Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 5

Speakers: Francois David

Collection: PSI 2019/2020 - Quantum Field Theory II (David)

Date: November 18, 2019 - 9:00 AM

URL: http://pirsa.org/19110014

$$S_I = \frac{g}{4!}\int d^d x\ \phi^4(x)$$

Feynman diagrams $\longrightarrow$ Integrals, combinatorial factors $\longrightarrow$ Green Functs

x_1 x_2 $\quad \hbar\, G(x_1 - x_2) \qquad G \longrightarrow \hat{G}(k) = \dfrac{1}{k^2 + m^2}$

propagator

$x \qquad -g \cdot \hbar^{-1}$

interaction vertex

- Perturbation theory of ϕ^4 QFT   Euclidean $\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S[\phi])$
- Continue on simplest examples (1^{st} order)
- semi-classical expansion $\hbar \Rightarrow$ topological expansion (Feynman diagrams)
- Generating Functionals $Z[j]$, $W[j]$ connected objects, $\Gamma(\varphi)$ effective action irreducible diagrams

Euclidean $\int D[\phi]\, \exp\left(-\frac{1}{\hbar} S[\phi]\right)$

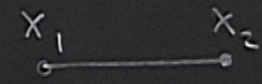
...nder) $\left\langle \exp -\frac{1}{\hbar} S[\phi] \right\rangle_0$ free theory

...ical expansion (Feynman diagrams)

...] connected objects, $\Gamma(\varphi)$ effective action irreducible diagrams

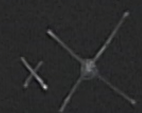
$S_I = \frac{g}{4!} \int d^dx \, \phi^4(x)$

Feynman diagrams $\longrightarrow$ Integrals, co...



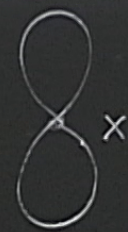
propagator

$\hbar\, G(x_1 - x_2)$ G ...

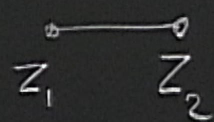


interaction vertex

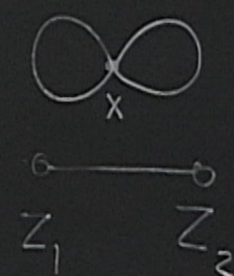
$-g \cdot \hbar^{-1}$

$\langle \phi^4 \rangle \longrightarrow$  $\times \ \frac{1}{8}(-g)\,\hbar$

2 pt function $\langle \phi(z_1)\phi(z_2)\rangle_{g\neq 0}$

$g^0 = 1$ 

2 external vertices

$g^1 \quad \langle \phi(z_1)\phi(z_2)\phi^4(x_1)\rangle_{g=0} \Rightarrow$ 

$(-g)\frac{1}{8}\hbar^2 +$

$S[\Phi])$

Free theory

irreducible diagrams

$$S_I = \frac{g}{4!}\int d^dx\, \phi^4(x)$$

$$\hat{\phi}(k) = \int d^dx\, e^{-ik\cdot x}\,\phi(x)$$

Feynman diagrams $\longrightarrow$ Integrals, combinatorial factors $\longrightarrow$ Green Functs

x_1 x_2

propagator

$\hbar\, G(x_1 - x_2)$

x

interaction vertex

$-g\cdot\hbar^{-1}$ integrate over x

position repr.

$$G \longrightarrow \hat{G}(k) = \frac{1}{k^2+m^2}$$

$$\langle \hat{\phi}(k_1)\,\hat{\phi}(k_2)\,\hat{\phi}(k_3)\,\hat{\phi}(k_4)\rangle_0$$

momentum representation

$S[\Phi])$

free theory

reducible diagrams

$$S_I = \frac{g}{4!}\int d^dx\ \phi^4(x)$$

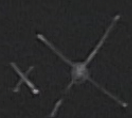
$$\hat{\phi}(k) = \int d^dx\, e^{-ik\cdot x}\phi(x)$$

$$\langle \hat{\phi}(k_1)\hat{\phi}(k_2)\rangle = (2\pi)^d\delta(k_1+k_2)\,\hat{G}(k)$$

Feynman diagrams $\longrightarrow$ Integrals, combinatorial factors $\longrightarrow$ Green Functions

propagator

$\hbar\, G(x_1 - x_2)$

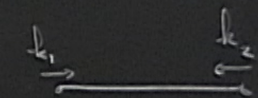


interaction vertex

$-g\cdot\hbar^{-1}$ integrate over x

position repr.

