Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 7

Speakers: Francois David

Collection: PSI 2019/2020 - Quantum Field Theory II (David)

Date: November 20, 2019 - 9:00 AM

URL: http://pirsa.org/19110017

Quantum effective action

$$S[\phi] = \int d^d x \left(\frac{1}{2} \phi(-\Delta + m^2)\phi + \frac{g}{4!}\phi^4 \right)$$

$$\Gamma[\varphi] = j \cdot \varphi - W[j] \quad ; \quad \varphi = \frac{\delta W[j]}{\delta j}$$

background field (classical)

connected functions

source term

Legendre Transform

Quantum effective action $\Rightarrow$ 1PI

$$S[\phi] = \int d^d x \left(\frac{1}{2} \phi(-\Delta + m^2)\phi + \frac{g}{4!}\phi^4 \right)$$

$$\Gamma[\varphi] = j \cdot \varphi - W[j] \quad : \quad \varphi = \frac{\delta W[j]}{\delta j}$$

background field (classical)

connected functions

source term

Legendre Transform

1 le

$\Gamma[$

1 loop

$$\Gamma[\varphi] = S[\varphi] + \hbar \frac{1}{2} \mathrm{Tr}\left[\log\left[S''[\varphi]\right]\right] + o(\hbar^2)$$

$$S''[\varphi]_{x,y} = \left[-\Delta_x + m^2 + \frac{g}{2}\varphi^2(x)\right]\delta(x-y) \quad ; \quad S''_0[\varphi] = -\Delta + m^2 ;$$

Kernel

Free theory

$g = 0$

$$\left(S''[\varphi]\cdot F\right)(x) = \left[-\Delta + m^2 + \frac{g}{2}\varphi^2(x)\right] f(x)$$

new function

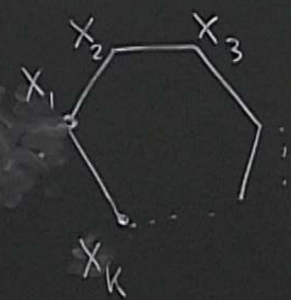
$(-\Delta+m^2)^{-1}\frac{g}{2}\varphi^2\big]\big)$

$\big]\big]$

In position space $\mathrm{Tr}\big[\underbrace{(\;)\cdots(\;)}_{k \text{ times}}\big] = \int d^dx_1 \int d^dx_2 \cdots \int d^dx_k$

$= \int d^dx_1 \cdots d^dx_k \; \varphi^2(x_1)$

$F^{1\,\text{loop}}_{(k)}(x_1,\dots,x_k) = G(x_1-x_2)\cdots G(x_k-x_1)$

$=$ 

1 loop Feynman diagram

ace

$$\mathrm{Tr}\Big[\underbrace{(\ )\cdots(\ )}_{K \text{ times}}\Big] = \int d^d x_1 \int d^d x_2 \cdots \int d^d x_k \Big[G(x_1 - x_2)\, \varphi^2(x_2)\, G(x_2 - x_3)\, \varphi^2(x_3) \cdots\cdots \cdots G(x_k - x_1)\, \varphi^2(x_1) \Big]$$

$$= \int d^d x_1 \cdots d^d x_k \; \underline{\varphi^2(x_1)} \cdots \varphi^2(x_k)\; F^{(1\,\text{loop})}_{(K)}(x_1 \cdots x_k)$$

$\cdots G(x_k - x_1)$

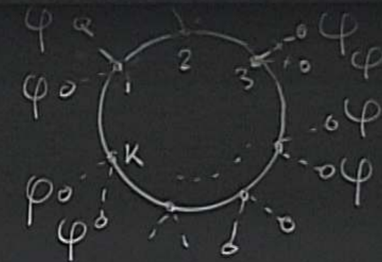
x_3

1 loop Feynman diagram

$k=1 \quad k$

k-times

$$\mathrm{Tr}\,\mathrm{Log}[\quad] = \sum_{k=1}^{\infty} g^k \frac{(-1)^{k-1}}{2^k \cdot k}$$



