

Title: The Cohomology of Groups (Johnson-Freyd/Guo) - Lecture 6

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R is a ring, M an R -module,
 ψ_m is rotation
if $rm = 0$.

Recall: If G, G' are gps, R a com ring. =-s.
 (trivial as a $G \times G'$ module)
 then there is a canonical R -bilinear map

$$(\star) \quad H^*(G; R) \otimes_R H^*(G'; R) \rightarrow H^*(G \times G'; R)$$

Subject to G, G' being not too large (e.g. finite) then

- if you replace $\otimes_R$ with derived $\otimes = \text{ie. Tor complex}$
 then $(\star)$ is true.

In particular,

$$R = \mathbb{Z},$$

$$\mathbb{Z}/N$$

$$H_p = \mathbb{Z}/p \text{ if prime}$$

replace $\otimes_R$ with derived $\otimes = \text{ie. Tor complex}$
 then $(\star)$ is an iso.

In particular, if $R = \mathbb{Z}$, only Tor_0 and Tor_1

$(\star)$ is an isomorphism w/ cokernel $\text{Tor}_1^{\mathbb{Z}}(H^*(G), H^*(G'))$.

If R is a field, then all R -modules are free and so there are no Tor_i ,
 and $(\star)$ is an iso.

If $G \cong G$, look at $\text{diag}: G \hookrightarrow G \times G$,

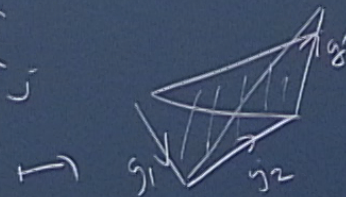
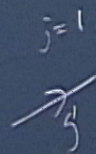
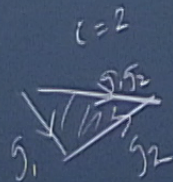
$$\begin{array}{ccc}
 H^*(G) \otimes H^*(G) & \longrightarrow & H^*(G \times G) \\
 \searrow \text{cup product} & & \downarrow \text{diag}^* \\
 & & H^*(G)
 \end{array}$$

$H^i(G)$

In the bar complex, Kunen also took

$$\langle g_1 | g_2 | \dots | g_i \rangle \otimes \langle g'_1 | \dots | g'_j \rangle \mapsto \langle g_1 | \dots | g_i | g'_1 | \dots | g'_j \rangle$$

$\cap$ $\cap$ $\cap$
 $C_i(G)$ C_j C_{i+j}



replace $\otimes$ with $\otimes_{\mathbb{R}}$ (or complex)
 then $(\otimes)$ is true.

12 $a \in C^i(G; R)$ $b \in C^j(G; R)$ cocycles in the bar complex,

then

$$(a \cup b)(g_1, g_2, \dots, g_{i+j})$$

$$= a(\overbrace{g_1 \dots g_i}^i) \cdot b(\overbrace{g_{i+1} \dots g_{i+j}}^j)$$

$\uparrow$
 mult in R

$(\dots, \dots, \dots)$
mult in R .

E.g. $G = \mathbb{Z}/2$, normalized bar complex R $H^*(\mathbb{Z}/2; R)$ has

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & & \\ R & \xrightarrow{0} & R & \xrightarrow{x_2} & R & \xrightarrow{0} & R \xrightarrow{x_2} R \rightarrow \dots \\ & & \text{basis } e_0, e_1, e_2, \dots & & & & \end{array}$$

bar
a cochain is normalized

if $a(g_1, \dots, g_n) = 0$ whenever some g_i is identity

$$e_i(g_1, \dots, g_i) = \begin{cases} 1 & \text{if } g_i = \text{identity} \\ 0 & \text{otherwise} \end{cases}$$

$$e_i \cup e_j = e_{i+j}$$

For $\mathbb{Z}/2$, a normalized cochain is $a(g_1, \dots, g_n) = \begin{cases} 0 & \text{if some } g_i = 0 \\ 1 & \text{if all } g_i = 1 \end{cases}$

If $R = \mathbb{Z}$, then the coh. is rep'd by e_{even}

$$H^*(\mathbb{Z}_2; \mathbb{Z}) = \mathbb{Z}, 0, \overset{t}{\mathbb{Z}_2} e_2, 0, \mathbb{Z}_2 e_4, \dots$$

$$e_4 = e_2^2$$

$$e_6 = e_2^3, \dots$$

deg 2.

