Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 9

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Collection: PSI 2019/2020 - Quantum Field Theory II (David)

Date: November 22, 2019 - 9:00 AM

URL: http://pirsa.org/19110019

- Renormalization 1 loop for ϕ^4
- "Running coupling constant"
- Renormalization Group
- Scale Invariance anomaly
- QFT renormalization $\longleftrightarrow$ Wilson. R. G

ϕ^4

for ϕ^4

nt"

ilson . R.G

ϕ^4 $d=4$ Euclidean $\hbar = 1$

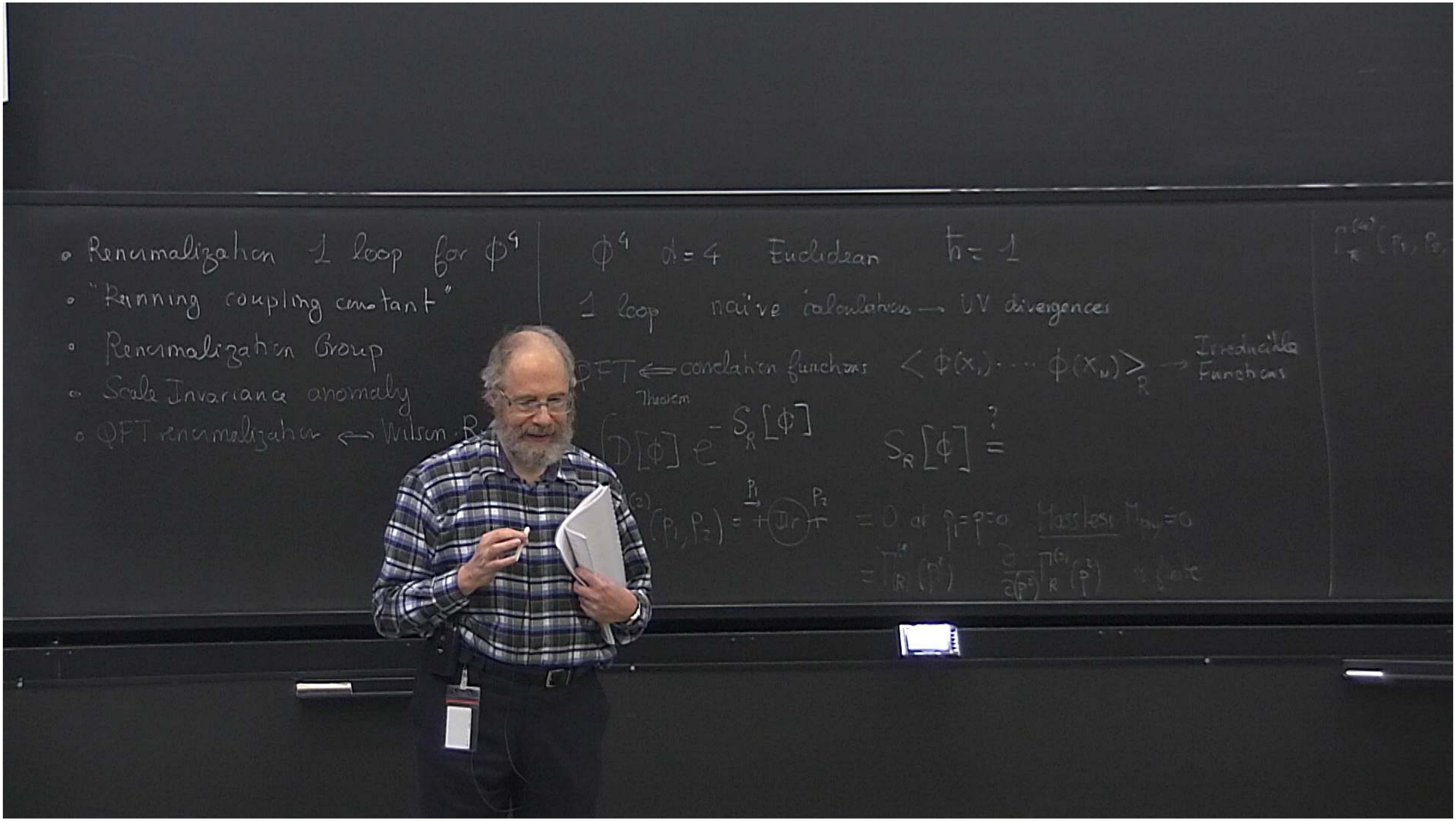
1 loop naïve calculations $\rightarrow$ UV divergences

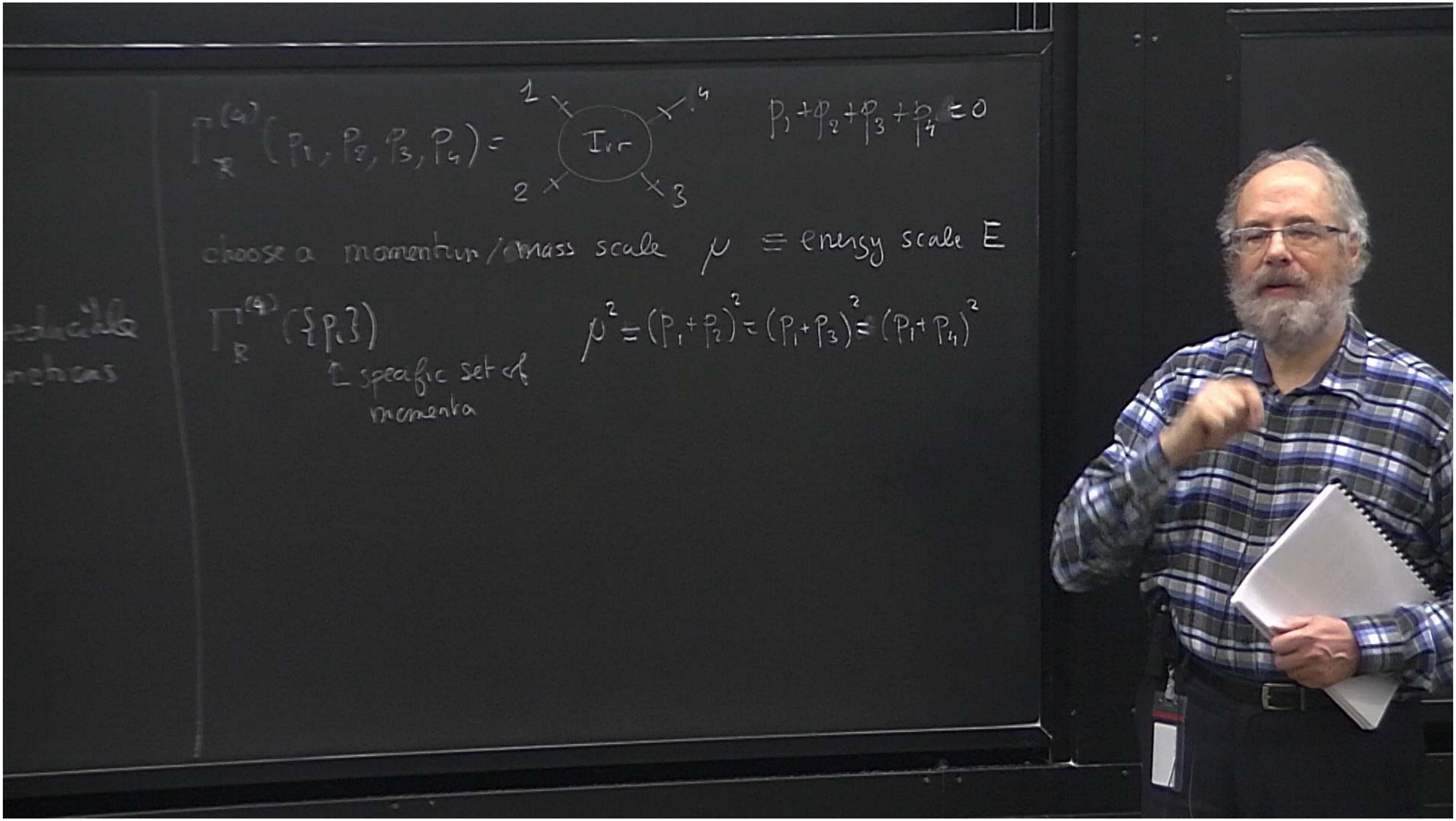
QFT $\underset{\text{Theorem}}{\Longleftarrow}$ correlation functions $\langle \phi(x_1)\cdots\phi(x_N)\rangle_R \rightarrow$ Irreducible Functions