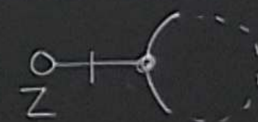
check that $\dfrac{1}{k \cdot 2^k}$ symmetry facto

1 loop - K vertices Feynman diagram

Standard Feynman Rules

$\underset{1 \qquad 2}{\bullet\!\!-\!\!-\!\!-\!\!\circ} = G(x_1 - x_2)$ $\qquad \underset{x}{\times}\,(-g)$

to each external verte



$\varphi($

+ new one $\quad \underset{z \qquad x}{\circ\text{- - -}\bullet} = \delta(z-x) = (-\Delta_z + m^2)\underset{z \qquad x}{\bullet\!\!-\!\!-\!\!\circ} = \underset{z \qquad 4}{\circ\!\!-\!\!+\!\!-\!\!\circ}$ "Truncated" propagator

$2K$ external vertices

so that $\varphi^2(x) = \int dz_1 \int dz_2\, \varphi(z_1)\varphi(z_2)\,\delta(z_1 - x)\,\delta(z_2 - x)$

=

gives a

$$\Gamma_{1\,loop}[\varphi] = \sum_{N=2K} \int d^d z_1 \cdots d^d z_N \frac{1}{N!} \varphi(z_1) \cdots \varphi(z_N) \Gamma_{1\,loop}(z_1 \cdots z_N)$$

$$\Gamma_{1\,loop}(z_1 \cdots z_N) = \sum_{\substack{\text{1 loop 1PI} \\ \text{diagrams } G}} \hbar^B (-g)^K \, C(G) \, I_G^{Irr}(z_1 \cdots z_N)$$

$B = 1$

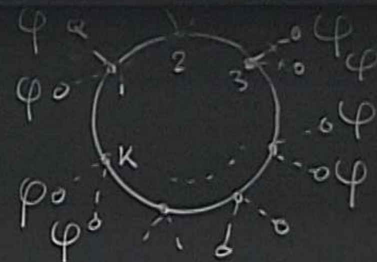
$C(G)$: $\frac{1}{K \cdot 2^{K+1}}$ symmetry factor

I_G^{Irr}: Feynman Rules for the diagrams with "truncated" or "amputated" external legs

$k=1$ $\quad K$ $\quad (2)$

K-times

$$\mathrm{Tr}\cdot\mathrm{Log}[\quad] = \sum_{K=1}^{\infty} g^K \frac{(-1)^{K-1}}{2^K\cdot K}$$



1 loop - K vertices

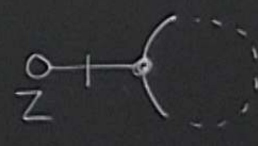
Feynman diagram

check that $\frac{1}{K\cdot 2^K}$ symmetry

Standard Feynman Rules

$\underset{1\qquad 2}{\bullet\!\!-\!\!-\!\!\circ} = G(x_1 - x_2)$ $\qquad \underset{x}{\times}\ (-g)$

to each external



2K external vertice

+ new one

$\underset{z\qquad x}{\circ\text{- - -}\bullet} = \delta(z-x) = (-\Delta_z + m^2)\ \underset{z\qquad x}{\bullet\!\!-\!\!-\!\!\circ} = \underset{z\qquad 4}{\circ\!-\!\!+\!\!-\!\circ}$ "Truncated" propagator

so that $\varphi^2(x) = \int dz_1 \int dz_2\, \varphi(z_1)\varphi(z_2)\,\delta(z_1 - x)\,\delta(z_2 - x)$

$$\Gamma_{1\,loop}[\varphi] = \sum_{N=2K} \int d^d z_1 \cdots d^d z_N \frac{1}{N!} \varphi(z_1)\cdots\varphi(z_N)\, \Gamma_{1\,loop}(z_1 \cdots z_N)$$

$$\Gamma_{1\,loop}(z_1 \cdots z_N) = \sum_{\substack{1\ \text{loop}\ 1PI \\ \text{diagrams}\ G}} - \hbar^B (-g)^K\, C(G)\, I^{Irr}_G(z_1 \cdots z_N)$$