$$= \mathbb{Z}[t] / (2t)$$

$$H^*(\mathbb{C}/N; \mathbb{C}/N) = \mathbb{C}/N, \mathbb{C}/N, \mathbb{C}/N, \dots$$

$$\stackrel{?}{=} \frac{1}{N} [x] \quad \text{[} \partial_0, 1 \text{]}$$

Wrong except when $N=2$.

$$H^*(\mathcal{C}_N; \mathcal{C}_N) = \overset{H^0}{\mathcal{C}_N^0}, \overset{H^1}{\mathcal{C}_N^1}, \overset{H^2}{\mathcal{C}_N^2}, \dots$$

$$\overset{?}{=} \mathcal{C}_N[x] \quad \text{deg 1.}$$

Why is it wrong?

U

is graded com. on H^*

$$\text{ie. } \overset{H^1}{a} \cup \overset{H^1}{b} = (-1)^{|a|} b \cup a.$$

revised guess. (turns out correct)

$$H^*(\mathcal{C}_N; \mathcal{C}_N) = \mathcal{C}_N[x, t] \quad \text{deg 1, deg 2}$$

$$x^i = \begin{cases} 0 & \text{if } N \text{ odd} \\ \frac{N}{2} & \text{if } N \text{ even} \end{cases}$$

Wrong except when $N=2$.

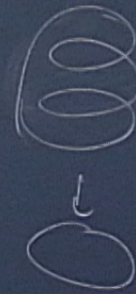
$$\Rightarrow \text{if } x \in H^1, \text{ then } xux = -xux \\ 2xux = 0. \Rightarrow xux = 0 \text{ if } N \text{ odd}$$

$$= \mathbb{Z}[t] / (t^2)$$

↖ deg 2

idea of
Pf of $(\star\star)$:

Consider the circle $S^1 \hookrightarrow \mathbb{C}^*$ by $\frac{2\pi}{N}$ rotation.
this action is free, quotient is also S^1 .



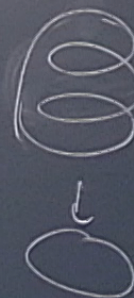
idea of
Pf of $(\star \star)$:



Consider the circle $S^1 \cong \mathbb{R}/\mathbb{Z}$ by $\frac{2\pi}{n}$ action.

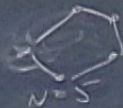
this action is free, quotient is also S^1 .

$$H^*(S^1; \mathbb{Z}) = \begin{matrix} 0 \\ \uparrow \\ \mathbb{Z} \end{matrix}, \begin{matrix} 1 \\ \uparrow \\ \mathbb{Z} \end{matrix}, 0, \dots = \mathbb{Z}[s]/(s^2) \quad \text{deg 1.}$$



If $\mathbb{Q} = \mathbb{Z}$, then the coh is rep'd by e_{even}

idea of
Pf of $(\star\star)$:

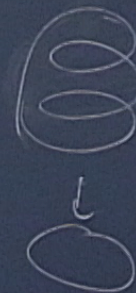


Consider the circle $S^1 \simeq \mathbb{Z}_N$ by $\frac{2\pi}{N}$ action.

This action is free, quotient is also S^1 .

$$H^*(S^1; \mathbb{Z}) = \begin{matrix} 0 \\ \uparrow \\ \mathbb{Z} \end{matrix}, \begin{matrix} 1 \\ \uparrow \\ \mathbb{Z} \end{matrix}, 0, \dots = \mathbb{Z}[s]/(s^2) \quad \text{deg 1.}$$

action of G on H^* is trivial.



action of G on H^* is trivial.