$$\int D[\phi]\, e^{-S_R[\phi]} \qquad S_R[\phi] \overset{?}{=}$$

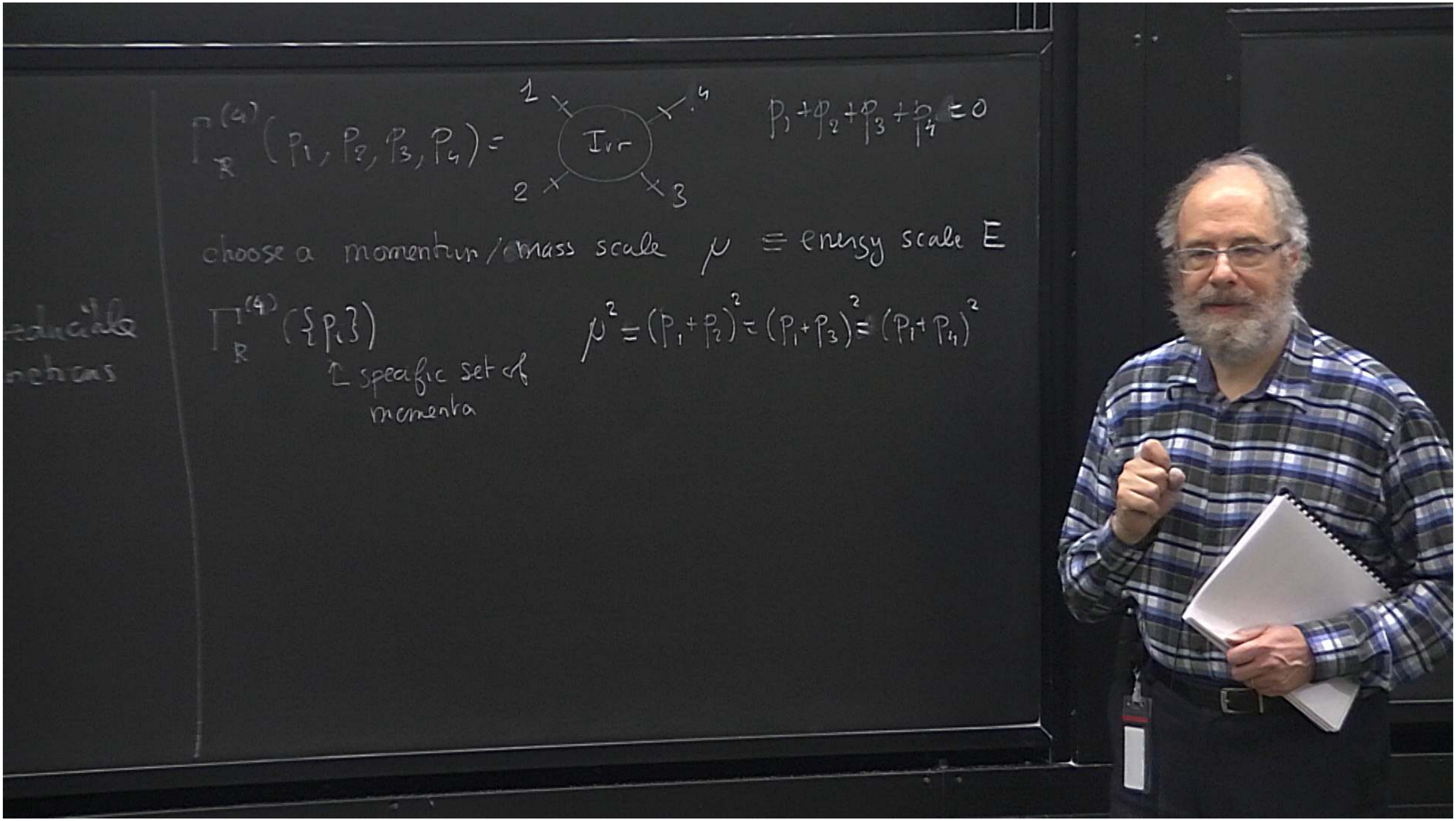
Condition 1 $\Gamma^{(2)}_{(R)}(p_1, p_2) = \overset{p_1}{\rightarrow}$ Jir $\overset{p_2}{}$ $= 0$ at $p_1 = p = 0$ Massless $M_{phy} = 0$

$= \Gamma^{(2)}_{(R)}(p^2)$ $\quad \frac{\partial}{\partial p^2}\Gamma^{(2)}_R(p^2)$ is finite


Renormalization 1 loop for φ⁴
"Running coupling constant"
Renormalization Group
Scale Invariance anomaly
QFT renormalization ⟷ Wilson R
φ⁴ d = 4 Euclidean ħ = 1
1 loop naive calculations → UV divergences
QFT ⟸ correlation functions ⟨φ(x₁) ⋯ φ(x_N)⟩_R → Irreducible Functions
Theorem
∫D[φ] e^{−S_R[φ]}
S_R[φ] =?
Massless M_phy = 0



$\Gamma^{(4)}_R(p_1, p_2, p_3, p_4) =$
1
4
Irr
2
3
$p_1+p_2+p_3+p_4=0$
choose a momentum/mass scale μ ≡ energy scale E
$\Gamma^{(4)}_R(\{p_i\})$
specific set of momenta
$\mu^2 = (p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2$



$\Gamma_R^{(4)}(p_1, p_2, p_3, p_4) =$
Irr
1
2
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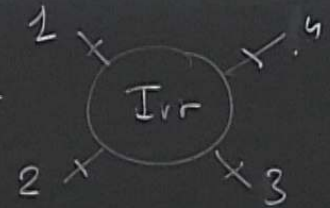
4
$p_1 + p_2 + p_3 + p_4 = 0$
choose a momentum/mass scale μ ≡ energy scale E
$\Gamma_R^{(4)}(\{p_i\})$
specific set of momenta
$\mu^2 = (p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2$

Irreducible Functions

$$\Gamma_R^{(4)}(p_1, p_2, p_3, p_4) =$$ 

$$p_1 + p_2 + p_3 + p_4 = 0$$

choose a momentum / mass scale μ $\equiv$ energy scale E

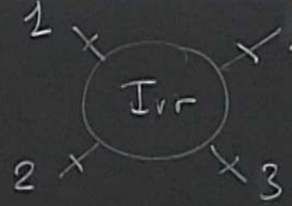
$$\Gamma_R^{(4)}(\{p_i\}) := g_R \qquad \mu^2 = (p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2$$

↑ specific set of momenta

reference point in momentum space

defines a coupling parameter g_R : the renormalized coupling constant

$\Gamma_R^{(4)}(p_1, p_2, p_3, p_4) =$



$p_1 + p_2 + p_3 + p_4 = 0$

choose a momentum / mass scale μ $\equiv$ energy scale E

$\Gamma_R^{(4)}(\{p_i\}) := g_R$ ← specific set of momenta

$\mu^2 = (p_1 + p_2)^2 = (p_1 + p_3)^2 = (p_1 + p_4)^2$

reference point in momentum space

defines a coupling parameter g_R : the renormalized coupling constant

Q Are all correlation functions UV finite for all value of the momenta

Answer: Conditions 1 & 2 Renormalization $\Rightarrow S_R[\phi] = \int d^4x \; \frac{\mathcal{A}}{2}(\partial_\mu \phi)^2 + \frac{\mathcal{B}}{2}\phi^2 + \frac{\mathcal{C}}{4!}\phi^4$

$$\mathcal{A} = 1 + O(g_R^2)$$

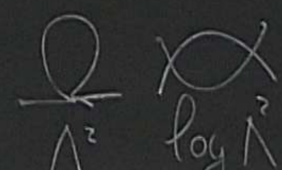
$$\mathcal{B} = -g_R \frac{1}{2}\frac{1}{(4\pi)^2}\Lambda^2 + O(g_R^2)$$

$$\mathcal{C} = g_R + g_R^2 \frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right) + O(g_R^3)$$

↑

"Counterterms"

"Magic" cancelled



$$\Gamma_R^{(2)}(p^2) = p^2 + O(g_R^2)$$

$$\Gamma_R^{(4)}(\{p_i\}) = g_R - g_R^2 \frac{1}{2}\frac{1}{(4\pi)}$$

$\Gamma_R^{(6)} =$

$+$ " "

correl. functions finite?

