

Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 8

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URL: <http://pirsa.org/19110018>

- Renormalization for ϕ^4 at 1 loop order
- Corrections to the mass
- " " coupling
- Renormalization "scale" & renormalization group

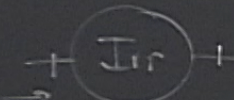
Classical

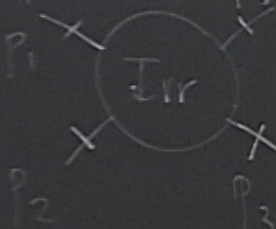
S

) Massless
Theory

Classical

$$S[\phi] = \int d^d x \quad \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4$$


 $= p^2 + m^2 + \frac{g}{2} \text{ (tadpole loop) } + O(g^2)$
 $M_{\text{phys}}^2 \text{ at order } g$


 $= \frac{g}{2} \left[\text{bubble diagram} + \text{other diagrams} \right] + O(g^3)$

$$M_{\text{phy}} = 0$$

Scale invariance (classical level)

Lorentzian

classical Massless
dimensional analysis

$$\int d^4x \quad \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4!} \phi^4$$

$$[x] = \text{length}$$

$$[\partial_\mu] = \text{length}^{-1}$$

$$[S] = \text{length}^0$$

$$[\phi] = \text{length}^{-1} \quad \text{in 4 dim.}$$

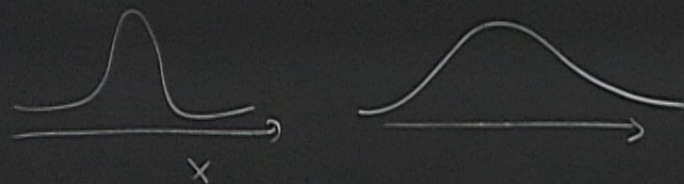
$$\text{length}^{\frac{-d+2}{2}} \quad \text{in d. dim}$$

$$\boxed{[g] = \text{length}^0 \quad \text{in 4 dim.}}$$
$$\text{length}^{d-4} \quad \text{in d. dim.}$$

Scale invariance (classical level)

any rescale by a factor λ

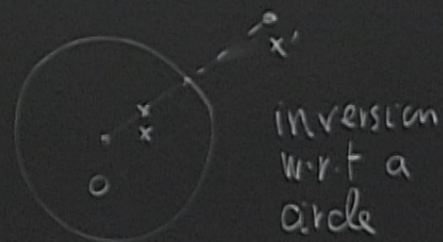
$$\phi(x) \longrightarrow \phi_\lambda(x) = \frac{1}{\lambda} \phi\left(\frac{x}{\lambda}\right)$$



$$S[\phi] = S[\phi_\lambda]$$

$$x \rightarrow \frac{x}{\lambda}$$

$$x \rightarrow \frac{x}{|x|^2}$$



Global Symmetry of the Classical Theory

Euclidean Sym + Scale Sym (+ Inversion) $\Rightarrow$ Conformal Group

$$\frac{x}{\lambda}$$

$$\frac{x}{|x|^2}$$

inversion
w.r.t. a
circle

Conformal Group

$$M_{\text{phys}}^2 = m^2 + \frac{g}{2} T(m; \Lambda)$$

U.V. momentum
cut-off

$$T(m; \Lambda) = \int_{|k| \leq \Lambda} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} = \frac{1}{(4\pi)^2} \Lambda^2 - \frac{m^2}{(4\pi)^2} \log\left(\frac{\Lambda^2}{m^2}\right) + \frac{m^4}{\Lambda^2} + \dots$$

expand $\frac{1}{k^2 + m^2} = \frac{1}{k^2} - \frac{m^2}{k^4} + \dots$
 $|k| \gg m$

quadratic
divergence

logarithmic
divergence

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{|k|^4}$$

$$\vec{k} = \vec{u} |k|$$

↑
hyper-spherical coordinates

$$|\vec{u}|=1 \in S_3$$


$$\int_{S_3} d\vec{u} = \int_0^\Lambda \frac{d|h| |h|^3}{|h|^4} \log \Lambda$$

$$Vol(S_3) =$$

$$\frac{Vol(S_3)}{(2\pi)^4}$$

$\frac{1}{h} = 1$
Euclidean

mass counterterm
so that

$M_{\text{phy}} = 0 \Rightarrow$ quantum theory scale
invariant
(possibly)

$$\dots \phi(z_N) >_R$$

$$\Lambda^2 - \frac{\beta}{(4\pi)^2} \log \frac{\Lambda^2}{\mu^2} + \dots$$

length / in d. dim.