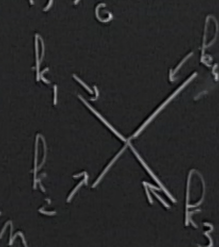
$$G \longrightarrow \hat{G}(k) = \frac{1}{k^2+m^2}$$



incoming momenta

$$\langle \hat{\phi}(k_1)\hat{\phi}(k_2)\hat{\phi}(k_3)\hat{\phi}(k_4)\rangle_0$$

$$= \delta^d(k_1+k_2+k_3+k_4)(2\pi)^d$$



momentum representation

...tion Theory of ϕ^4 QFT   Euclidean $\int D[\phi]\, e\mathrm{xp}\left(-\frac{1}{\hbar} S[\phi]\right)$

... simplest examples (1^{st} order) $\left\langle \exp -\frac{1}{\hbar} S[\phi] \right\rangle_0$ Free theory

...cal expansion $\hbar \Rightarrow$ topological expansion (Feynman diagrams)

Functionals $Z[j]$, $W[j]$ connected objects , $\Gamma(\varphi)$ effective action   irreducible diagrams

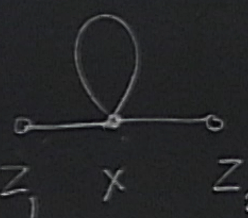
T + Parity invariance $\hat{G}(k) = \hat{G}(-k)$

Translation inv

$\frac{1}{4!} 4 \cdot 3$

$$I_\ell(z_1, z_2) = \int d^d x \, G(z_1 - x)\, G(x - z_2) \cdot G(\underset{0}{\underset{\|}{x - x}})$$

$$(-g)\frac{1}{8}\hbar^2 \; + $$



$$(-g)\,\hbar^2 \cdot \frac{1}{2}$$

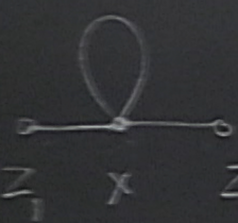
$$I_\ell(p_1, p_2)$$

↑

external momenta

"fixed"

$\frac{1}{4!}\, 4\cdot 3$



$(-g)\,\hbar^2\cdot\frac{1}{2}$

$$I_\ell(z_1,z_2) = \int d^dx\, \boxed{G(z_1-x)\,G(x-z_2)}\cdot\boxed{\underset{\substack{\shortparallel\\ 0}}{G(x-x)}}$$

FT

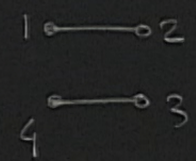
$$I_\ell(p_1,p_2) = (2\pi)^d\,\delta(p_1+p_2)\,\boxed{\int d^dk\,\frac{1}{k^2+m^2}}\ \boxed{\frac{1}{p_1^2+m^2}\cdot\frac{1}{p_2^2+m^2}}$$

↑ external momenta "fixed"