$\Gamma_{(R)}(p_1,p_2) = + (\text{Ir}) + \;= 0$ at $p_1=p=0$ Massless $M_{phy}=0$

$\left[= \Gamma^{(2)}_{(R)}(p^2) \quad \frac{\partial}{\partial(p^2)}\Gamma^{(2)}_{R}(p^2) \text{ is finite} \right]$

all value of the momenta (at least at 1 loop order) and 2

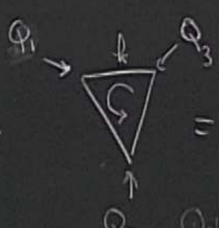
$\frac{1}{2}(\partial_\mu\phi)^2 + \frac{B}{2}\phi^2 + \frac{g}{4!}\phi^4$

YES, IT WORKS (at 1 loop)

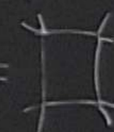
$$\Gamma^{(2)}_R(p^2) = p^2 + O(g_R^2)$$

$$\Gamma^{(4)}_R(\{p_i\}) = g_R - g_R^2\,\frac{1}{2}\,\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{(p_1+p_2)^2} + \log\frac{\mu^2}{(p_1+p_3)^2} + \log\frac{\mu^2}{(p_1+p_4)^2}\right] + O(g_R^3)$$

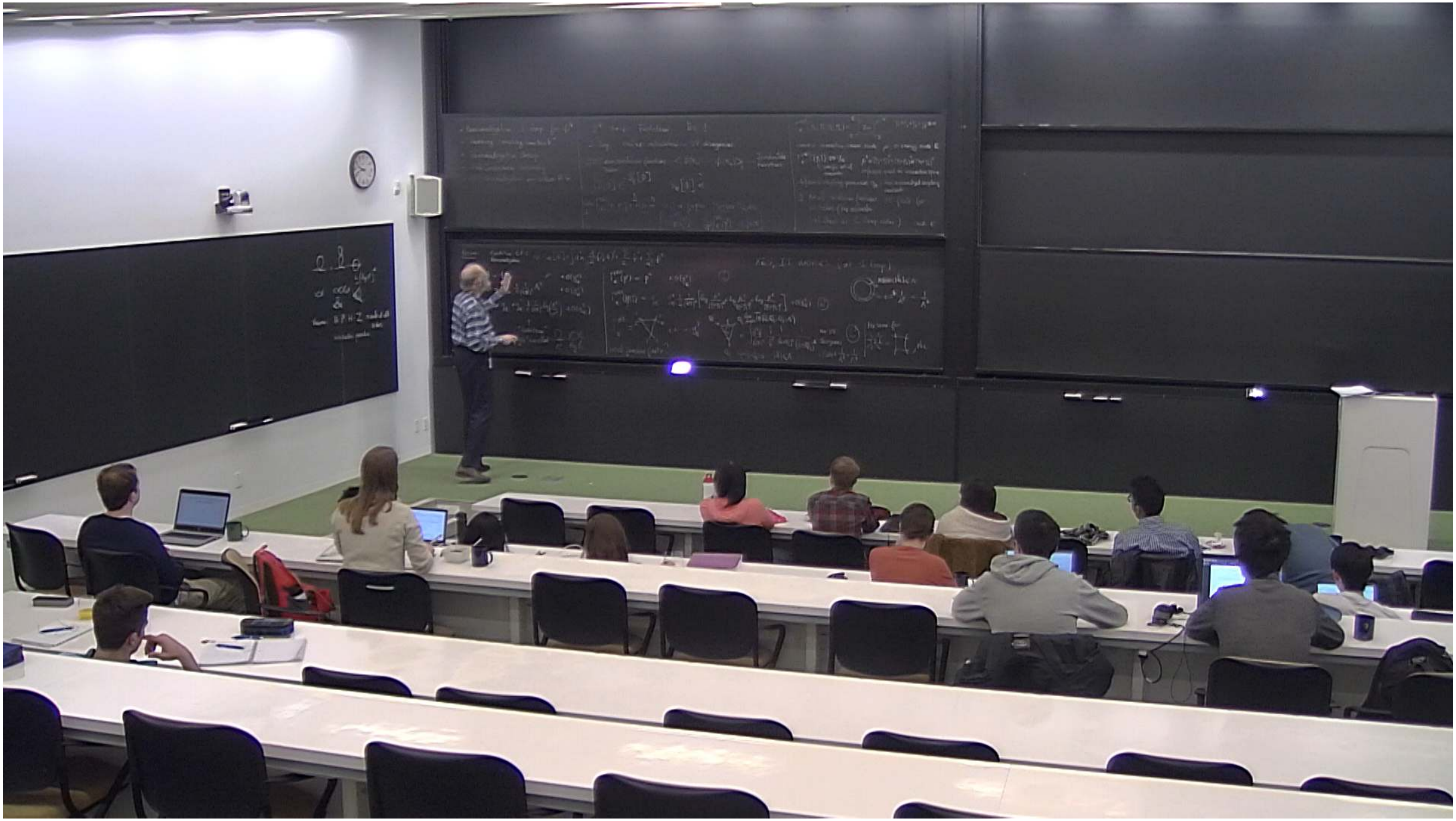
$\Gamma^{(6)}_R =$  $+ \ldots \; g_R^3$,

 $= \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\,\frac{1}{(k+Q_1)^2}\,\frac{1}{(k-Q_3)^2}$ no UV divergences

$Q_1+Q_2+Q_3=0$

the same for  $=$ , etc

correl. functions finite?



$\Gamma^{(2)}_{(R)}(p_1, p_2) = \ldots = 0$ at $p_1 = p_2 = 0$ Massless $M_{phy} = 0$

$\left[= \Gamma^{(2)}_{(R)}(p^2) \quad \frac{\partial}{\partial (p^2)} \Gamma^{(2)}_R (p^2) \text{ is finite} \right]$

all value of the momenta

(at least at 1 loop order) cond. 2

$(\partial_\mu \phi)^2 + \frac{B}{2}\phi^2 + \frac{C}{4!}\phi^4$

YES, IT WORKS (at 1 loop)

$\Gamma^{(2)}_R (p^2) = p^2 + O(g_R^2)$

$\Lambda(1-\epsilon) < |k| < \Lambda$

$\epsilon \Lambda^4 \frac{1}{\Lambda^6} \sim \frac{1}{\Lambda^2}$

$$\Gamma^{(4)}_R(\{p_i\}) = g_R - g_R^2 \frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{(p_1+p_2)^2} + \log\frac{\mu^2}{(p_1+p_3)^2} + \log\frac{\mu^2}{(p_1+p_4)^2}\right] + O(g_R^3)$$

$\lim_{\Lambda\to\infty} T_R(Q_1, Q_2, Q_3; \Lambda)$

$$\Gamma^{(6)}_R = \ldots + \text{''} \; \text{''} \; g_R^3, \quad \ldots = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(k+Q_1)^2} \frac{1}{(k-Q_3)^2}$$