The theory of spectral sequences $\Rightarrow$

there exists a differential $\delta \leftarrow \text{derivation}$
 $\delta(a \cup b) = \delta a \cup b + (-1)^{\deg a} a \cup \delta b$

$$H^*(G, \mathbb{Z}) \otimes H(s'; \mathbb{Z}) \quad (+ \text{ triv, but vanish here } H^*(s') = \mathbb{Z}, \mathbb{Z}, 0, \dots)$$

$$\text{st. } H^*(H^* \otimes H, \delta)$$

$$= H^*(s', \mathbb{Z}) = H^*\left(\frac{s'}{G}, \mathbb{Z}\right)$$

$$\begin{array}{c|cccc}
 & z_s & z_{s_1} & z_{s_2} & z_{s_3} \\
 \hline
 H'(S', \mathbb{Z}) = & z_s & z_{s_1} & z_{s_2} & z_{s_3} \\
 & z_1 & z_2 & z_3 & z_4
 \end{array}$$

$$H(G, \mathbb{Z})$$

must have

$$\delta s = t_1, \quad \delta(t_1) = 0.$$

$$\delta(st_1) = t_2$$

$$t_1 \cdot t_1$$

Sach: $H^1(\text{elementary abelian } \mathbb{F}_p)$

$$\begin{array}{c} E \\ \uparrow \\ \text{"elementary"} \end{array} = (\mathbb{Z}/2)^K$$

$$\begin{array}{c} \uparrow \\ (\mathbb{Z}/p)^H \end{array}$$

$p^5)$
 $\mathbb{Z}/p)^{\#}$

recollection:

1	p^1	$\mathbb{Z}/p.$	
2	p^2	$(\mathbb{Z}/p)^2, \mathbb{Z}/p^2.$	$H^2(\mathbb{Z}/p, \mathbb{Z}/p)$
5	p^3	$(\mathbb{Z}/p)^3, \mathbb{Z}/p \times \mathbb{Z}/p^2, \mathbb{Z}/p^3$	$= \mathbb{Z}/p.$

and two unimodular reps:
 $p=2: D_8, Q_8$

so p extensions
 but only 2 $\overline{\text{tors}}$.

and two nonabelian ones. but only 2 is abelian.

$$p=2: D_8, Q_8$$

$p \neq 2$: one where some elts have order p^2
 Here $\rightarrow$ [other all elts have order p .

5.00m $\mathbb{F}_1$ elementary abelian p -groups

$$\begin{array}{c} E = (\mathbb{Z}/2)^k \\ \uparrow \\ \text{"elementary"} \end{array}$$

$$\uparrow (\mathbb{Z}/p)^{\mathbb{F}}$$

recollection

$$\begin{array}{ccc} 1 & P^1 & \mathbb{Z}/p \\ 2 & P^2 & (\mathbb{Z}/p)^2 \\ 5 & P^3 & (\mathbb{Z}/p)^3 \end{array}$$

$$H^*(E, \mathbb{F}_2) = \left(H^*(\underbrace{\mathbb{Z}/2, \mathbb{F}_2}_{\parallel \mathbb{F}_2[x] \xrightarrow{\text{deg } 1}}) \right)^{\oplus k} = \mathbb{F}_2[x_1, \dots, x_k]$$

Koszul over a field is easy.

th:

is noncanonically iso to $(\mathbb{Z}/2)^{\#}$
 ring str do it canonically.

depends on e, k

For E elementary
 x_2 is 0 on $H^*(E; \mathbb{Z})$

LES of cdh

$$\mathbb{Z} \rightarrow \mathbb{F}_2 \rightarrow 0$$

$$H^i(E; \mathbb{Z}) \xrightarrow{x_2} H^i(E; \mathbb{Z}) \xrightarrow{\text{mod } 2} H^i(E; \mathbb{F}_2)$$

$$H^{i+1}(E; \mathbb{Z}) \xrightarrow{x_2} \dots \xrightarrow{\beta_2} H^i(E; \mathbb{F}_2)$$

"integral Bockstein"

$$H^*(E, \mathbb{Z})?$$

Kunneth:

this is noncanonically iso to $(\mathbb{Z}/2)^{\#}$

Today:

ring sds do it canonically.

Recall: SES $\Rightarrow$ LES.