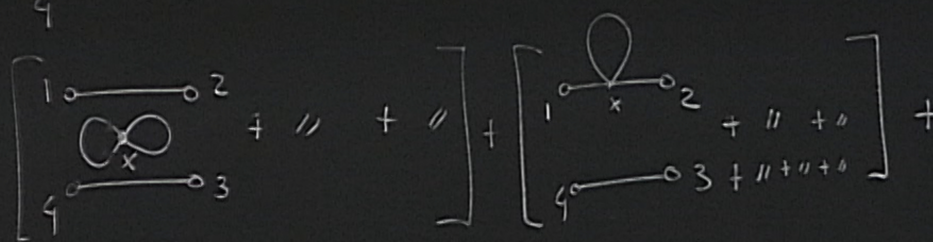
A number $= G(0)$

4pt. Function $\langle \phi(z_1)\,\phi(z_2)\,\phi(z_3)\,\phi(z_4) \rangle_g$

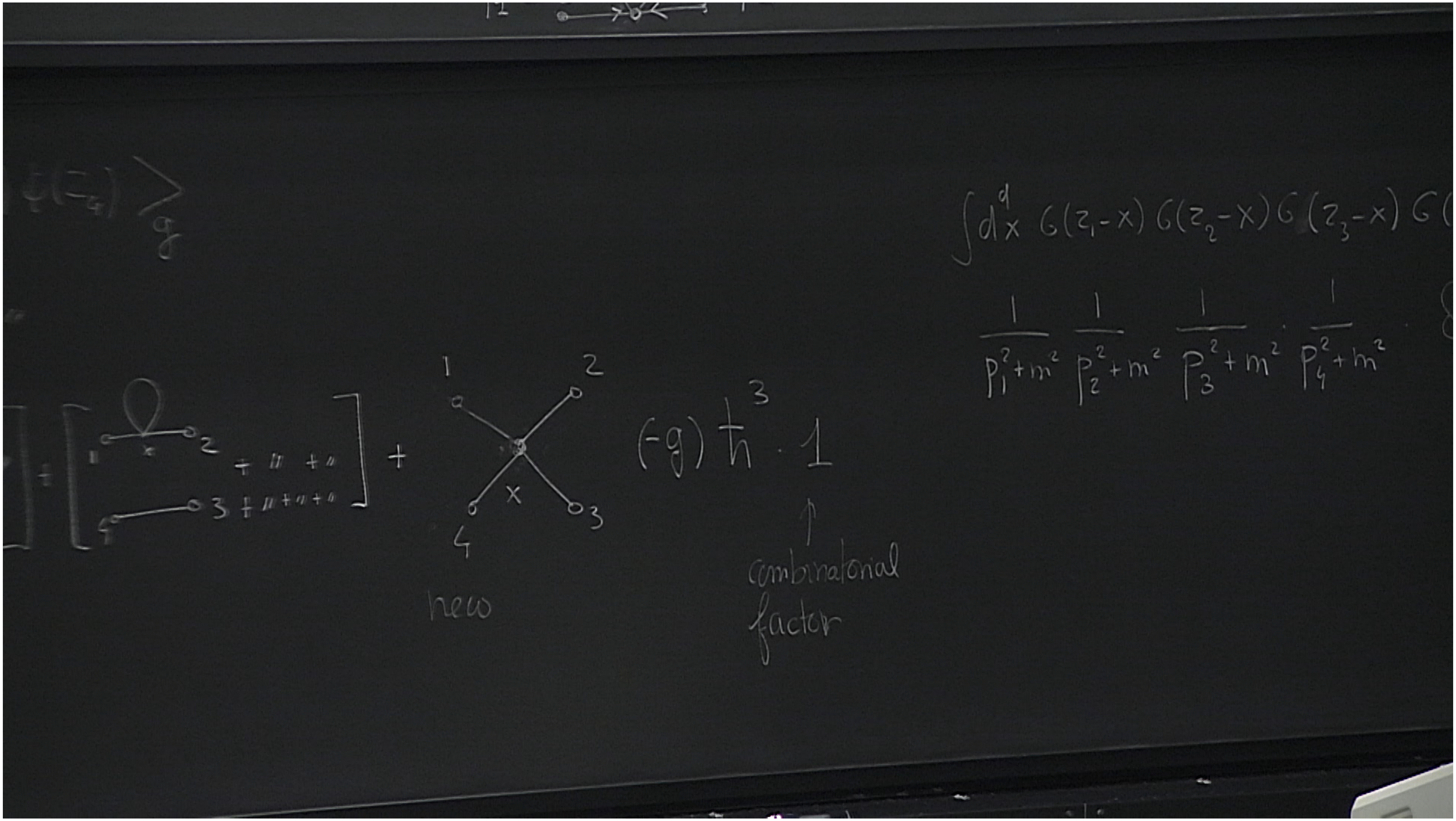
order g^0 $\Longrightarrow$



order g^1 $\Longrightarrow$



$\langle \phi(z_1)\,\phi(z_2)\,\phi(z_3)\,\phi(z_4)\,\phi^4(x) \rangle$



$\int d^d x\, G(z_1-x)\, G(z_2-x)\, G(z_3-x)\, G($
$\frac{1}{p_1^2+m^2}\,\frac{1}{p_2^2+m^2}\,\frac{1}{p_3^2+m^2}\cdot\frac{1}{p_4^2+m^2}$
$(-g)\,\hbar^3\cdot 1$
combinatorial factor
new