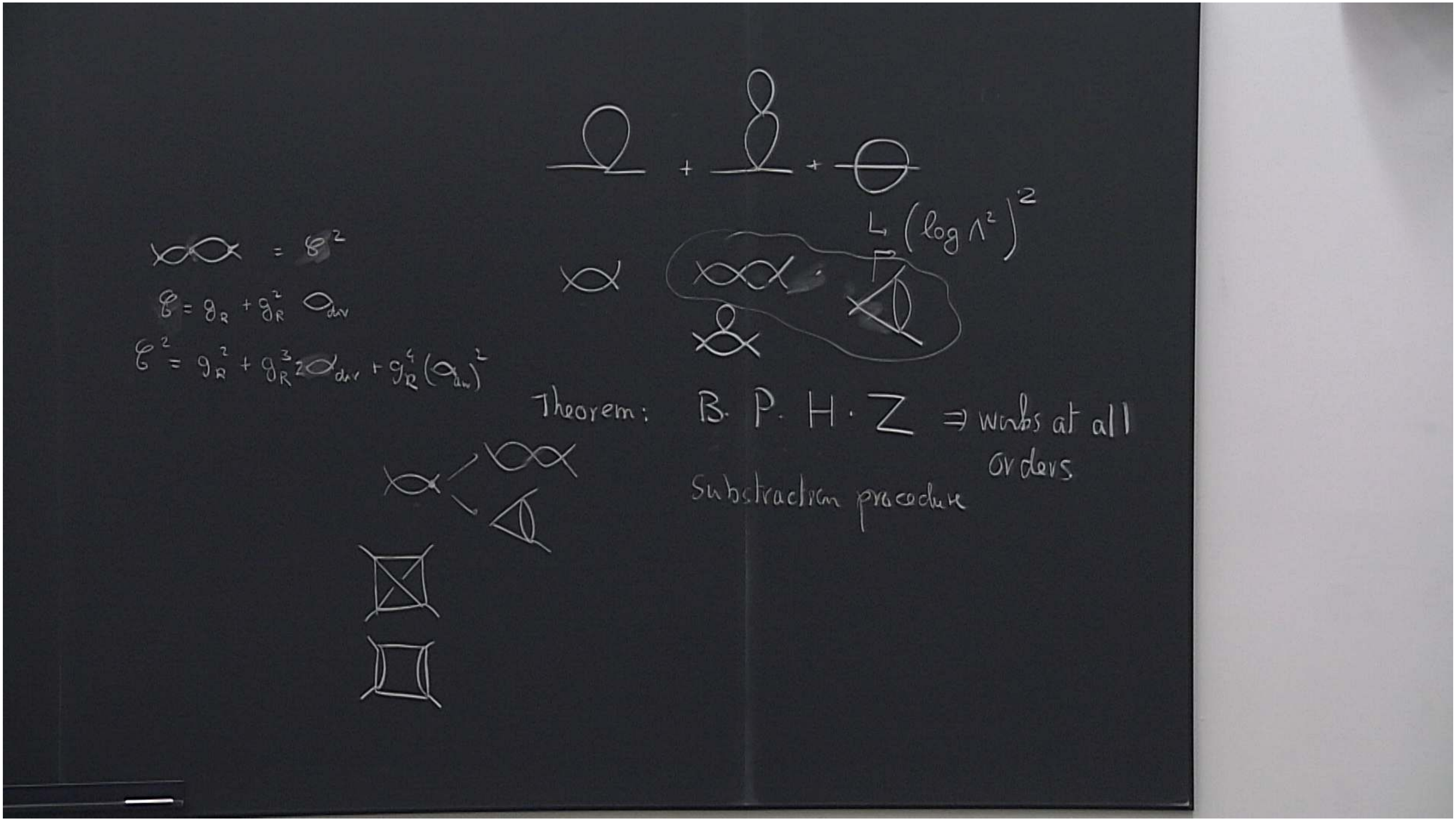
$Q_1 + Q_2 + Q_3 = 0 \quad |k| < \Lambda$

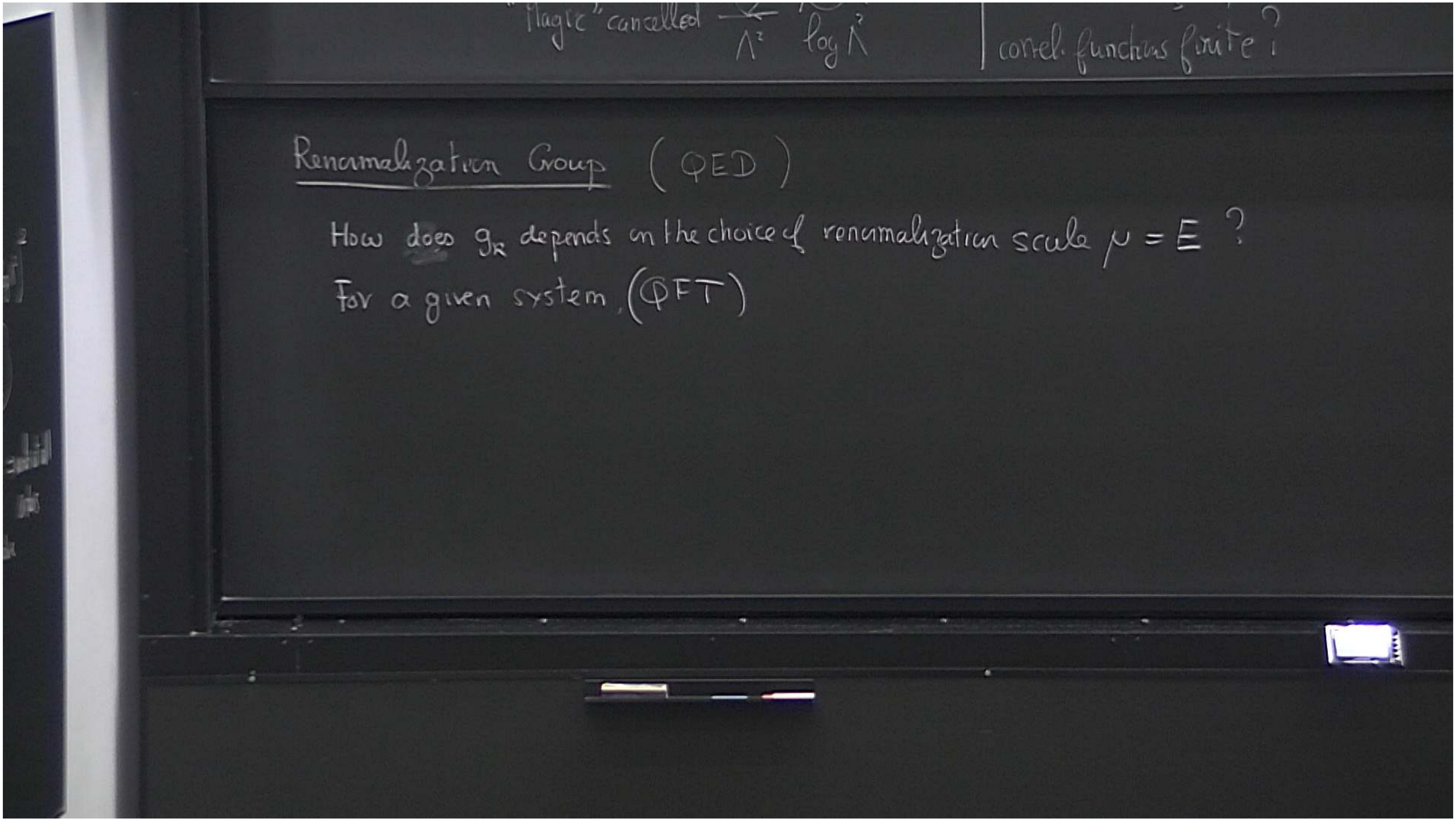
no UV divergences

$= \text{Finite} + \frac{1}{\Lambda^2} + \frac{1}{\Lambda^4}$

the same for $\ldots = \ldots$, etc

correl. functions finite?


(log Λ²)²
Theorem: B. P. H. Z ⇒ works at all orders
Substraction procedure


"Magic" cancelled Λ^2 $\log \Lambda$
correl. functions finite?
Renormalization Group (QED)
How does g_R depends on the choice of renormalization scale $\mu = E$?
For a given system, (QFT)

Renormalization Group (QED)

How does g_R depends on the choice of renormalization scale $\mu = E$?

For a given system, (QFT)

different choice of μ

$\downarrow$

different value for g_R

$$\Gamma^{(4)}_R(\{p_i\}; g_R, \mu) = \Gamma^{(4)}_R(\{p_i\}; g'_R, \mu')$$

$\{p_i\}$: physical arguments of the function

g_R, μ: parameters label the theory

choose $\mu \Rightarrow g_R$; choose $\mu' \Rightarrow g'_R$

correl. functions finite

Renormalization Group (QED)

How does g_R depends on the choice of renormalization scale $\mu = E$?

