

Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 10

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Date: November 27, 2019 - 9:00 AM

URL: <http://pirsa.org/19110020>

Dirac Field $\Leftarrow$ QED . Grassmann Numbers (Berezin calculus)

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Spinors $s = \frac{1}{2}$ & Fermions Fermi-Dirac Statistics

Spin-Statistics Theorem - spin $\frac{1}{2} \leftarrow$ Fermi-Dirac spin integer $\leftarrow$ Bose-Einstein
 $s = n + \frac{1}{2}$ $s = n$

locality & unitarity

- 1) Berezin calculus $\leftarrow$ QFT, non-relativistic Q.T., Fermi Liquid, -
- 2) Application to Dirac Field

Complex
Grassman

"numbers"

$\mathbb{C}_N$

$i = 1, N$

$\theta_i, \bar{\theta}_i$

generators of an (abstract) algebra on $\mathbb{C}$

↑ "conjugates"

$2N$ generators

all the θ_i and $\bar{\theta}_i$ anticommute $\{\theta_i, \theta'_j\} = \theta_i \cdot \theta'_j + \theta'_j \cdot \theta_i = 0$

$\theta_i^2 = 0, \bar{\theta}_i^2 = 0$, nilpotent

$$\theta_i \theta_j = -\theta_j \theta_i, \quad \bar{\theta}_i \bar{\theta}_j = -\bar{\theta}_j \bar{\theta}_i, \quad \theta_i \bar{\theta}_j = -\bar{\theta}_j \theta_i$$

($i \neq j$) (i, j)

on $\mathbb{C}$

elements $g \in \mathbb{G}_N$

$$g = \sum_{\substack{I = \{1 \leq i_1 < i_2 < \dots < i_K \leq N\} \\ J = \{1 \leq j_1 < j_2 < \dots < j_H \leq N\}}} c_{I,J} \underbrace{\theta_{i_1} \theta_{i_2} \dots \theta_{i_K}}_{K \text{ } \theta\text{'s}} \underbrace{\overline{\theta}_{j_1} \overline{\theta}_{j_2} \dots \overline{\theta}_{j_H}}_{H \text{ } \overline{\theta}\text{'s}}$$

$$0 \leq K \leq N, \quad 0 \leq H \leq N$$

I = set of K ordered integers in $[1, N]$

J " H " " " "

$c_{I,J}$ is a complex number

g = polynomial in $\theta, \overline{\theta}$ with complex coefficients

conjugation $*$ in G_N $* \cdot * = Id$

C complex number $C^* = \bar{C}$ ordinary complex conjugate

θ_i generator $\theta_i^* = \bar{\theta}_i$

$\bar{\theta}_i$ " $\bar{\theta}_i^* = \theta_i$

$g_1, g_2 \in G_N$ $(g_1 g_2)^* = g_2^* g_1^*$ like for $*$ = Transpose for real matr.
matrices $*$ = T cc for complex matr.

$N=1$ $g = a + b\theta + c\bar{\theta} + d\theta\bar{\theta}$ $a, b, c, d \in \mathbb{C}$ $\dim_{\mathbb{C}} G_N = 4$

$g' = a' + b'\theta + c'\bar{\theta} + d'\theta\bar{\theta}$

$g \cdot g' = aa' + (ab' + ba')\theta + (ac' + ca')\bar{\theta} + (ad' + da' + bc' - cb')\theta\bar{\theta}$

$$g^* = \bar{a} + \bar{c} \theta + \bar{b} \cdot \bar{\theta} + \bar{d} \cdot \theta \bar{\theta}$$

$$N > 1 \quad \dim_{\mathbb{C}} \mathbb{B}_N = 2^{2N}$$

$$g = \left(\sum_{\underbrace{\theta \bar{\theta} \dots}_{\text{even } \#}} \right) + \left(\sum_{\underbrace{\theta \dots \bar{\theta} \dots}_{\text{odd number}}} \right)$$

$$g^* = \bar{a} + \bar{c} \theta + \bar{b} \bar{\theta} + \bar{d} \theta \bar{\theta}$$