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{F}_2 \rightarrow 0$$

LES of cth

$$H^i(E, \mathbb{Z}) \xrightarrow{\times 2}$$

$$H^{i+1}(E, \mathbb{Z})$$

deg i polys in
 $x_1, \dots, x_n$

$$H^i(E, \mathbb{Z}) \xrightarrow{(\text{mod } 2)} H^i(E, \mathbb{F}_2) \xrightarrow{\beta_{\mathbb{Z}}} H^{i+1}(E, \mathbb{Z}) \rightarrow 0$$

$$H^i(E, \mathbb{Z}) \xrightarrow{(\text{mod } 2)} H^i(E, \mathbb{F}_2)$$

sub ring

$\mathbb{F}_2[x_1, \dots, x_n]$

it is the kernel of $\beta_{\mathbb{Z}}$.
 So just need to understand $\beta_{\mathbb{Z}}$.

$C^i(E; \mathbb{F}_2)$ i -cocycle for $H^i(E; \mathbb{F}_2)$.

choose $\tilde{a} \in C^i(E; \mathbb{Z})$

e.g. in the bar complex,

with $\tilde{a}(g_1, \dots, g_i) = a(g_1, \dots, g_i)$

$\mathbb{Z} \supseteq \mathbb{F}_2$

$a \in C^i(E; \mathbb{F}_2)$ i -cocycle for $H^i(E; \mathbb{F}_2)$

choose $\tilde{a} \in C^i(E; \mathbb{Z})$

e.g. in the bar complex,
could $\tilde{a}(g, \dots, g)$

$\xrightarrow{\text{cocycle}} \partial \tilde{a} \in C^{i+1}(E; \mathbb{Z})$
 $\leftarrow \text{cocycle boundary}$

Since a was a cocycle, $\partial \tilde{a} \equiv 0 \pmod{2}$,

so " $\frac{1}{2} \partial \tilde{a}$ " makes sense in $C^{i+1}(E; \mathbb{Z})$

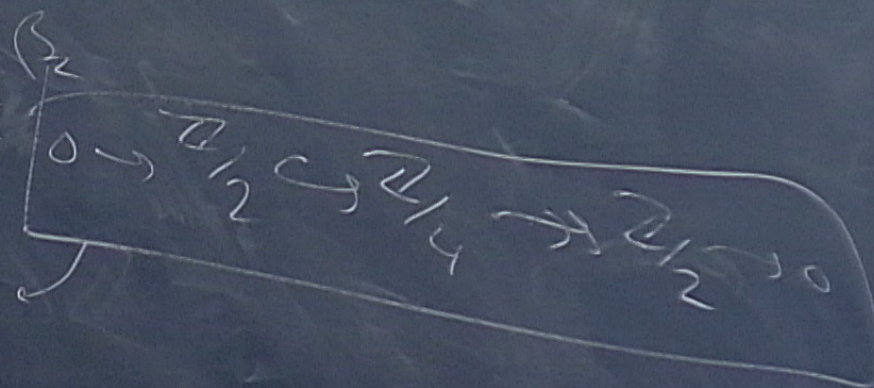
$\beta_{\mathbb{Z}} a$

and " $\frac{1}{2} \partial \tilde{a}$ " is a cocycle

$$d(a \cup b) = d a \cup b \pm a \cup d b$$

$$\beta_2(a \cup b) = \beta_2 a \cup b + a \cup \beta_2 b$$

Defn:
mod 2 bookstem
 is the connectivity map
 for LES for



$(\mathbb{F}_p)^5$

$(\mathbb{Z}/p)^4$

recollection:

1	p^1	$\mathbb{Z}/p$	
2	p^2	$(\mathbb{Z}/p)^2, \mathbb{Z}/p^2$	$H^2(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$
5	p^3	$(\mathbb{Z}/p)^3, \mathbb{Z}/p \times \mathbb{Z}/p^2, \mathbb{Z}/p^3$	so p extensions but only 2 $\mathbb{Z}/p$

and two unabelian ones:

$p=2: D_8, Q_8$
 $p \neq 2: S_4$

Here $S_4 \rightarrow$ (other all extensions of p)

$(\mathbb{F}_2)^5 = \mathbb{F}_2[x_1, \dots, x_4]$ deg 1

$\mathbb{F}_2[x]$ deg 1

If E elem ab, then

$$H^1(E, \mathbb{Q}) \xrightarrow{\text{mod } 2} H^1(E, \mathbb{F}_2) \rightarrow \text{inj}$$

$$\hookrightarrow \text{Ker}(\beta_{\mathbb{Q}}) = \text{Ker}(\beta_2)$$

i.e. we have:

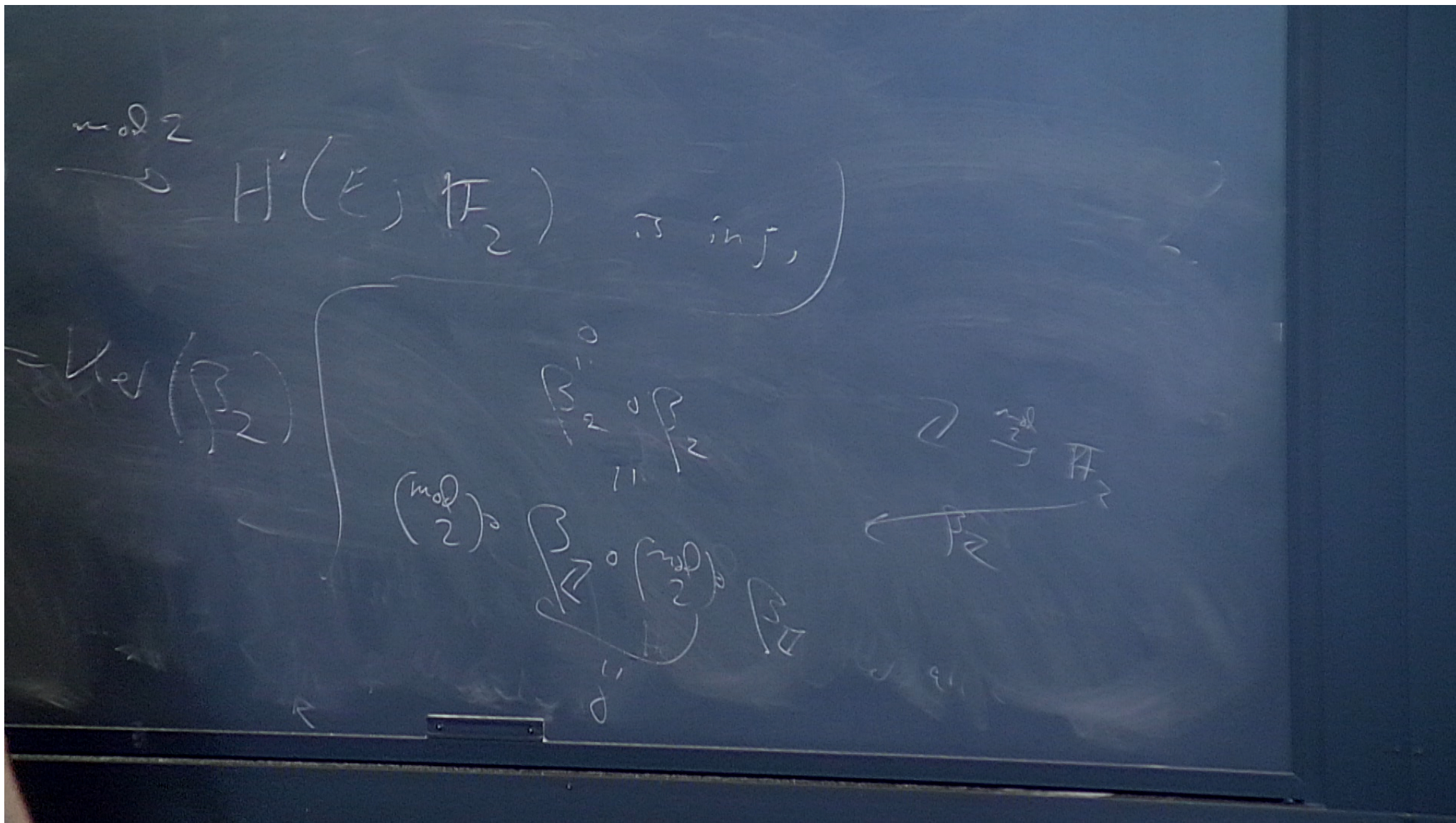
$$H^*(E, \mathbb{F}_2) \xrightarrow{\text{deg 1 map } \beta_2}$$

β_2 is a derivation.

$$\parallel$$
$$\mathbb{F}_2[x_1, \dots, x_n]$$

and $H^*(E; \mathbb{Q})$ is the kernel of β_2 .

so β_2 is a derivation.



i.e. we have:

$$H^*(E; \mathbb{F}_2) \xrightarrow{\text{deg 1 map}} \beta_2$$

$$\parallel$$

$$\mathbb{F}_2[x_1, \dots, x_n]$$

and $H^*(E; \mathbb{Q})$ is the kernel of β_2 .