For a given system, (QFT)

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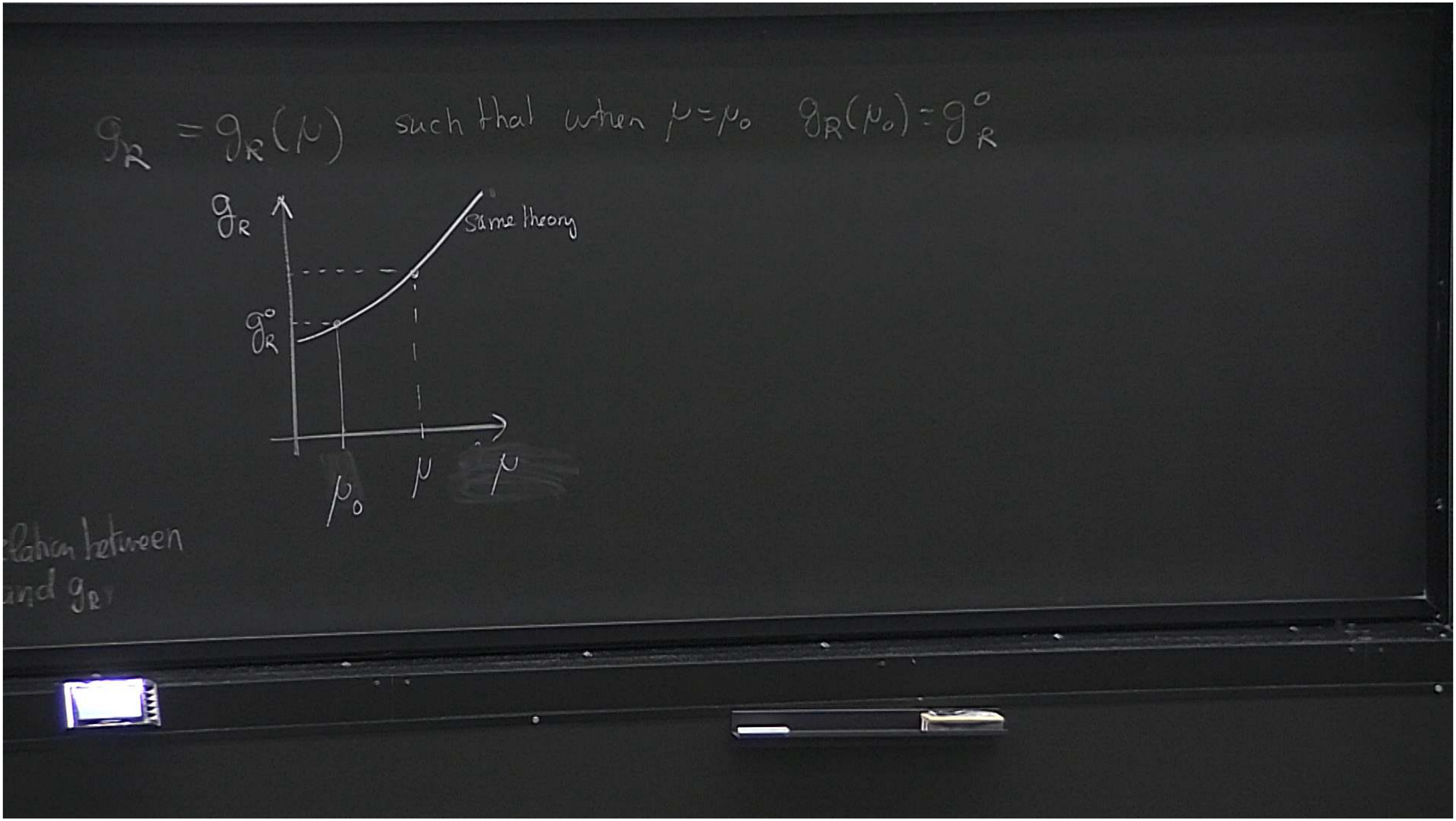
diffent choice of μ

$\downarrow$

different value for g_R

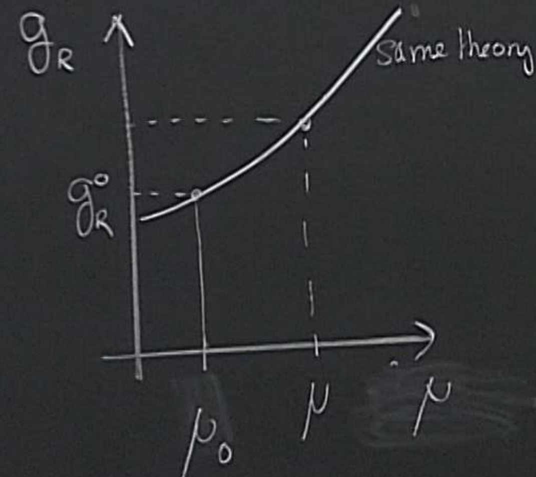
choose $\mu \Rightarrow g_R$; choose $\mu' \Rightarrow g'_R$

there is a relation between g_R and $g_{R'}$



g_R = g_R(μ) such that when μ = μ_0 g_R(μ_0) = g_R^0
g_R
same theory
g_R^0
μ_0
μ
μ
lation between
and g_R

$g_R = g_R(\mu)$ such that when $\mu = \mu_0$ $g_R(\mu_0) = g_R^0$: which relation-

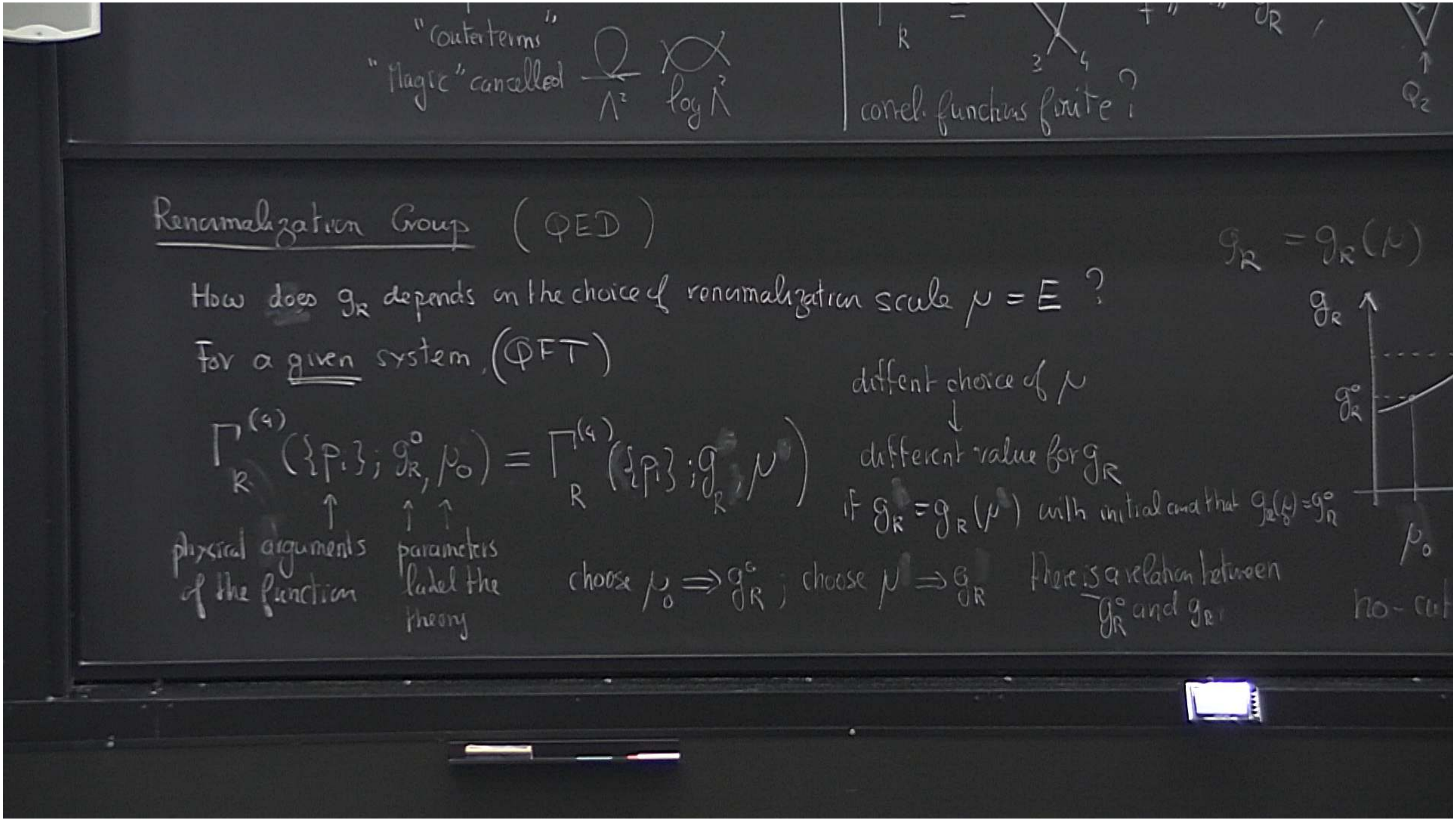


$$g_R^0 - g_R^{0\,2}\,\frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu_0^2}{''} + '' + ''\right] + O(g_R^{0\,3})$$