$$N > 1 \quad \dim_{\mathbb{C}} \mathbb{G}_N = 2^N$$

$$g = \left(\sum_{\substack{\text{even } \# \\ K+H = \text{even}}} \theta \bar{\theta} \dots \right) + \left(\sum_{\substack{\text{odd number} \\ K+H = \text{odd}}} \theta \bar{\theta} \dots \right)$$

$$\mathbb{G}_N = \mathbb{G}_N^{\text{even}} \oplus \mathbb{G}_N^{\text{odd}} \text{ is } \mathbb{Z}_2 \text{ graded algebra}$$

elements in $\mathbb{G}_N^{\text{even}}$ commutes with any elements of $\mathbb{G}_N$

elements in $\mathbb{G}_N^{\text{odd}}$ anticommutes with $\mathbb{G}_N^{\text{odd}}$

$$\mathbb{G}_N^{\text{odd}} \cdot \mathbb{G}_N^{\text{odd}} = \mathbb{G}_N^{\text{even}}$$

$g \in \mathbb{G}_N \approx$ "function of the θ_i and $\bar{\theta}_i$ "

Derivation and integration

$$\frac{\partial}{\partial \theta_i} = ?$$

use anticommutate
to move θ_i to the left, then

$$\frac{\partial}{\partial \bar{\theta}_i} = \text{similar}$$

example: $g = a + b\theta + c\bar{\theta} + d\theta\bar{\theta}$

$$\frac{\partial}{\partial \theta} g = b + d\bar{\theta}$$

$$\frac{\partial}{\partial \bar{\theta}} g = c - d\theta$$

$$\frac{\partial}{\partial \theta_i} \text{ complex nb} = 0$$

$$\frac{\partial}{\partial \theta_i} \theta_j = 0 \quad \text{if } j \neq i$$

$$\frac{\partial}{\partial \theta_i} \theta_i = 1$$

$$\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \bar{\theta}_j} \text{ anticommute}$$

Integration $\Leftarrow$ functional integral

Goal $\int d\theta_i$ such that $\int d\theta_i \frac{\partial}{\partial \theta_i} = 0$

Trivial solution

$$\int d\theta_i = \frac{\partial}{\partial \theta_i}$$

$$\int d\theta_i \quad \mathbb{C}_N \rightarrow \mathbb{C}_N$$

$\int d\theta_i \quad \theta_i = 1$, the rest is zero

$$\int d\bar{\theta}_i = \frac{\partial}{\partial \bar{\theta}_i}$$

Gaussian Integrals

g is polynomial in $\theta, \bar{\theta}$ with complex coefficients

$$g' = a' + b'\theta + c'\bar{\theta} + d'\theta\bar{\theta}$$

$$g \cdot g' = aa' + (ab' + ba')\theta + (ac' + ca')\bar{\theta} +$$

now

$$\int \prod_{i=1}^N d\bar{\theta}_i d\theta_i \exp(-\bar{\theta} \cdot A \cdot \theta) = \det[A]$$

$\triangle$ order is important

as many θ 's and $\bar{\theta}$'s as $d\theta$'s and $d\bar{\theta}$'s

$$= \sum_{\sigma} \left(\prod_{i=1}^N \theta_{i\sigma(i)} \right) \epsilon(\sigma)$$

permutations

signature = ± 1

anticommutative rule

$$\int d\bar{\theta}_2 d\theta_2 d\bar{\theta}_1 d\theta_1 \quad \underbrace{\theta_1 \bar{\theta}_2}_{1}$$

$$\det(A)$$

$$\theta_1 \bar{\theta}_1 + (A_{21}A_{12} - A_{11}A_{22}) \theta_1 \bar{\theta}_2 \theta_2 \bar{\theta}_1$$

only the last term

(i, j)

(i, j)

g: polynomial in coefficients

Gaussian Integrals

Hermitian Matrix $A = (A_{ij})$ $A_{ji} = \overline{A_{ij}}$

quadratic form $\overline{\theta} \cdot A \cdot \theta = \sum_{i,j=1}^N \overline{\theta}_i A_{ij} \theta_j = \sum_{i,j} -A_{ij} \theta_j \overline{\theta}_i$