So as always (except $n=0$)

β_2 is a derivation

So $H^*(E; \mathbb{Q})$

is a $\mathbb{Q}$

$$\begin{array}{c} x^2 \\ || \\ \beta_2(x) = 10 \\ \hline \deg 2 \end{array}$$

$$\beta_2(x^n) = \begin{cases} 0 & n \text{ even} \\ x^{n+1} & n \text{ odd} \end{cases}$$

$$n \cdot x^{n+1}$$

$$\boxed{\beta_2 = x^2 \frac{\partial}{\partial x}}$$

$$x^2$$

$$\uparrow$$

$$\downarrow$$

(x kept in $\deg d$)

$\deg 2$

For general $E = (\mathbb{P}^1)^k$

$$H^0(E, \mathcal{O}_E) = \mathbb{C}[x_1, \dots, x_k]$$

$$\beta_2 = \sum_{i=1}^k x_i^2$$

$$E = (\mathbb{Z}/2)^4$$

$$) = \mathbb{F}_2[x_1, \dots, x_4]$$

$$\beta_2 = \sum_i x_i^2 \frac{\partial}{\partial x_i}$$

Remark

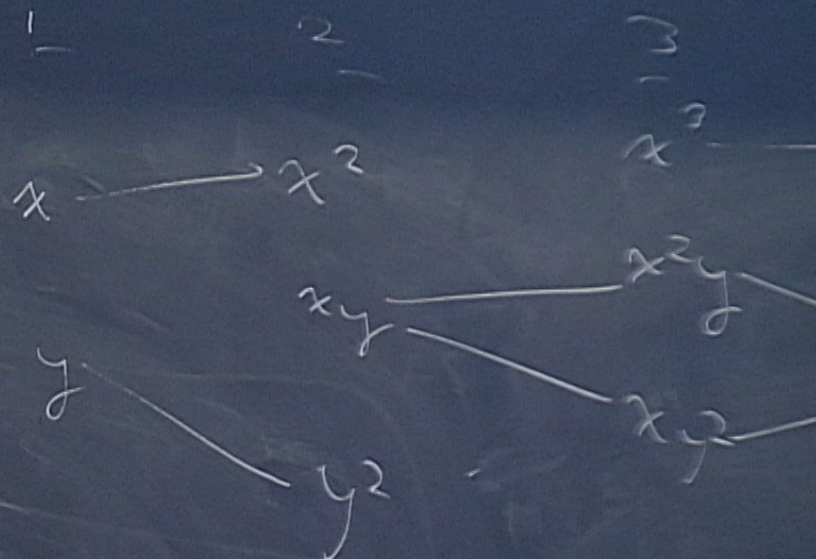
"Freshman's Dream"

$$(x+y)^2 = x^2 + y^2$$

$$E = (\mathbb{Z}/2)^2$$

"Klein 4"