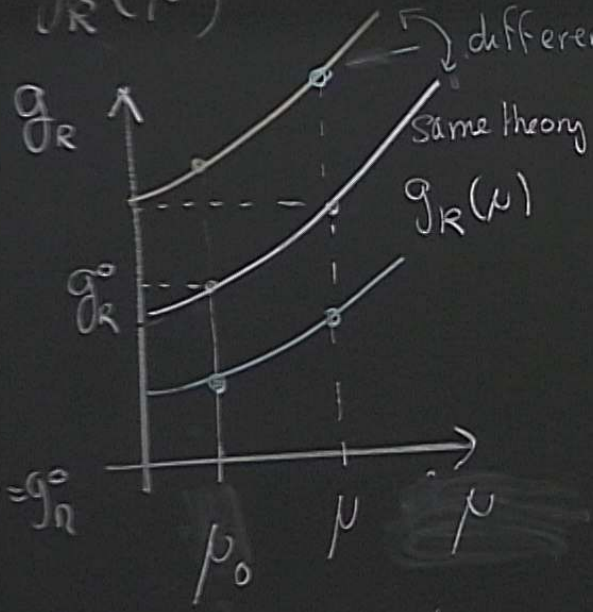
$$= g_R - g_R^2\,\frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{''} + '' + ''\right] + O(g_R^3)$$

$$\boxed{g_R(\mu) - g_R^2(\mu)\,\frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\mu^2}{\mu_0^2}\right) = g_R^0} + O(g_R^{0\,3})$$

elation between
and g_R


"Counterterms"
"Magic" cancelled
Λ^2 $\log\Lambda^2$
correl. functions finite?
Renormalization Group (QED)
$g_R = g_R(\mu)$
How does g_R depends on the choice of renormalization scale $\mu = E$?
For a given system (QFT)
$\Gamma^{(4)}_R(\{p_i\}; g^0_R, \mu_0) = \Gamma^{(4)}_R(\{p_i\}; g_R, \mu)$
physical arguments of the function
parameters label the theory
diffent choice of μ
different value for g_R
if $g_R = g_R(\mu)$ with initial cond that $g_R(\mu_0) = g^0_R$
choose $\mu_0 \Rightarrow g^0_R$; choose $\mu \Rightarrow g_R$
There is a relation between g^0_R and g_R
g_R
g^0_R
μ_0

$g_R = g_R(\mu)$ such that when $\mu = \mu_0$ $\quad g_R(\mu_0) = g^0_R$: which relation-



$$g^0_R - g^{0\,2}_R \, \frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu_0^2}{''} + '' + ''\right] + O(g^{0\,3}_R)$$

$$= g_R - g^2_R \, \frac{1}{2}\frac{1}{(4\pi)^2}\left[\log\frac{\mu^2}{''} + '' + ''\right] + O(g^3_R)$$

$$\boxed{g_R(\mu) - g^2_R(\mu)\,\frac{3}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\mu^2}{\mu_0^2}\right) = g^0_R} + O(g^{0\,3}_R)$$

hat $g_R(\xi) = g^0_R$

etween

no-cut-off !

Better to write it in a differential form

Beta function:

$$g_R^0, \mu_0 \longrightarrow g_R(\mu), \mu$$

$$\boxed{\beta(g_R(\mu)) = \mu \frac{d}{d\mu} g_R(\mu)} = \frac{3}{(4\pi)^2} g_R^2(\mu) + \dots$$

$g_R(\mu_0) = g_R^0$

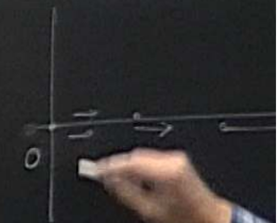
Flow equation generated by a vector field in space of g

calculation

The β function is a function of $g_R(\mu)$ only!