$\exp(-\overline{\theta} \cdot A \cdot \theta) = \sum_{k=0}^N \frac{1}{k!} (-\overline{\theta} \cdot A \cdot \theta)^k$ it is a polynomial

example,

$N=1$ $\exp(-\overline{\theta} A \theta) = 1 + A \cdot \theta \overline{\theta}$

$N=2$ A is a 2×2 matrix $\exp(-\overline{\theta} \cdot A \cdot \theta) = 1 + \sum_{i,j} A_{ij} \theta_i \overline{\theta}_j + (A_{12} A_{21} - A_{11} A_{22}) \theta_1 \theta_2 \overline{\theta}_1 \overline{\theta}_2$
some algebra only the last term

$\det(A)$
||

now $\int \prod_{i=1}^N d\overline{\theta}_i d\theta_i \exp(-\overline{\theta} \cdot A \cdot \theta)$

Δ order is important as many θ 's as $d\theta$ and $\overline{\theta}$

$\int d\overline{\theta}_2 d\theta_2 d\overline{\theta}_1 d\theta_1 \underbrace{\theta_1 \overline{\theta}_1 \theta_2 \overline{\theta}_2}_{1}$

$$g' = a' + b'\theta + c'\bar{\theta} + d'\theta\bar{\theta}$$

$$g \cdot g' = aa' + (ab' + ba')\theta + (ac' + ca')\bar{\theta} + (ad' + da' + bc' - cb')\theta\bar{\theta}$$

$$[A] = \sum_{\sigma} \left(\prod_{i=1}^N \theta_{i, \sigma(i)} \right) \epsilon(\sigma)$$

permutations $\rightarrow$ signature = ± 1

anticommutative rule

NB for real Grassman numbers

$$A = -A^T \text{ antisymmetric}$$

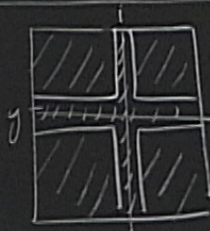
$$\text{Pfaffian}[A] \approx \sqrt{\det(A)}$$

ordinary complex number $z_i, \bar{z}_i \quad i=1, N$

$$\int_{\mathbb{C}^N} \prod_{i=1}^N d\bar{z}_i dz_i \exp(-\bar{z}_i A_{ij} z_j) = \det[A]^{-1}$$

if A is real $N \times N$ matrix

$$\int_{\mathbb{R}^N} \prod_{i=1}^N dx_i \exp(-x_i A_{ij} x_j) = (\det[A])^{-1/2}$$



$$A \text{ } N \times N \quad \text{Minor}[A]_{ij} = \det[\text{Minor matrix } A_{kl}]_{k,l \neq i,j}$$

"Correlation function" 2 phs cancel

like an action

observable

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\begin{bmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{bmatrix} = (\bar{A}^{-1})$$

$$\langle g \rangle = \frac{\int \prod_i d\bar{\theta}_i d\theta_i \exp(-\bar{\theta} \cdot A \cdot \theta) g}{\int \prod_i d\bar{\theta}_i d\theta_i \exp(-\bar{\theta} \cdot A \cdot \theta)}$$

↑
expectation value

$$g = \theta_i \bar{\theta}_j \quad \langle \theta_i \bar{\theta}_j \rangle$$

$$N=1 \quad \langle \theta \bar{\theta} \rangle = \frac{1}{A} \quad \langle \theta \cdot \theta \rangle = \langle \bar{\theta} \cdot \bar{\theta} \rangle = 0$$

$$\langle \bar{\theta} \theta \rangle = -\frac{1}{A} \quad (-1)^{1/2}$$

$$N=2 \quad \langle \theta_i \bar{\theta}_j \rangle = \frac{(\text{Minor } A)_{ij}}{\det A} = (\bar{A}^{-1})_{ij}$$

for any N

$$\langle \theta_i \bar{\theta}_j \rangle = (\bar{A}^{-1})_{ij}$$

$$\langle \bar{\theta}_i \theta_j \rangle = -(\bar{A}^{-1})_{ji}$$

$$\langle \theta_i \theta_j \rangle = 0$$

$$\langle \bar{\theta}_i \bar{\theta}_j \rangle = 0$$

same rule

$$\langle \theta_i \theta_j \bar{\theta}_k \bar{\theta}_l \rangle$$

acten

-observable

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \begin{bmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{bmatrix} \quad (A^{-1})$$

$$\begin{bmatrix} A_{11} A_{22} - A_{12} A_{21} \end{bmatrix} =$$

4 pls funchen

$$\langle \theta_i \bar{\theta}_j \theta_k \bar{\theta}_e \rangle = \bar{A}_{ij}^{-1} \bar{A}_{ke}^{-1} - \bar{A}_{ie}^{-1} \bar{A}_{jk}^{-1}$$

$$= \langle \theta_i \bar{\theta}_j \rangle \langle \theta_k \bar{\theta}_e \rangle - \langle \theta_i \bar{\theta}_e \rangle \langle \theta_k \bar{\theta}_j \rangle$$

