

Title: PSI 2019/2020 - Condensed Matter (Wang) - Lecture 5

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Collection: PSI 2019/2020 - Condensed Matter (Wang)

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Today: Berry phase & Hall conductivity.

- Adiabatic Thm & Berry phase. (Your favorite Q.M. Book)
- Berry phase in Band Theory (Hasan & Kane, "Topological insulators".
Book by Bernevig & Hughes)

Adiabatic Thm.

$$H(t) |\psi_n(t)\rangle = E_n(t) |\psi_n(t)\rangle$$

$$H |\bar{\psi}(t)\rangle = i \partial_t |\bar{\psi}(t)\rangle$$

$$|\psi(t)\rangle = \sum_n C_n(t) e^{-i \int_0^t E_n(t') dt'} |\psi_n(t)\rangle$$

$$\Rightarrow \dot{C}_n(t) = -C_n(t) \langle \psi_n(t) | \partial_t | \psi_n(t) \rangle - \sum_{m \neq n} C_m(t) \frac{\langle \psi_n(t) | \dot{H} | \psi_m(t) \rangle}{E_m(t) - E_n(t)} e^{i \int_0^t [E_n(t') - E_m(t')] dt'}$$

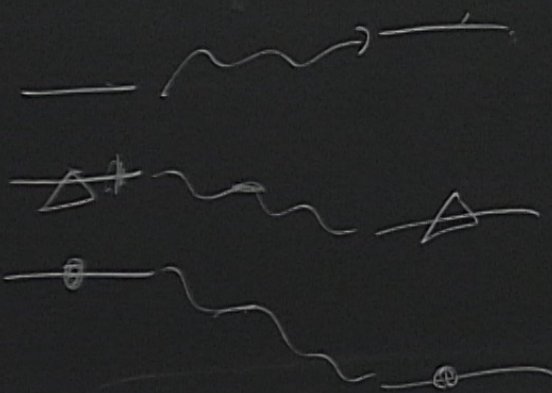
$$\text{Adiabatic limit: } \frac{1}{\tau} \ll |E_n - E_m| \text{ for all } m$$

$$\Rightarrow \dot{C}_n(t) = -C_n(t) \langle \psi_n | \partial_t | \psi_n \rangle \Rightarrow C_n(t) = C_n(0) e^{i \gamma_n(t)}, \quad \gamma_n(t) = \int_0^t \langle \psi_n(t') | \partial_{t'} | \psi_n(t') \rangle dt'$$

$$\text{If } |\psi(t=0)\rangle = |\psi_n(t=0)\rangle$$

$$|\psi(t)\rangle = e^{i\delta_n} |\psi_n(t)\rangle$$

$$\frac{1}{t} \sim \Delta E \ll (E_n - E_m)$$



$$H|\psi(t)\rangle = i\partial_t|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-i\int_0^t E_n(t') dt'} |\psi_n(t)\rangle$$

$$i\partial_t|\psi(t)\rangle = \sum_{m \neq n} C_m^{(0)} \frac{\langle \psi_n^{(0)} | H | \psi_m^{(0)} \rangle}{E_m^{(0)} - E_n^{(0)}} e^{i\int_0^t [E_n(t') - E_m(t')] dt'} \sim e^{i(E_m - E_n)t}$$

$$\frac{1}{E_n} \ll |E_n - E_m| \text{ for all } m$$

$$\langle \psi_n | i\partial_t | \psi_n \rangle \Rightarrow C_n(t) = C_n(0) e^{i\gamma_n(t)}, \quad \gamma_n(t) = \int_0^t \langle \psi_n(t') | i\partial_{t'} | \psi_n(t') \rangle dt'$$

Berry phase
Geometric

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = i\partial_t |\psi(t)\rangle$$

$$i \int_0^t [E_n(t') - E_m(t')] dt' \sim e^{i(E_m - E_n)t}$$

$$\gamma_n(t) = \int_0^t \langle \psi_n(t') | i\partial_{t'} | \psi_n(t') \rangle dt'$$

Berry phase
Geometric phase

Gauge transform:

$$|\psi_n(t)\rangle \rightarrow e^{-i\alpha(t)} |\psi_n(t)\rangle$$

$$\gamma_n(t) \rightarrow \gamma_n(t) + \alpha(t) - \alpha(0)$$

$$\text{Unless } |\psi_n(t)\rangle \sim |\psi_n(0)\rangle$$

$$H(t) \sim H(0)$$

$$\gamma_n \text{ gauge-invariant}$$

$$\alpha(t) = \alpha(0)$$

Many-body states connected by adiabatic evolution
can be considered "similar".