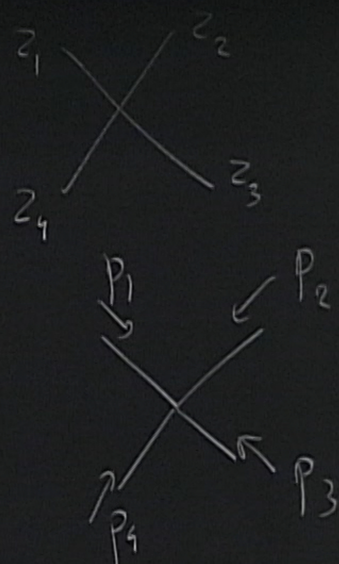
$$\int d^d x \, G(z_1 - x)\, G(z_2 - x)\, G(z_3 - x)\, G(z_4 - x)$$

F.T.

$$\frac{1}{p_1^2+m^2}\,\frac{1}{p_2^2+m^2}\,\frac{1}{p_3^2+m^2}\cdot\frac{1}{p_4^2+m^2}\cdot\delta^d(p_1+p_2+p_3+p_4)$$

$) \hbar^3 \cdot 1$

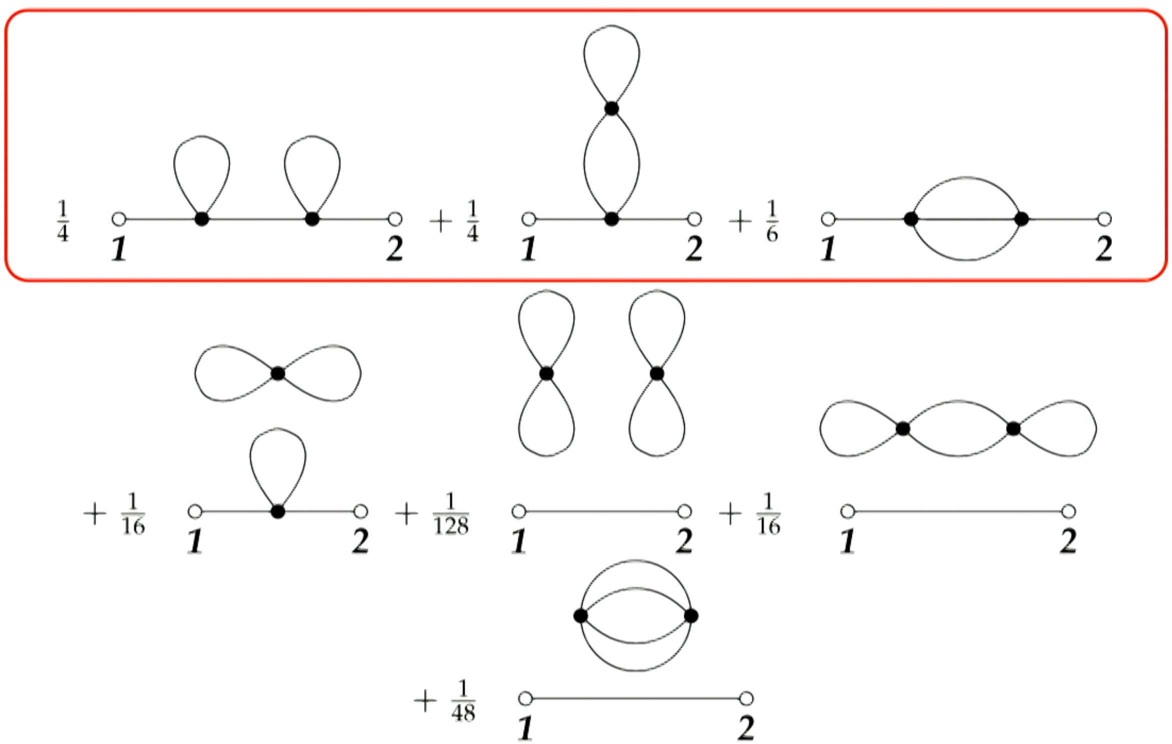
combinatorial factor



2 points diagrams (continued)