Wick Theorem for Dirac Fields

here no spin, but Fermi statistics

$$\begin{matrix} \theta_i & \theta_j & \theta_k & \theta_l \\ \text{---} & & \text{---} & \\ & \text{---} & & \end{matrix} +$$

$$\begin{matrix} & \text{---} & & \\ \text{---} & & \text{---} & \end{matrix} -$$

and so on for general

2m point functions

$$\langle \underbrace{\theta \bar{\theta} \dots \theta \bar{\theta}}_m \rangle$$

and so on for general
 $2m$ point functions

$$\langle \underbrace{\theta \bar{\theta} \dots \theta \bar{\theta}}_m \rangle$$

$m \theta$'s and $m \bar{\theta}$'s

$$\Rightarrow \neq 0$$

Proof: add Grassmannian source terms
 J_i and $\bar{J}_i$

$$\int d\bar{\theta}_i d\theta_i \exp(-\bar{\theta}_i A_{ij} \theta_j + \bar{J}_i \bar{\theta}_i + \theta_i J_i)$$

$\nwarrow \nearrow$
 cancelling -

der A

For Dirac Fields

Dirac Equ $\rightarrow$ Action

Dirac Field $\Psi(x), \bar{\Psi}(x)$

$x = (t, \vec{x})$

in 4d Ψ has 4 components $\Psi = (\Psi^a; a=1,4)$
 $a = \text{Dirac Indices}$ $\bar{\Psi} = (\bar{\Psi}^a; a=1,4)$

$$\text{Action } S[\bar{\Psi}, \Psi] = \int d^4x \bar{\Psi}(x) (i \not{\partial} - m) \Psi(x)$$

$$\not{\partial} = \gamma^\mu \partial_\mu$$

$\mathcal{L}_{\text{Dirac Mah}}$

$$\gamma^\mu = (\gamma^\mu_a)$$

4×4 matrices

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$$

$$= -2\eta^{\mu\nu}_{\text{real coord}}$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1,1,1 \end{pmatrix}$$

$\downarrow$

$\leftarrow \text{West Coast}$

$$ds^2 = -dt^2 + d\vec{x}^2$$

$$\frac{\delta S}{\delta \bar{\Psi}(x)} \Rightarrow (i \not{\partial} - m) \Psi = 0$$

$$ds^2 = -dt^2 + d\vec{x}^2$$

$$\frac{\delta S}{\delta \bar{\Psi}(x)} \Rightarrow (i \not{\partial} - m) \Psi = 0$$

Dirac Particles

Ψ^a are real functions

↓
Dirac Field

$\Psi^a, \bar{\Psi}^a \Rightarrow$ operators acting
on a Fock Space

$\Psi^a, a=1,4$

$\bar{\Psi}^a, a=1,4$

$$h^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1_{3 \times 3} \end{pmatrix}$$

↓

$\Psi(x)$

Recipe for Functional Integral

Promote each $\psi^a(x), \bar{\psi}^a(x)$ to Grassman Numbers
~~~~~

$$a = 1, 4$$

$$x \in \mathbb{M}_4$$



$$\frac{\delta S}{\delta \psi(x)} \Rightarrow (i \not{\partial} - m) \psi = 0$$

$\psi^a$  are real functions

$\psi^a, \bar{\psi}^a \Rightarrow$  operators acting  
on a Fock Space

Recipe for Functional Integral

Promote each  $\psi^a(x), \bar{\psi}^a(x)$  to Grassman Numbers

$$a = 1, 4$$

$$x \in \mathbb{M}_4$$

$$i = 1, N \Rightarrow (a, x)$$