— Gapped states remain gapped under
(insulator)

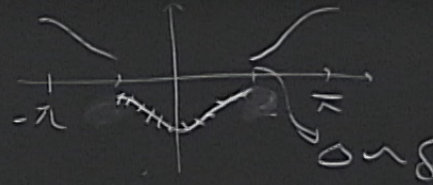
$$\delta \gg t.$$

$$GS, \langle C_{2j}^+ C_{2j} \rangle = 1.$$

$$\langle C_{2j+1}^+ C_{2j+1} \rangle = 0$$



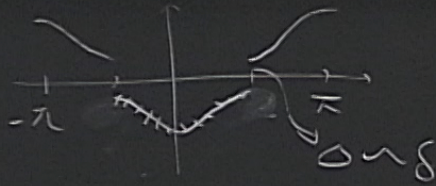
$$\delta \ll t.$$



$$\delta(\tau) = \delta \frac{\tau}{T} (\delta)$$

$$\delta(\tau) = \begin{cases} \tau=0, \delta=100t. \\ \tau=T \Rightarrow \frac{1}{t\delta}, \delta \approx 0.1t \end{cases}$$

$$\delta \ll t$$



$$\delta(\tau) = \delta(0) + \frac{\tau}{T} (\delta(T) - \delta(0))$$

$$\delta(0) \gg t$$

$$0 < \delta(T) \ll t$$

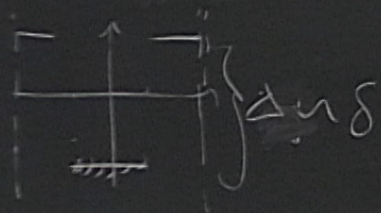
$$\delta(\tau) = \delta(0) + \frac{\tau}{T} (\delta(T) - \delta(0))$$

$$\tau = T \gg \frac{1}{t\delta}, \delta \approx 0.1t$$

$$\delta \gg t.$$

$$GS, \langle c_{2j}^\dagger c_{2j} \rangle = 1.$$

$$\langle c_{2j+1}^\dagger c_{2j+1} \rangle = 0$$

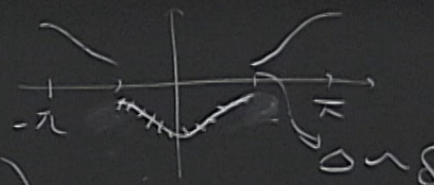


Atomic limit
Atomic insulator

$$\delta(\tau) = \begin{cases} \tau=0, & \delta = 100t \\ \tau=T \Rightarrow \frac{1}{t\delta}, & \delta \approx 0.1t \end{cases}$$

$$\tau=T \Rightarrow \frac{1}{t\delta}, \delta \approx 0.1t$$

$$\delta \ll t.$$



$$\delta(\tau) = \delta(0) +$$

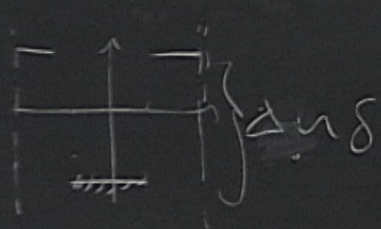
$$\delta(0) \gg t$$

$$0 < \delta(T) \ll t$$

$$\delta \gg t.$$

$$GS, \langle c_{2j}^+ c_{2j} \rangle = 1.$$

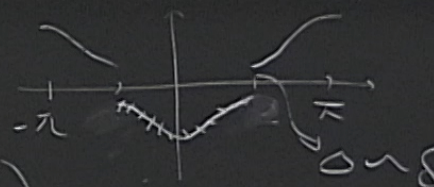
$$\langle c_{2j+1}^+ c_{2j+1} \rangle = 0$$



Atomic limit
Atomic insulator

$$\delta(\tau) = \begin{cases} \tau=0, & \delta = 100t \\ \tau=T \Rightarrow \frac{1}{t\delta}, & \delta \approx 0.1t \end{cases}$$

$$\delta \ll t.$$



$$\delta(\tau) = \delta(0) +$$

$$\delta(0) \gg t$$

$$0 < \delta(T) \ll t$$

It's often useful to view a complicated state as an adiabatic deformation of some simpler state.

$\gamma - \delta(0)$

Gauge transform:

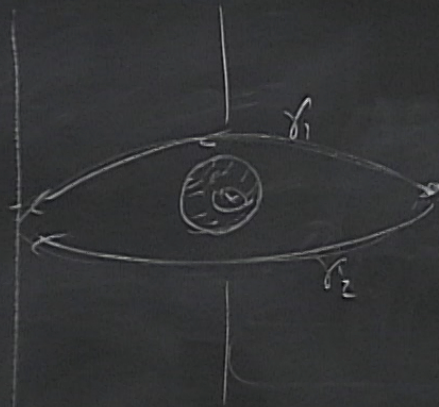
$$|\psi(\vec{\lambda})\rangle = e^{-i\alpha(\vec{\lambda})} |\psi(\vec{\lambda})\rangle$$

$$\Rightarrow \vec{A} \rightarrow \vec{A} + \partial_{\vec{\lambda}} \alpha.$$

$\vec{A}$: a vector potential in $\vec{\lambda}$ space.
a.k.a. Berry connection.