$N = 2,\ K = 2$

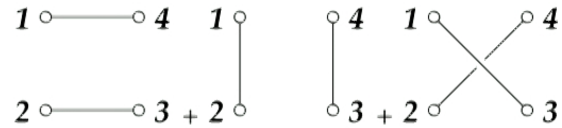
$$\frac{1}{2!\,(4!)^2}\int_{x_1}\int_{x_2}\langle\Phi(z_1)\Phi(z_2)\Phi^4(x_1)\Phi^4(x_2)\rangle_0 \ =$$



4 points diagrams

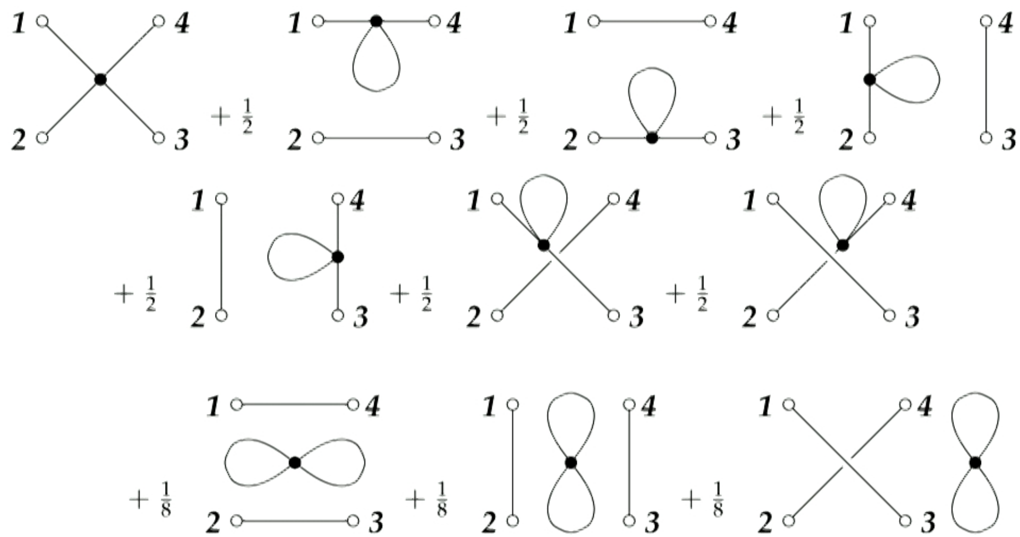
$N = 4, \ K = 0$

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle_0 \ =$$



$N = 4, \ K = 1$

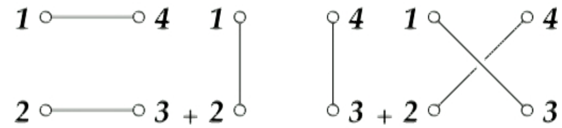
$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\Phi^4(x_1)\rangle_0 \ =$$