$$\boxed{\beta(g_R) = \frac{3}{(4\pi)^2} g_R^2 + \dots}$$

for the ϕ^4 theory



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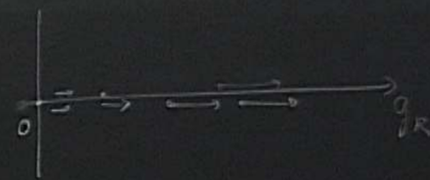
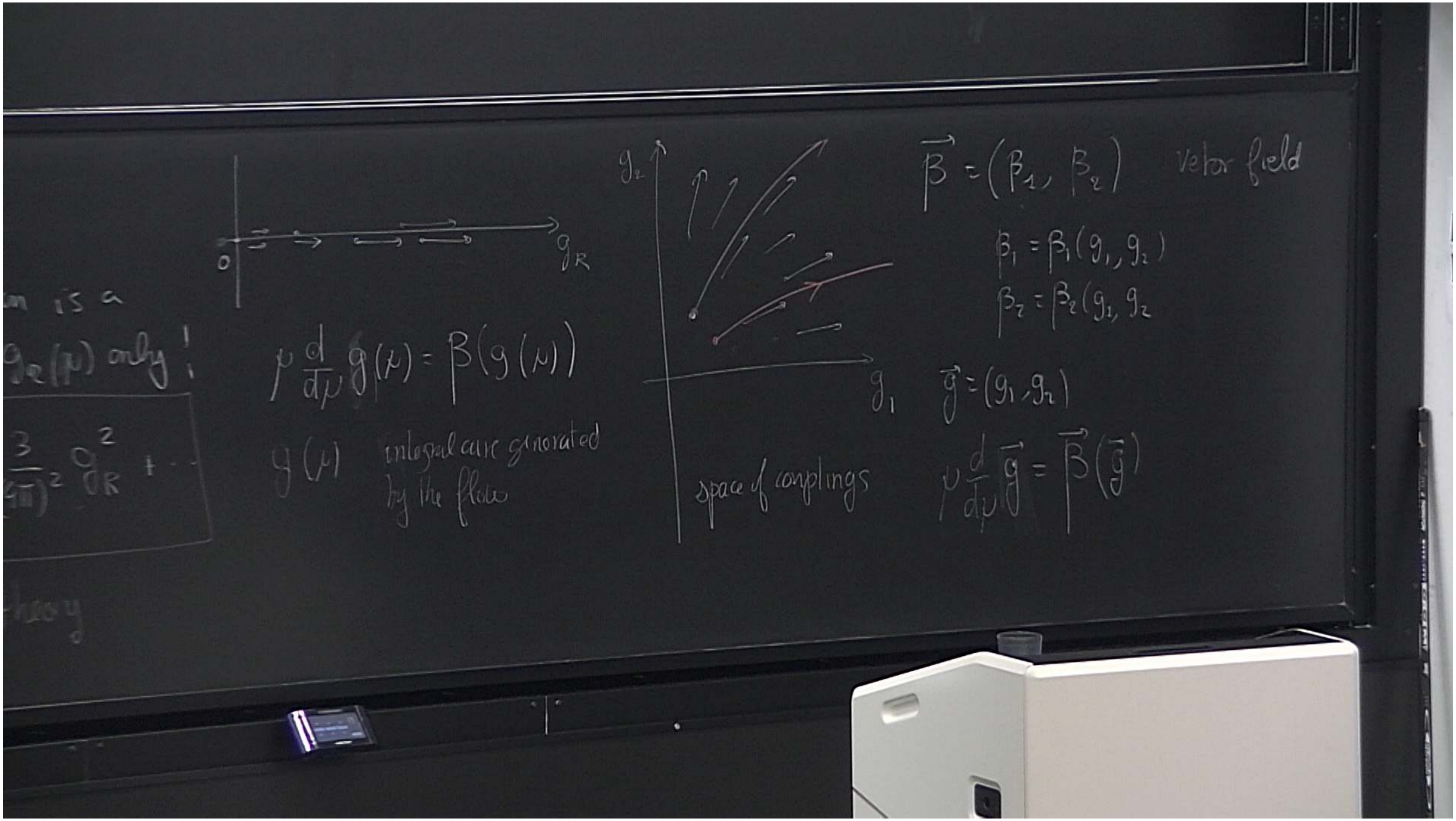
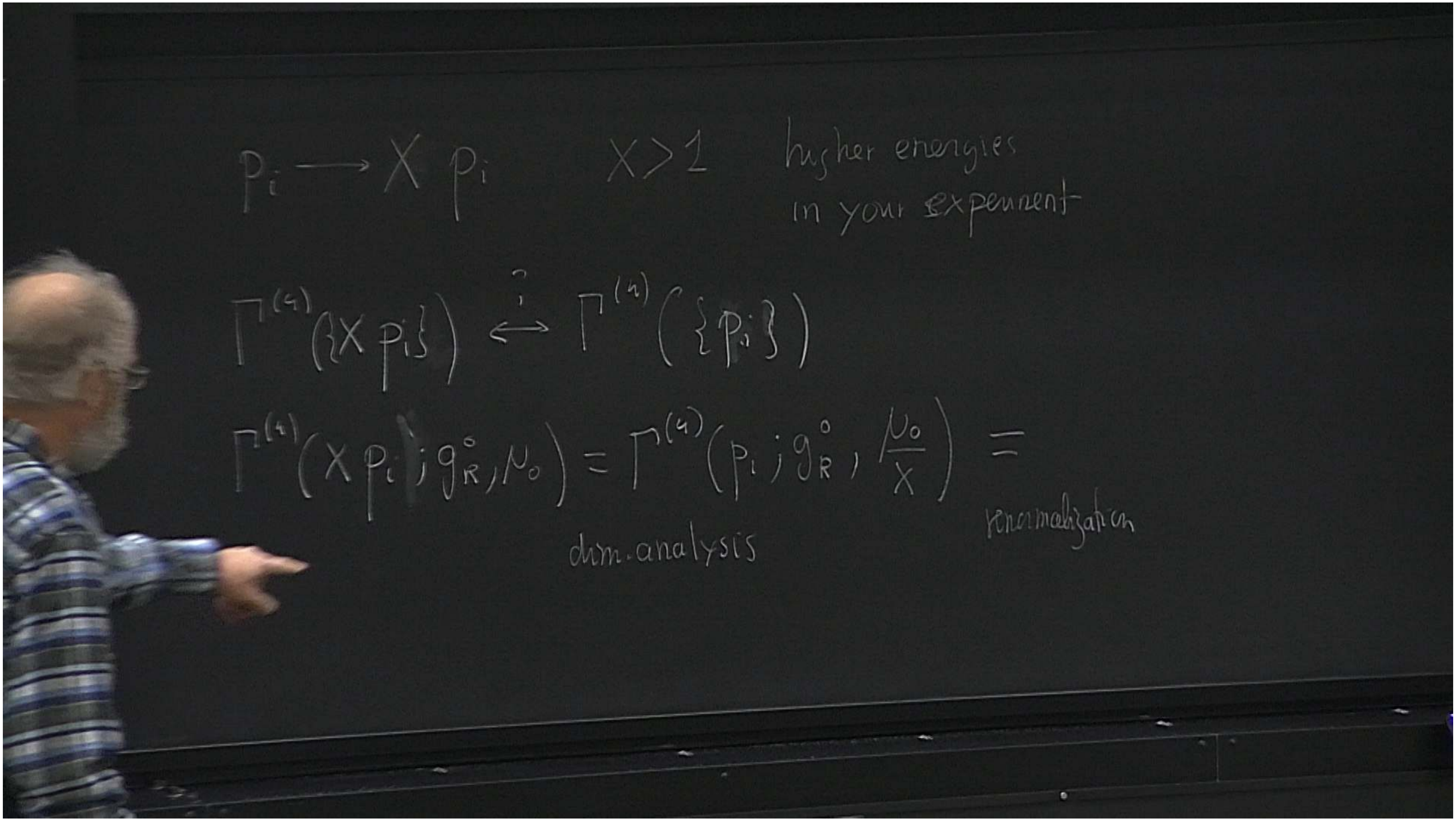


$$\mu \frac{d}{d\mu} g(\mu) = \beta(g(\mu))$$

$g(\mu)$ integral curve generated by the flow


is a
$g_R(\mu)$ only !
theory
g_R
0
$\mu \frac{d}{d\mu} g(\mu) = \beta(g(\mu))$
$g(\mu)$ integral curve generated by the flow
g_2
g_1
space of couplings
$\vec{\beta} = (\beta_1, \beta_2)$ vector field
$\beta_1 = \beta_1(g_1, g_2)$
$\beta_2 = \beta_2(g_1, g_2$
$\vec{g} = (g_1, g_2)$
$\mu \frac{d}{d\mu} \vec{g} = \vec{\beta}(\vec{g})$


$p_i \longrightarrow X\, p_i \qquad X > 1$
higher energies
in your experiment
$\Gamma^{(4)}(\{X p_i\}) \overset{?}{\longleftrightarrow} \Gamma^{(4)}(\{p_i\})$
$\Gamma^{(4)}(X p_i ; g_R^\circ, \mu_0) = \Gamma^{(4)}(p_i ; g_R^\circ, \frac{\mu_0}{X}) =$
dim. analysis
renormalization

field in space of g

$p_i \longrightarrow X\, p_i \qquad X>1$ higher energies in your experiment

what is the relation between $g_R(X)$ and X?