$$\vec{A} \cdot d\vec{\lambda}$$

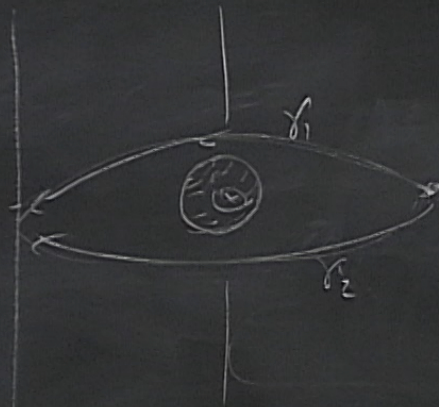
Aharonov-Bohm effect.



$$\Delta\gamma = \int_{r_1} \vec{A} \cdot d\vec{r} - \int_{r_2} \vec{A} \cdot d\vec{r}$$

$$= \oint_{r_1-r_2} \vec{A} \cdot d\vec{r} = \int B \, ds = \Phi$$

(AB)
Aharonov-Bohm effect.



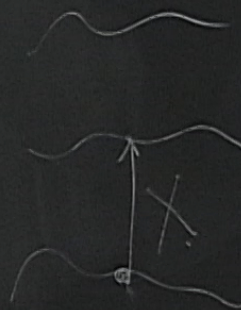
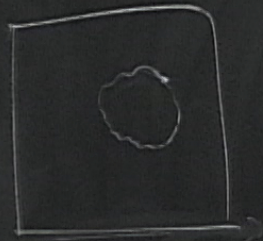
$$\Delta\gamma = \int_{r_1} \vec{A} \cdot d\vec{r} - \int_{r_2} \vec{A} \cdot d\vec{r}$$

$$= \oint_{r_1-r_2} \vec{A} \cdot d\vec{r} = \int B \, ds = \Phi$$

$$\Phi_0 = 2\pi = \frac{hc}{e} \cdot \text{flux-quantum.}$$

Berry phase in Band Theory.

$$\{\vec{\lambda}\} = \{\vec{k} \in \text{B.Z.}\}, \quad |\psi_n(\vec{\lambda})\rangle \approx |\psi_n(\vec{k})\rangle.$$



Consider a time-dependent $|\vec{k}(t)\rangle$, $(|u_{\vec{k}(t)}\rangle)$

Berry phase. $\gamma = \oint \underbrace{\langle \vec{k} | i \partial_{\vec{k}} | \vec{k} \rangle}_{\vec{A}(\vec{k})} \cdot d\vec{k}$

$$= \oint \vec{A}(\vec{k}) \cdot d\vec{k}$$

$$\text{let's do 2D} = \int \underbrace{\vec{\partial} \times \vec{A}}_b d^2k = \int b d^2k$$

b = Berry curvature $\rightarrow$ A "magnetic field" in momentum space.

Reminder: in real space, $\vec{B}$ modifies equation of motion.

$$\left. \begin{aligned} \dot{\vec{x}} &= \frac{\partial H(\vec{B}=0)}{\partial \vec{p}} \\ \dot{\vec{p}} &= -\frac{\partial H(\vec{B}=0)}{\partial \vec{x}} + \dot{\vec{x}} \times \vec{B} \end{aligned} \right\} \rightarrow \begin{aligned} \dot{\vec{k}} &= \frac{\partial H}{\partial (-\vec{x})} \\ -\dot{\vec{x}} &= -\frac{\partial H(\vec{B}=0)}{\partial \vec{k}} + \dot{\vec{k}} \times \vec{b} \end{aligned}$$

Now treat $\vec{k}$ as "coordinate", $-\vec{x}$ is now "momentum".

$$i\hbar = [\vec{x}, \vec{p}] = [\vec{p}, -\vec{x}]$$

Extra current (2d)

$$\delta \vec{J} = \int \frac{d^2 k}{(2\pi)^2} f(\vec{k}) b(\vec{k}) (\vec{E} \times \hat{z})$$

$$\delta J_z = \left[\int \frac{d^2 k}{(2\pi)^2} \theta(1 - \xi_k) b(\vec{k}) \right] \epsilon_{ij} E_j$$

$$f = f^{(0)} = \theta(1 - \xi_k)$$

$$\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$J_x = \sigma_{xy} E_y$$

$$J_y = -\sigma_{xy} E_x$$

Anomalous

Hall conductivity

Totally filled band. ($\sigma_{xx} = 0$).

$$\sigma_{xy} = \int \frac{d^2k}{(2\pi)^2} b(\vec{k}) \quad \text{may not be zero.}$$

non-zero: Chern insulator/
Integer quantum Hall

$$= \frac{1}{2\pi} n.$$

Consider
Berry