$B = 1$

⚠ Additional − sign

$C(G) = \dfrac{1}{K \cdot 2^{K+1}}$ Symmetry factor

additional $\frac{1}{2}$

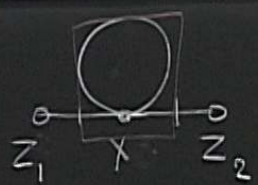
$\Uparrow$

$\frac{1}{2} \mathrm{Tr} \,\mathrm{Log}(\ )$

I^{Irr}_G: Feynman Rules for the diagrams with "truncated" or "amputated" external legs

K = 1

"Tadpole"

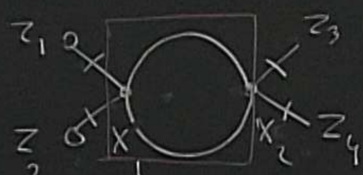


$$\frac{1}{2} \times \int dx\, \delta(z_1 - x)\, \delta(z_2 - x)\, G(0) = \frac{1}{2}\, \delta(z_1 - z_2)\, G(0)$$

↑ I'll check

K = 2

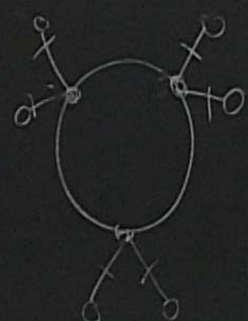
"Bubble"



$$\delta(z_1 - z_2)\, \delta(z_3 - z_4)\, G(z_1 - z_3)^2$$

K = 3

"Triangle"

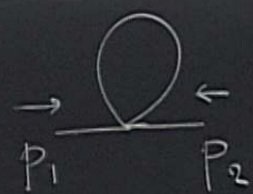


F. Transform

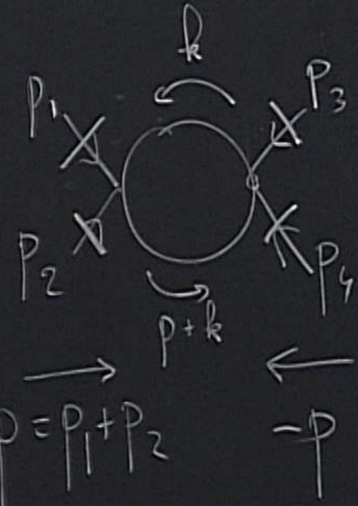
$\Rrightarrow$

$$\frac{1}{p^2+m^2}$$

$$1$$



$$\delta(p_1+p_2)\cdot\ G(0)$$



$$\delta(p_1+p_2+p_3+p_4)\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+m^2}\cdot\frac{1}{(p+k)^2+m^2}$$

etc...

1^{st} term $\quad K=0 \quad$ Free Theory

ground state of harmonic oscillator

$$\hbar \frac{1}{2} \mathrm{Tr} \cdot \mathrm{Log}(-\Delta + m^2) \xrightarrow[F \cdot T]{} \hbar \frac{1}{2} \int d^d z \int \frac{d^d k}{(2\pi)^2} \log(k^2 + m^2)$$

convenient representation

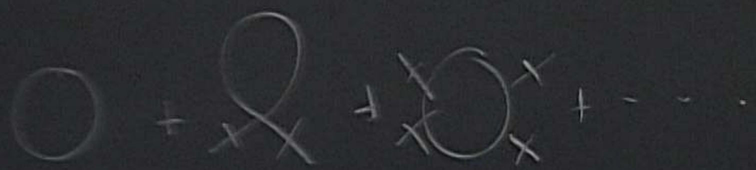
quantum correction to
the vacuum energy of the Free Fied