Use a "renormalized action" in functional integral

$\hbar = 1$
Euclidean

mo
so
M_p

$$S_R[\phi] = \int d^4x \frac{A}{2} (\partial_\mu \phi)^2 + \frac{B}{2} \phi^2 + \frac{C}{4!} \phi^4$$

$$\int \mathcal{D}[\phi] \exp(-S_R[\phi]) \longrightarrow \text{Green functions } \langle \phi(z_1) \dots \phi(z_N) \rangle_R$$

$A=1$ makes sense at 1 loop

$$C = g + ?$$

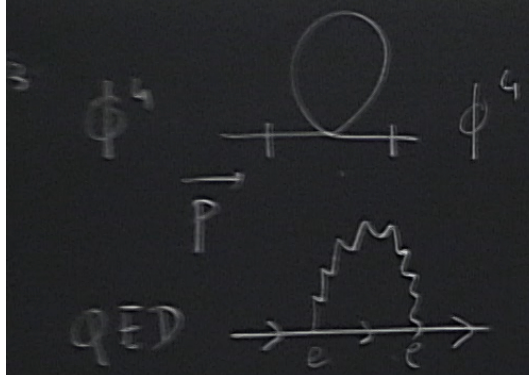
$$B = 0 - g \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2$$

$$M_{\text{phys}}^2 = \left[B + C \frac{1}{2} \left(\frac{1}{(4\pi)^2} \Lambda^2 - \frac{B}{(4\pi)^2} \log \frac{\Lambda^2}{B} + \dots \right) \right]$$

$\Rightarrow M_{\text{phys}}^2 = 0$ at order g

Finite $\Lambda \rightarrow \infty, = 0$

order g^2



Λ^2 indep on the momentum p
 no field renormalization

$\not\propto \log \Lambda$ divergence

$$\bar{\psi} \not\propto \psi \left(1 + e^2 \log \Lambda \right)$$

Field or wave function
 renormalization

interterm

quantum theory scale invariant (possibly)

4 point Function
at order g^2

$\mathcal{L} = g + g^2 \dots ?$
coupling constant $\int$
renormalization



$$B(p; \Lambda) = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(p+k)^2}$$

$p = p_1 + p_2$ $|k| < \Lambda$

when $\Lambda \gg |p|$ $= \log \Lambda^2$
UV divergent

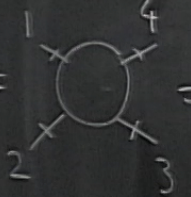
$$\frac{1}{k^2 + B} \rightarrow \frac{1}{k^2}$$

because $B = O(g)$

↓
correction $O(g^3)$

$$\frac{1}{k^2 - g \Lambda^2} = \frac{1}{k^2} + \frac{1}{k^4} g \Lambda^2 + \dots$$

$$B(p, \Lambda) = \underbrace{\frac{1}{(4\pi)^2}}_{\text{universal}} \log\left(\frac{\Lambda^2}{p^2}\right) + \underbrace{\text{cst}}_{\substack{\uparrow \\ \text{depends on the} \\ \text{regularization}}} + \underbrace{\frac{p^2}{\Lambda^2} + \dots}_{\text{subdominant}}$$

$$\Gamma_{(4)}^R = \text{diagram} = \mathcal{E} - \mathcal{E}^2 \frac{1}{2} \left[\log \frac{\Lambda^2}{(p_1+p_2)^2} + \frac{\log \Lambda^2}{(p_1+p_3)^2} + \frac{\log \Lambda^2}{(p_1+p_4)^2} + 3 \text{cst} \right]$$


$$\times (2\pi)^4 \delta(p_1+p_2+p_3+p_4) \quad \text{Finite} \quad \mathcal{E} = g + g^2 \frac{3}{2} \log \Lambda^2$$

$\Lambda \rightarrow \infty$

$$B(p, \Lambda) = \underbrace{\frac{1}{(4\pi)^2}}_{\text{universal}} \log\left(\frac{\Lambda^2}{p^2}\right) + \underbrace{\text{cst}}_{\substack{\uparrow \\ \text{depends on the} \\ \text{regularization}}} + \underbrace{\frac{p^2}{\Lambda^2}}_{\text{subdominant}} + \dots$$

we must introduce a new moment scale
 let us denote it $\mu = \text{renormalization scale}$ $\Rightarrow$

$$\Gamma_{(4)}^R = \text{diagram} = \mathcal{O} - \mathcal{O} \frac{1}{(4\pi)^2} \left[\log \frac{\Lambda^2}{(p_1+p_2)^2} + \frac{\log \Lambda^2}{(p_1+p_3)^2} + \frac{\log \Lambda^2}{(p_1+p_4)^2} + 3 \text{cst} \right] = g + g^2 \frac{3}{2} \log \Lambda^2 = (g + g^2) \frac{1}{2} \left(\frac{\log \Lambda^2}{(p_1+p_2)^2} + \dots \right) \quad g^2 + \mathcal{O}(g^3)$$

