \qquad \underline{\underline{D_{p+1}}} \otimes C_q \to \underline{\underline{D_p}} \otimes C_q \to \underline{\underline{D_{p-1}}} \otimes C_q$$

free resolution of $\mathbb{Z}$ over $\mathbb{Z}[Q]$

$$E^1 = D_p \otimes H_*(C_\bullet)$$

$$\downarrow$$

$$D_{p-1} \otimes H_*(C_\bullet)$$

$$E^2 = H_p(Q, H_q(C_\bullet))$$

$$\|$$

$$H_p(Q, H_q(H, M))$$

$$\Rightarrow H_{p+q}(TC) = H_{p+q}(G, M).$$

(G,M)

$$E' = H_*(Q, \underline{C_q}) \overset{\text{claim}}{=\!=} \begin{cases} H_q(G,M) & *=0 \\ 0. & *>0 \end{cases}$$

$$E^\infty \qquad \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} M = \mathbb{Z}[G/H]\otimes M.$$

$$H_{p+q}(TC)$$

q

$H_0(Q, C_q) = H_q(G,M)$

$p=0$ $\qquad p$

$$C_q = F_q \otimes_{\mathbb{Z}[H]} M.$$

$$F_q = \oplus\, \mathbb{Z}[G]$$

$$= \oplus\, \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} M$$

$$= \oplus\, \mathbb{Z}[G/H]\otimes M' = \oplus\, \mathbb{Z}[Q]\otimes M'$$