$$H^1(E, \mathbb{F}_2) = 1$$



$$P_2(x, y) = a x^{a+1} y^b + b x^a y^{b+1}$$

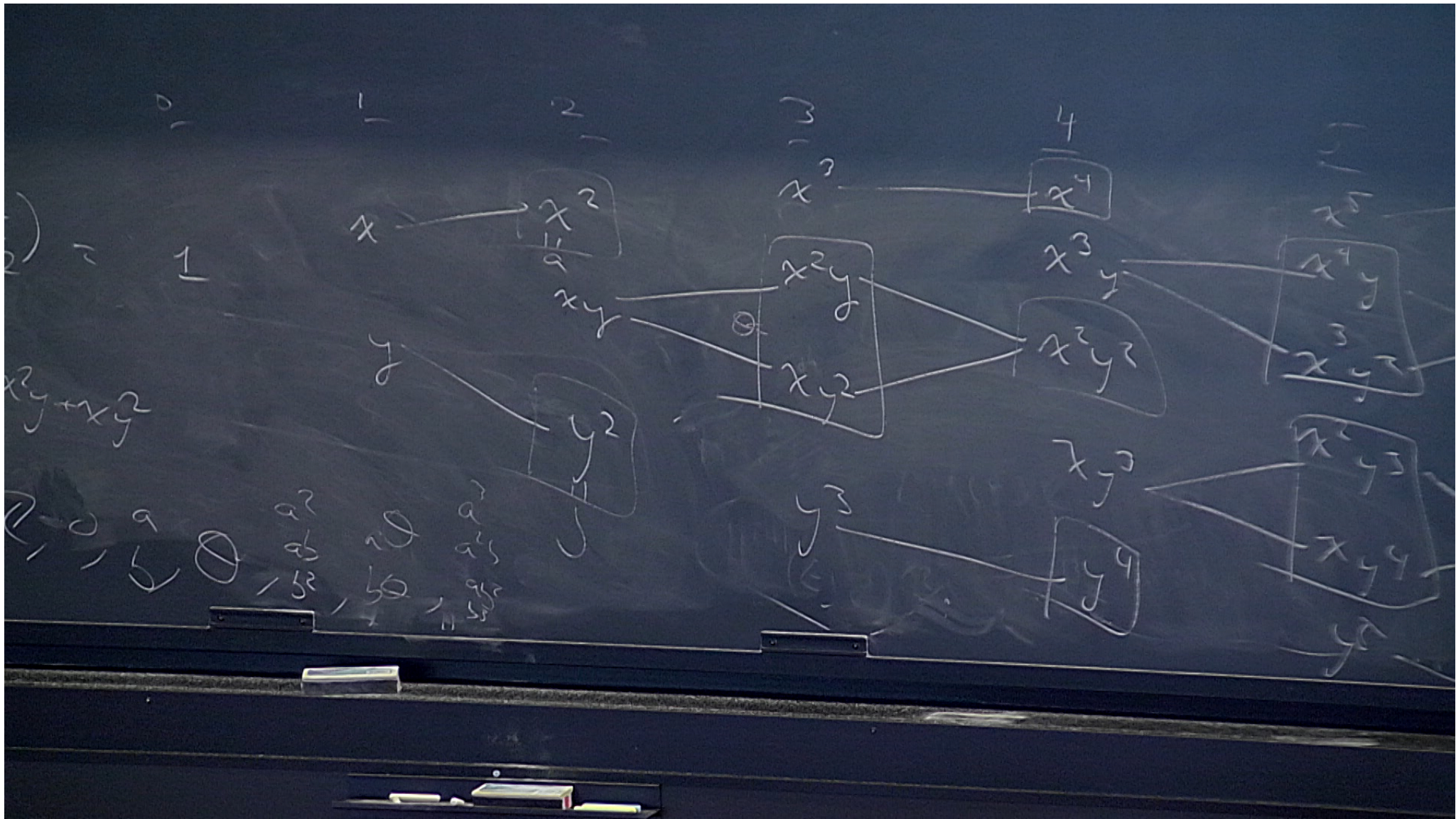
have: E an arbitrary TS

$(\mathbb{F}_2)$ $\hookrightarrow$ deg 1 map
 β_2

β_2 is a derivation, differential

so $(H^*(E; \mathbb{F}_2), \beta_2)$
 is a dga.

$(E; \mathbb{Q})$ is the kernel of β_2 .
 as $\beta_2 = 0$ (except $n=2$)



$$E = (\mathbb{Q}/2)^2$$

"Klein 4"