$-\hbar \frac{1}{2}$ 

↑
my choice

closed loop
with no vertex

$$\int \frac{d^d k}{(2\pi)^d} \log \frac{1}{k^2 + m^2}$$



$$\int d^dx \, \frac{1}{2}(\partial\phi)^2 + V(\phi) \qquad V(\phi) = \underset{\substack{\uparrow \\ \text{classical energy density}}}{E_0} + \frac{m^2}{2}\phi^2 + \cdots$$

$$S[\phi = 0] = \int d^dx \cdot E_0 = E_0 \times \text{Volume of Space-Time}$$

$$\mathrm{Tr}\left[\mathrm{Log}\left(-\Delta + m^2\right)\right] = \int d^dx \left[\mathrm{Log}\left(-\Delta + m^2\right)\right]_{x,x}$$

$$\left[\left(-\Delta + m^2\right)^{-m}\right]_{xx} = \int \frac{d^dk}{(2\pi)^d}\left[\frac{1}{k^2+m^2}\right]^m \qquad \text{any } m \in \mathbb{N}$$

correct result

$$-\frac{d}{dm}\left(\;\right)\Big|_{m=0} = \left[\log\left(-\Delta + m^2\right)\right]_{xx} = \int \frac{d^dk}{(2\pi)^d}\log\left(k^2+m^2\right)$$

$\underset{k=1}{} \; k \quad (2)$ ... K-times

Quantum effective action $\Longrightarrow$ U.V divergences

$$S[\phi] = \int d^d x \left(\frac{1}{2} \phi(-\Delta + m^2)\phi + \frac{g}{4!}\phi^4 \right)$$

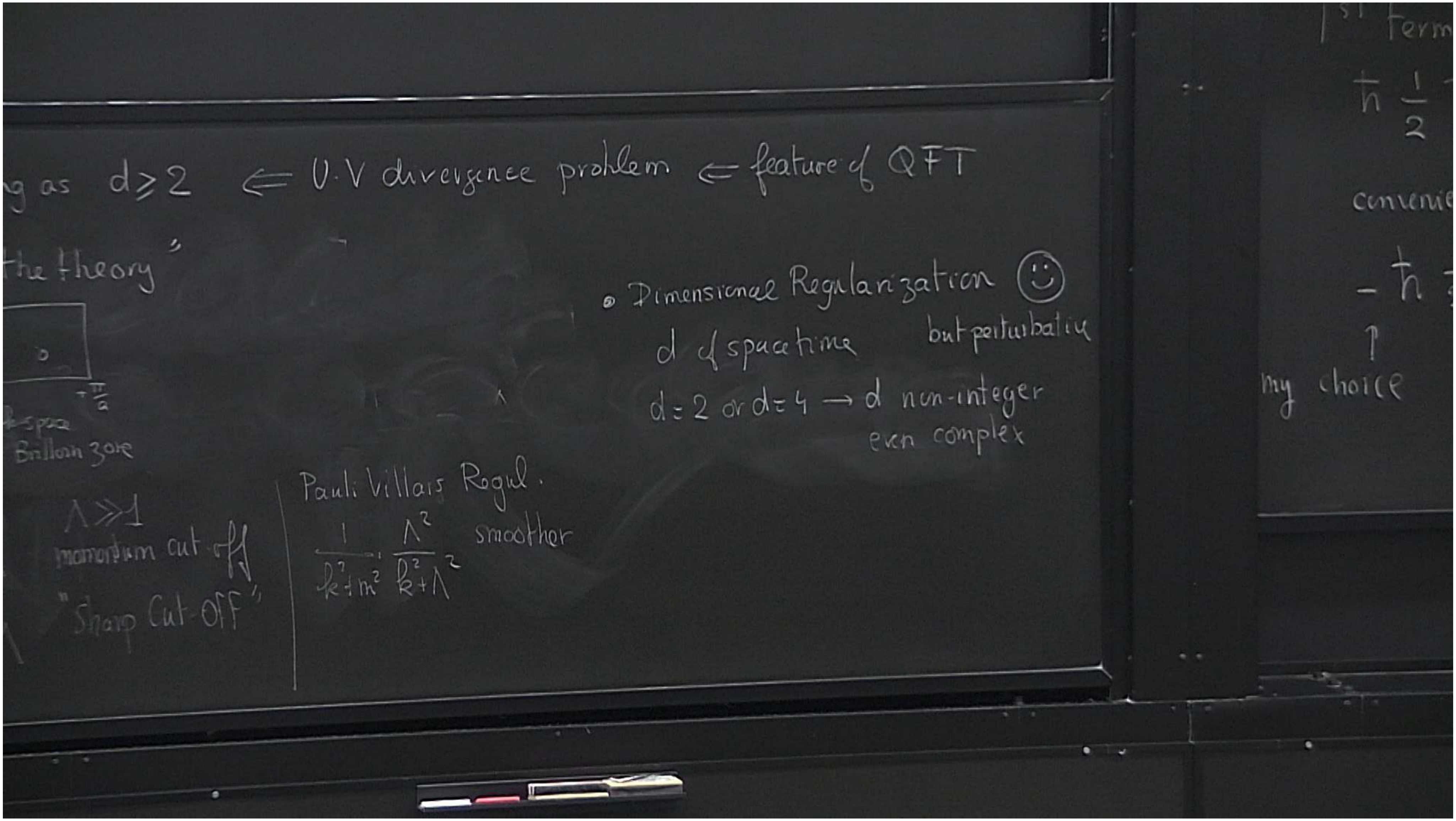
1 P.I 2 point function $\quad \Gamma^{(2)} = (-\Delta + m^2) + \hbar \frac{g}{2} \, \Omega + \ldots$