$$\times (2\pi)^4 \delta(p_1+p_2+p_3+p_4)$$

we have a c. constant counterterm

$$\text{Finite } \Lambda \rightarrow \infty \quad \mathcal{O} = g + g^2 \frac{3}{2(4\pi)^2} \log\left(\frac{\Lambda^2}{p^2}\right)$$

this does the job

we must introduce a new moment scale
 let us denote it μ = renormalization scale

$$\Rightarrow \Gamma_{(4)}^R = \text{diagram} = g - g^2 \frac{1}{2} \frac{1}{(4\pi)^2} \left[\log\left(\frac{\mu^2}{(p_1+p_2)^2}\right) + \dots \right] + \mathcal{O}(g^3)$$

↑
finite

$$\left[\frac{\log \Lambda^2}{(p_1+p_2)^2} + 3 \text{ct} \right] = g + g^2 \frac{3}{2} \log \frac{\Lambda^2}{\mu^2} = (g + g^2 \cdot \frac{3}{2} \log \frac{\Lambda^2}{\mu^2}) + \mathcal{O}(g^3)$$

this does the job

introduce a new momentum scale

let it μ = renormalization scale

$$\Rightarrow \Gamma_{(4)}^R = \text{bubble diagram} = g - g^2 \frac{1}{2} \frac{1}{(4\pi)^2} \left[\log \left(\frac{\mu^2}{(p_1 + p_2)^2} \right) + \dots \right]$$

Renormalized 4pt function

Finite when $\Lambda \rightarrow \infty$

↑
finite

+ $O(g^3)$

$$\frac{3}{2} \log \frac{\Lambda^2}{\mu^2} = (g + g^2) \frac{1}{2} \left(\log \frac{\Lambda^2}{(p_1 + p_2)^2} + \dots \right)$$

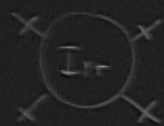
$g^2 + O(g^3)$

renormalization
counterterm

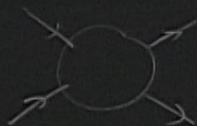
$$\Lambda \rightarrow \infty$$

$$0 \leq \nu \leq 2(4\pi) \nu(\rho)$$

What does this μ corresponds to?



$\Rightarrow$ scattering
amplitude



$\Rightarrow$ measure the
coupling of the theory

experiments at a given Energy scale

$$E \Rightarrow g = g(E)$$



coupling
measured at
energy E

massless bosons

$$E' \Rightarrow g' = g(E')$$

μ analog of the energy scale
at which the coupling is probed

choose a reference point in momentum space p_1, p_2, p_3, p_4

$$(p_1 + p_2)^2 = E^2, \quad (p_2 + p_3)^2 = E^2, \quad (p_1 + p_4)^2 = E^2$$

$$\Gamma_R^{(4)}(\{p\}) = g - g^2 \cdot \frac{1}{2} \frac{1}{(4\pi)^2} \log\left(\frac{\mu^2}{E^2}\right) + \dots = g \quad \text{if } E = \mu$$

at this reference
point

reference energy

$$E = \mu$$

$$g = \text{Value of } \Gamma_{(4)}^R[\{p_i\}] \text{ when } \{p_i\} \text{ are chosen at this reference point } E = \mu$$

fix g , and I choose $\mu \iff$ choose the Energy $E = \mu \Rightarrow g$ is the coupling measured at this scale μ
where I measure the coupling

g we shall denote it g_R renormalized coupling

$$A = 1$$

$$B = -g_R \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2$$

$$C = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi)} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

change simultaneously
 g_R and $\mu \equiv E$ energy scale
so that $\mathcal{L}$ does not change

g we shall denote it g_R renormalized coupling

$$A = 1$$

$$B = -g_R \frac{1}{2} \frac{1}{(4\pi)^2} \Lambda^2$$

$$C = g_R + g_R^2 \frac{3}{2} \frac{1}{(4\pi)} \log\left(\frac{\Lambda^2}{\mu^2}\right)$$

change simultaneously
 g_R and $\mu \approx E$ energy scale
so that $\mathcal{G}$ does not change

$\langle \phi(z_1) \cdots \phi(z_N) \rangle_R$ do not change

 S. Ampl // // //

fix g , and choose μ where I measure the coupling $\rightarrow$ measured at this scale μ

$\Rightarrow$ Effective "Running" coupling with the E
Renormalization Group Flows; Beta Functions, ...

$\downarrow$
Stat. Mech

do not change

// // //