4 points diagrams

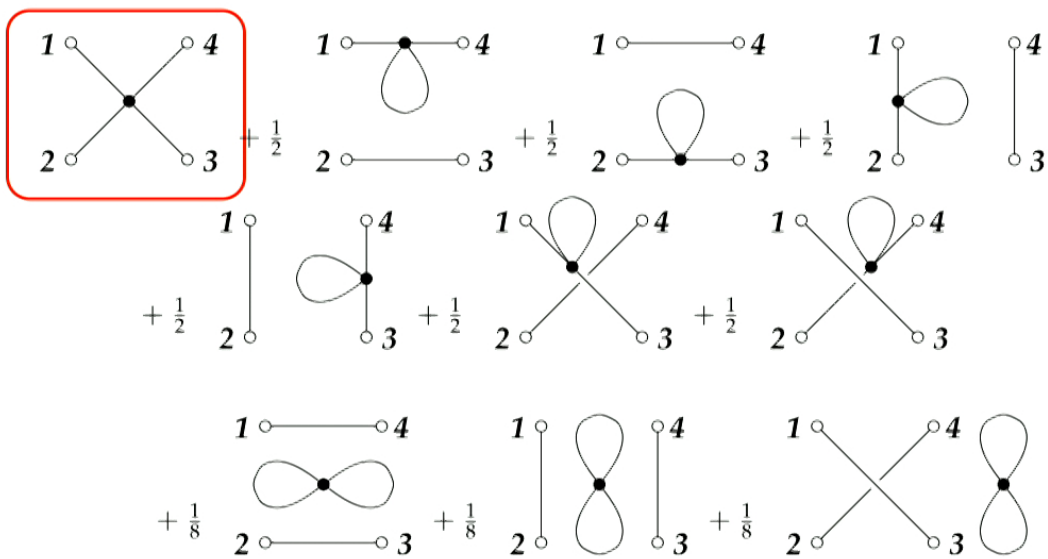
$$N = 4,\ K = 0$$

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle_0 \ =$$

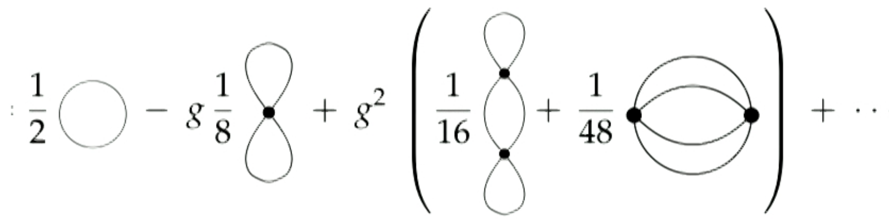


$$N = 4,\ K = 1$$

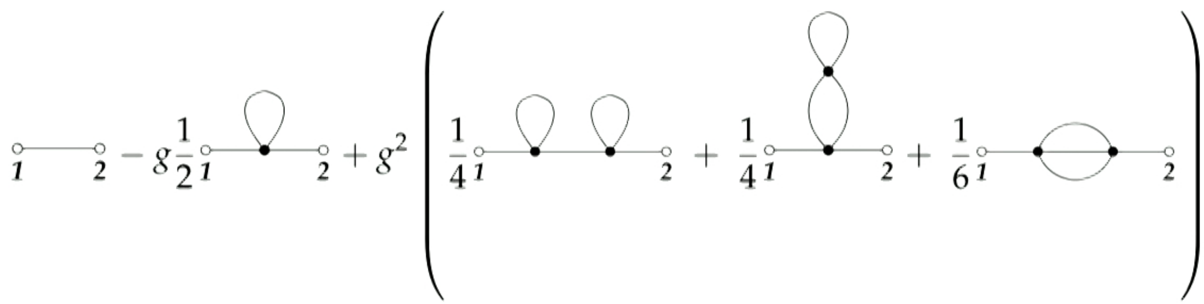
$$\frac{1}{4!}\int_{x_1} \langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\Phi^4(x_1)\rangle_0 \ =$$



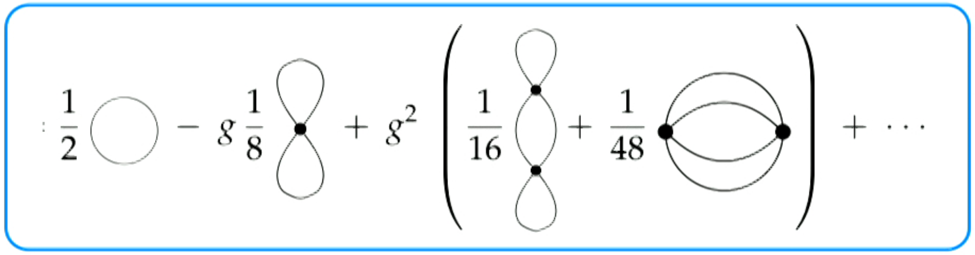
Connected vacuum diagrams



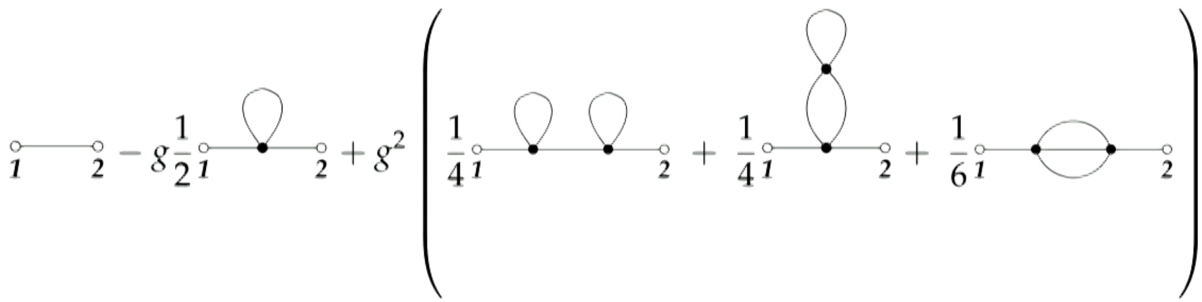
2 points function = connected 2 points function