momentum representation

$$\hbar = 1 \qquad \overset{p_1}{\rightarrow} \boxed{\Gamma} \overset{p_2}{\leftarrow} = (2\pi)^d \delta(p_1 + p_2) \left[p_1^2 + m^2 + \frac{g}{2} T \right]$$

1 loop order $\qquad T = T(m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \qquad p_1 = p$

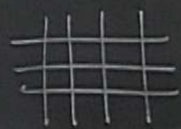
T(m


g as d ≥ 2 ⇐ U.V divergence problem ⇐ feature of QFT
the theory"
• Dimensional Regularization ☺
d of spacetime
but perturbative
d = 2 or d = 4 → d non-integer
even complex
Brillouin zone
Λ ≫ 1
momentum cut-off
"Sharp Cut-Off"
Pauli Villars Regul.
smoother
1st term
convenie
my choice

$T(m)$ is ∞ as long as $d \geq 2$ $\Leftarrow$ U.V divergence pr

Trick : "regularize the theory"

- Lattice



$\overleftrightarrow{a}$

positive space

$-\frac{\pi}{a}$ $+\frac{\pi}{a}$

k-space

= Brillouin zone

Poincaré

- Modify the propagator

$$\hat{G}(k) = \frac{1}{k^2 + m^2} \quad |k| \leq \Lambda$$

$$= 0 \quad |k| > \Lambda$$

$\Lambda \gg 1$

momentum cut-off

"Sharp Cut-Off"

Pauli Villars Regul.

$$\frac{1}{k^2 + m^2} \cdot \frac{\Lambda^2}{k^2 + \Lambda^2}$$

smoother

Unitarity

$\frac{g}{2} T \;]$

as long as $d \geq 2$ $\Leftarrow$ U.V divergence problem $\Leftarrow$ feature of QFT

larize the theory"

Poincaré

$-\frac{\pi}{a}$ $+\frac{\pi}{a}$

k-space

= Brillouin zone

$\Lambda \gg 1$

$|k| \leq \Lambda$ momentum cut-off

$|k| > \Lambda$ "Snap Cut-Off"

Pauli Villars Regul.