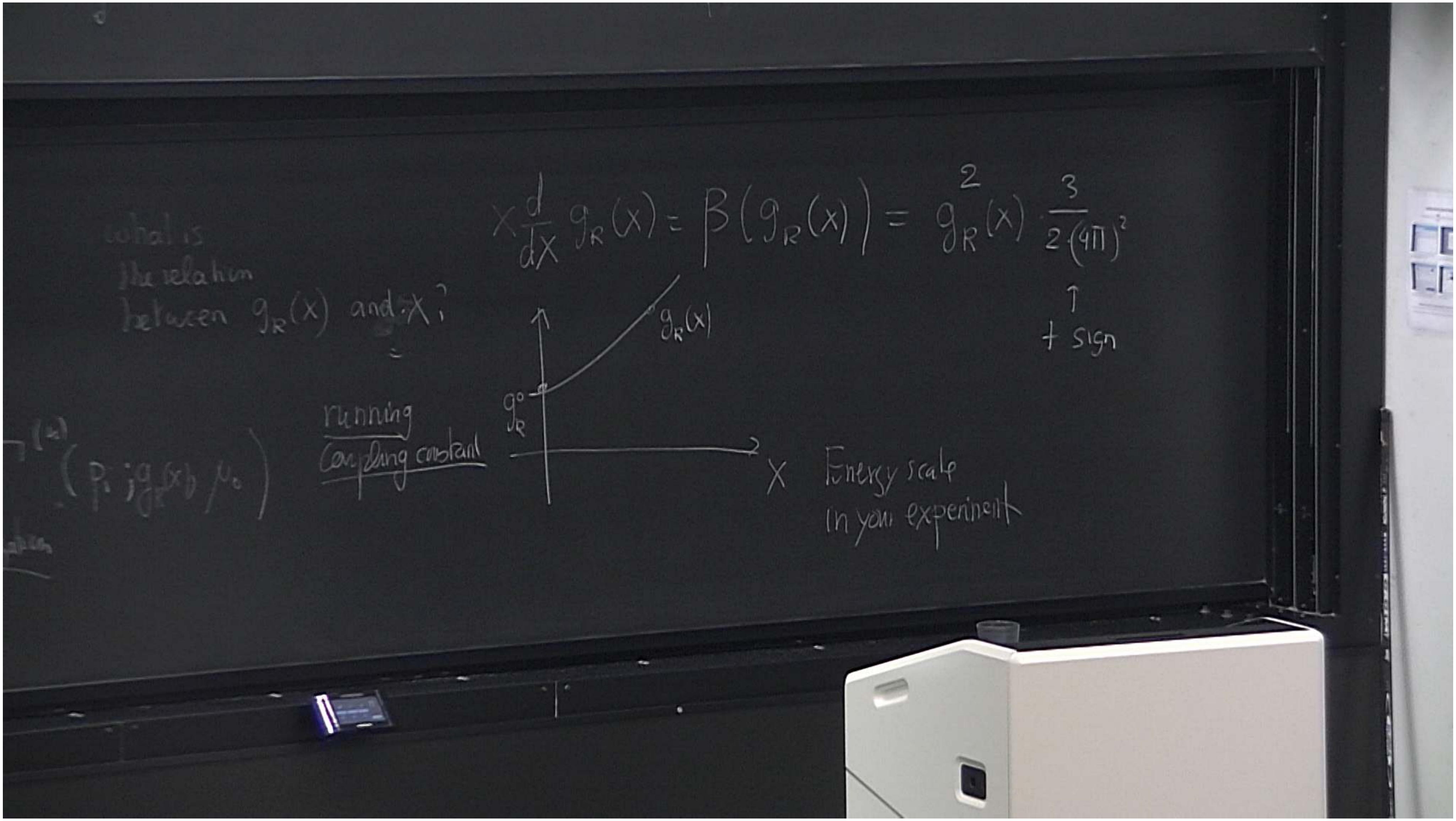
$X \frac{d}{dX}$

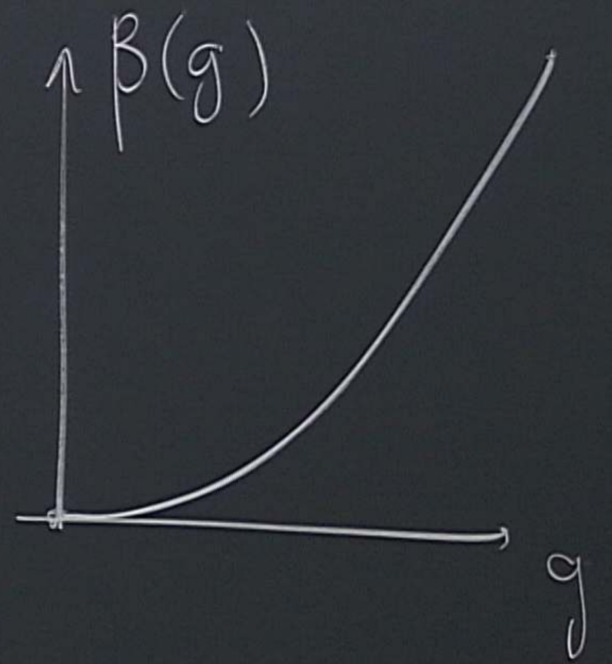
$$\Gamma^{(4)}(\{X p_i\}) \overset{?}{\longleftrightarrow} \Gamma^{(4)}(\{p_i\})$$

$$\Gamma^{(4)}(X p_i ; g_R^0, \mu_0) = \Gamma^{(4)}\left(p_i ; g_R^0, \frac{\mu_0}{X}\right) = \Gamma^{(4)}(p_i ; g_R(X), \mu_0)$$

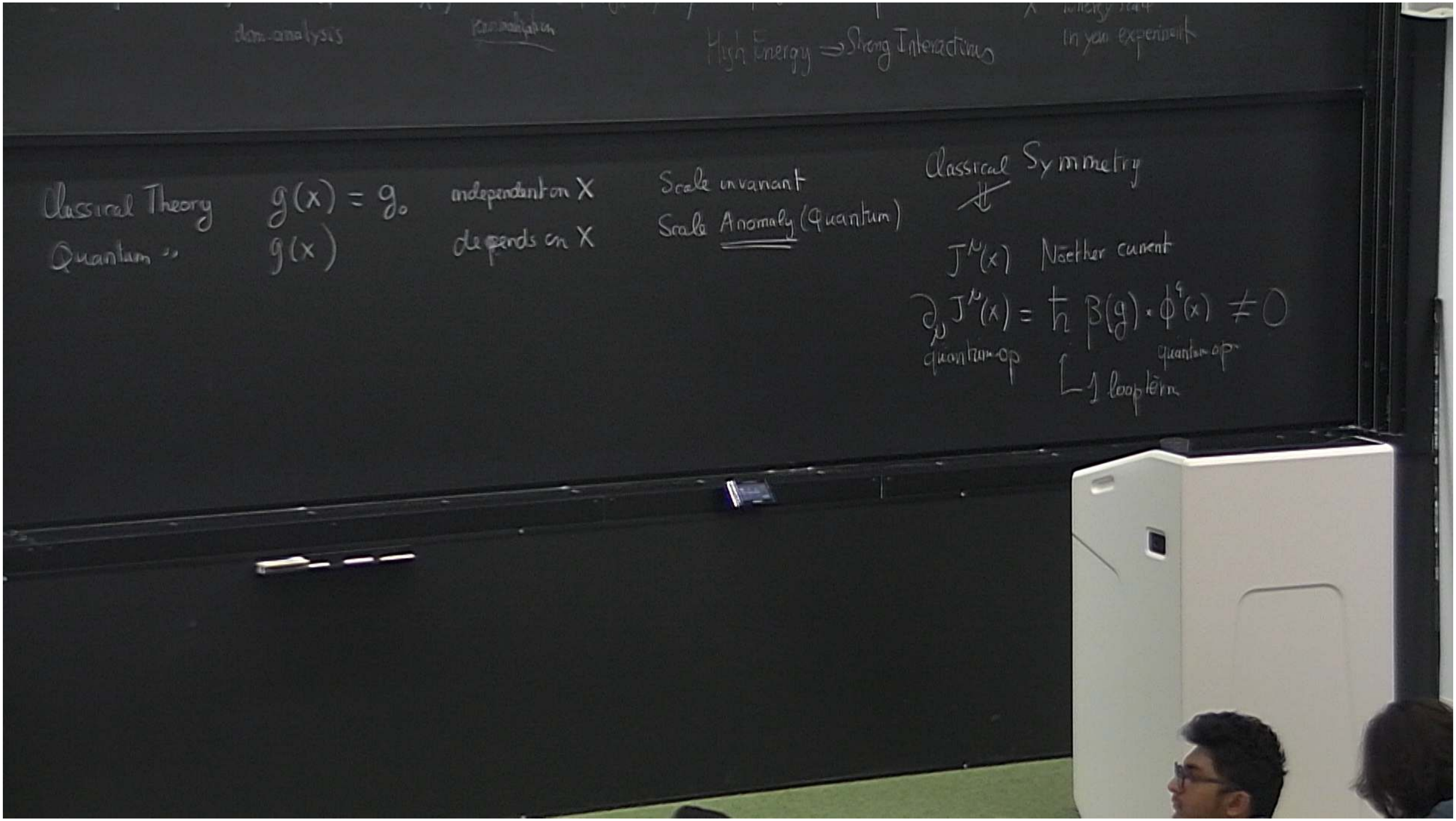
dim. analysis

renormalization


$x \frac{d}{dx} g_R(x) = \beta(g_R(x)) = g_R^2(x) \frac{3}{2(4\pi)^2}$
+ sign
What is the relation between $g_R(x)$ and x?
$g_R(x)$
g_R^0
running coupling constant
X Energy scale in your experiment



$$\beta(g) = b_0 g^2 \qquad b_0 > 0$$


dim. analysis
renormalization
High Energy ⇒ Strong Interactions
in your experiment
Classical Theory $g(x) = g_0$ independent on X
Quantum : $g(x)$ depends on X
Scale invariant
Scale Anomaly (Quantum)
Classical Symmetry
$J^\mu(x)$ Noether current
$\partial_\mu J^\mu(x) = \hbar\, \beta(g) \cdot \phi^4(x) \neq 0$
quantum op
quantum op
1 loop term