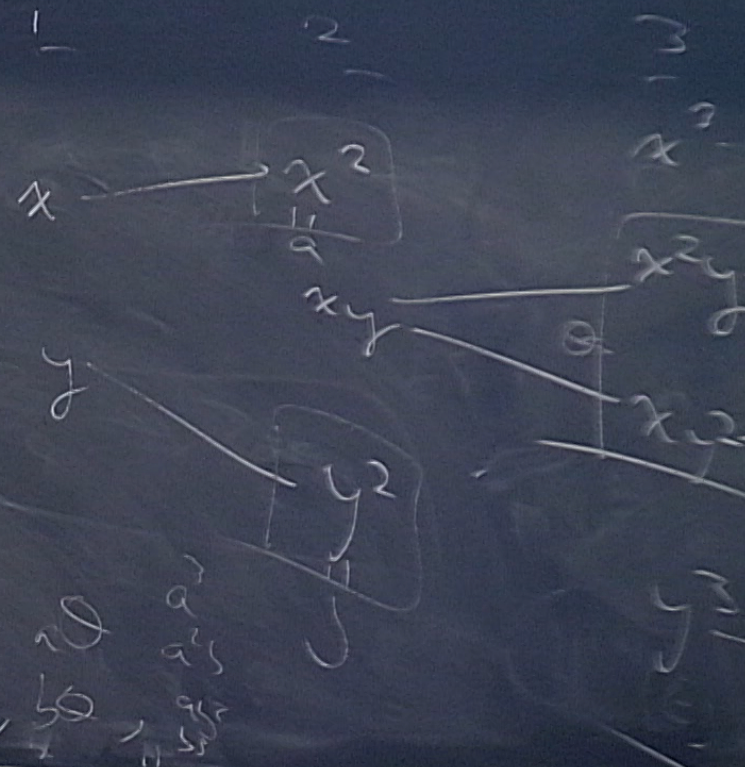
$$\Theta^2 = (x^2y + y^2x)^2$$

$$= x^4y^2 + x^2y^4$$

$$H^1(E, \mathbb{F}_2) = 1$$

$$\Theta = x^2y + xy^2$$

$$H^1(E, \mathbb{Q}) = \mathbb{Q}, 0, 0, \Theta$$



$$H^*(\mathbb{P}^2, \mathbb{Z})$$

$$= \mathbb{Z} [a, b, \theta]$$

$\delta_{\mathbb{Z}^2}$ $\delta_{\mathbb{Z}^3}$
 $\swarrow \quad \searrow$ $\downarrow$
 a, b, θ

$$2a = 2b = 2\theta = 0$$

$$\theta^2 = a^2b + ab^2$$

Can repeat the story for $\mathbb{Z}/N$.

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times N} \mathbb{Z} \xrightarrow{\text{mod } N} \mathbb{Z}/N \rightarrow 0$$

$$\swarrow \beta_2$$

$$\mathbb{Z}/N \rightarrow \mathbb{Z}/N^2 \rightarrow \mathbb{Z}/N^3$$

$$\swarrow \beta_3$$

mod N back to $\mathbb{Z}$.

SES

$$0 \rightarrow H^0(\mathbb{P}_N, \mathbb{Z}) \xrightarrow[\text{reduction}]{\text{mod } N} H^0(\mathbb{P}_N, \mathbb{Z}_N) \xrightarrow{\beta_2} H^2(\mathbb{P}_N, \mathbb{Z}_N)$$

$$\frac{\mathbb{Z}[t]}{(Nt)}$$

N backside

$$\frac{m_8}{N} \frac{\partial}{\partial t} [x, t]$$

$$x^2 = \frac{0}{2} t$$

N odd

N even

Remark:

← fixed by graded co.

← fixed by graded even

& restricts to $\frac{\partial}{\partial t} \subset \frac{\partial}{\partial t}$

Repeat the analysis:

$$H'(\varphi_N; \varphi_N) \stackrel{\text{reg}}{=} \frac{\varphi}{N} [x, t]$$

and $\beta_N(x) = t$, $\beta_N = t \frac{\partial}{\partial x}$

$$\beta_N(t) \approx 0,$$

ad N) backstein ring is

aded can ring $\mathbb{Z}/N$ w/ a
dgca

derivation " β_N "

deg 1,

$$\beta_N \circ \beta_N = 0$$

$$\text{and } \sum_{i=0}^N \beta_N(x) = x^2$$

if x has deg 1.

x has $\deg 1$.

$$x \in H^1(G; \mathbb{Z}/n)$$

$H^1(G; \mathbb{Z}/n)$ is free on a single gen
as a bockstein- ring .

$H^1(G; \mathbb{Z}/n)$ is a bockstein- ring .

x has $\deg 1$
||

$$x \in H^1(G; \mathbb{Z}/n)$$

$$\cong H^1(G; \mathbb{Z}/n)$$

So since $\square$ holds
 $n \rightarrow n$, holds $\square$.