$\frac{1}{k^2+m^2} \cdot \frac{\Lambda^2}{k^2+\Lambda^2}$ smoother

Unitarity

- Dimensional Regularization ☺

d of spacetime — but perturbation

$d=2$ or $d=4 \rightarrow d$ non-integer, even complex

Sensible S.T picture

- Work with a cut-off
- Control UV problem $\Leftarrow$ renormalization
- Take continuum limit $\Lambda \to \infty$, check does

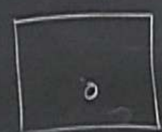
$T(m)$ is ∞ as long as $d \geq 2$ $\Leftarrow$ U.V divergence problem $\Leftarrow$ feature of QFT

Trick : "regularize the theory"

- Lattice

$\overleftrightarrow{a}$

position space



$-\frac{\pi}{a}$ $\quad +\frac{\pi}{a}$

k-space
= Brillouin zone

Poincaré

- Modify the propagator

$$\hat{G}(k) = \frac{1}{k^2+m^2} \quad |k| \leq \Lambda$$

$$= 0 \quad |k| > \Lambda$$

$\Lambda \gg 1$

momentum cut-off

"Sharp Cut-Off"

Unitarity, Causality, Locality

Pauli Villars Regul.

$$\frac{1}{k^2+m^2} \cdot \frac{\Lambda^2}{k^2+\Lambda^2} \quad \text{smoother}$$

Unitarity

- Dimensional Regularizati

d of spacetime but

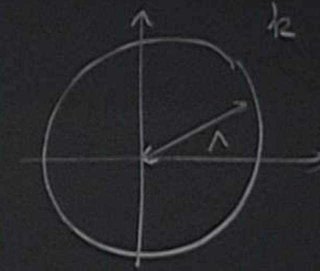
$d=2$ or $d=4 \rightarrow d$ non-

even co

Sensible S.T picture

- Work with a cut-off
- Control UV problem $\Leftarrow$ re
- Take continuum limit $\Lambda \to \infty$

Sharp cut off $\quad T(m;\Lambda)\ \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d}\,\frac{1}{k^2+m^2}$



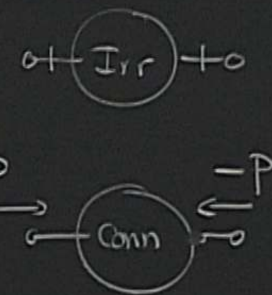
$\Lambda \gg m$

$$T(m;\Lambda) = \cdot\ \Lambda^{d-2} + \cdot\ m^2\,\Lambda^{d-4} + \dots$$

$\boxed{d=4}$

compute these coefficients

$= p^2 + m^2 + \frac{g}{2} T(m;\Lambda)$



$= \text{FT of } \langle \phi(x_1)\phi(x_2)\rangle_{\text{int theory}}$

$$C^2(p) = \frac{1}{p^2 + m^2 + \frac{g}{2} T(m,\Lambda)}$$

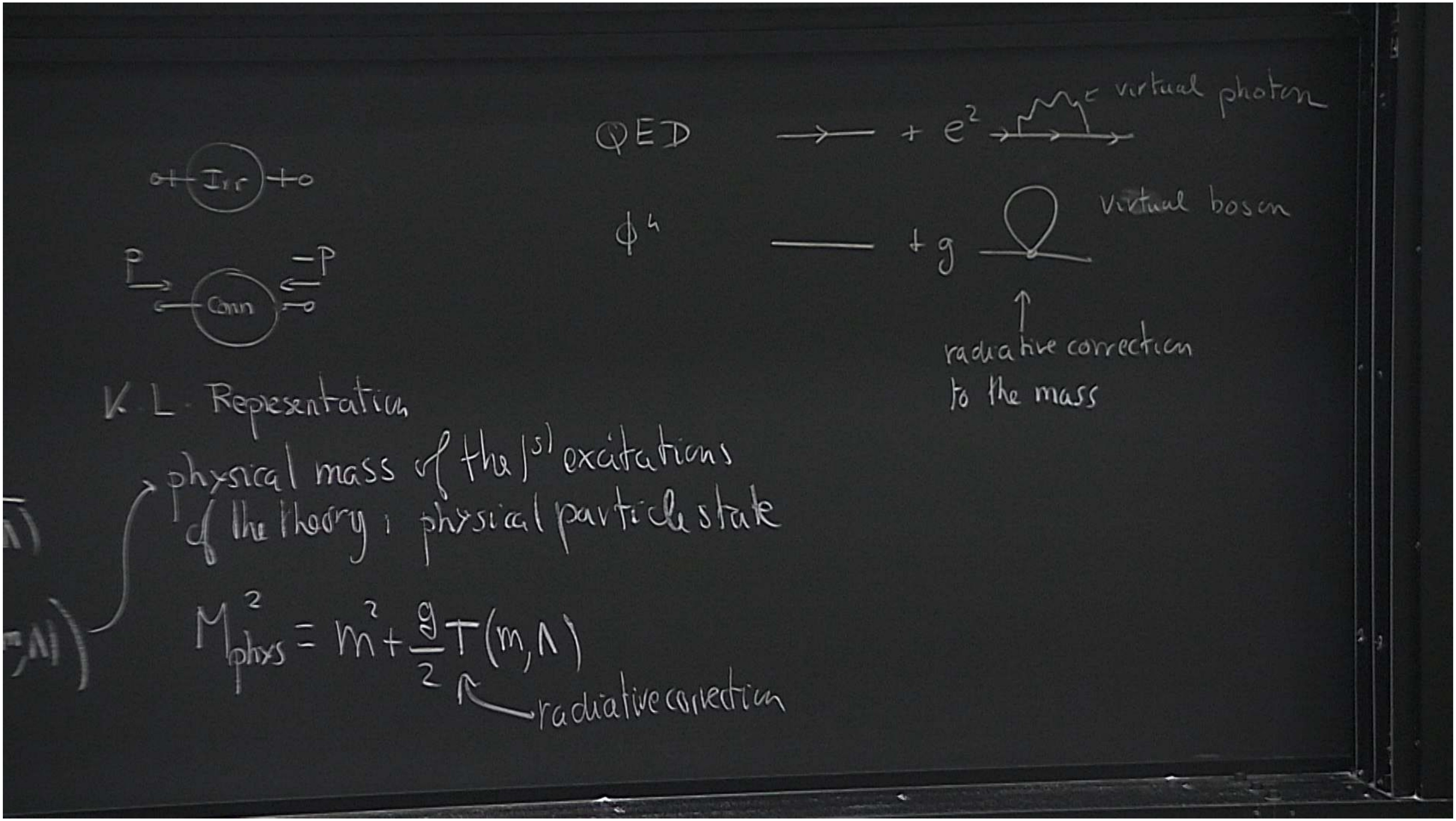
pole at $p^2 = -\left(m^2 + \frac{g}{2} T(m,\Lambda)\right)$

K. L. Representation

physical mass of the 1st excitations of the theory: physical particle state

$$M^2_{phys} = m^2 + \frac{g}{2} T(m,\Lambda)$$

radiative correction


QED
+ e² ← virtual photon
φ⁴
+ g
virtual boson
radiative correction
to the mass
Irr
P
−P
Conn
K.L. Representation
physical mass of the 1st excitations
of the theory: physical particle state
M²phys = m² + g/2 T(m,Λ)
radiative correction