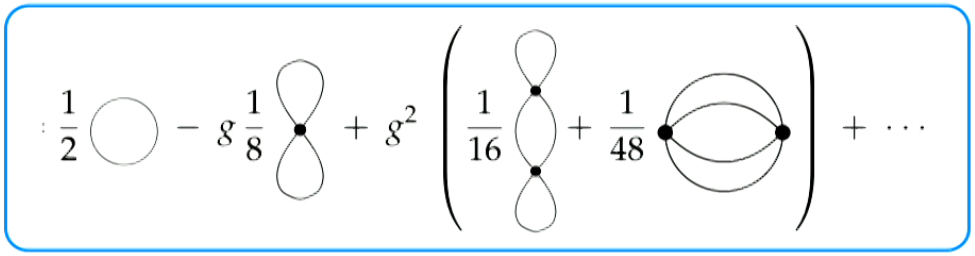
Connected vacuum diagrams



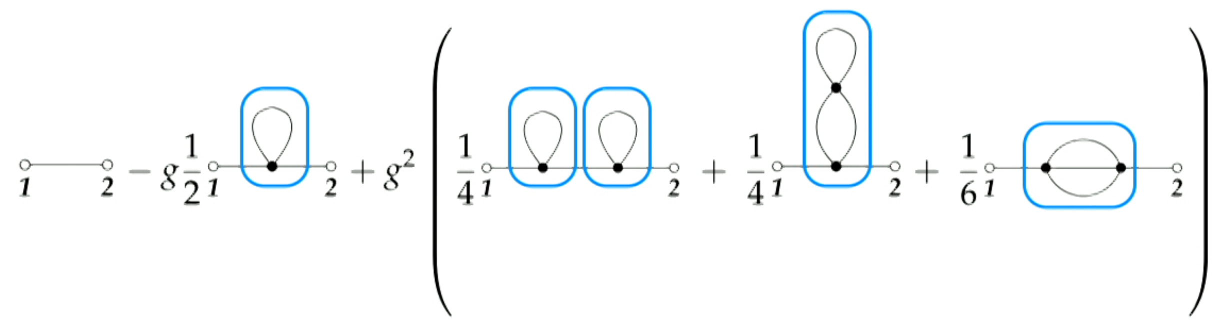
2 points function = connected 2 points function



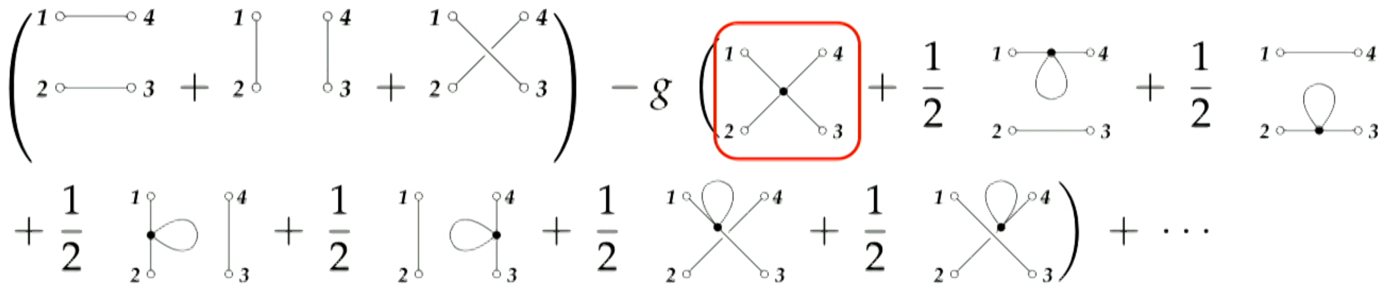
Connected vacuum diagrams



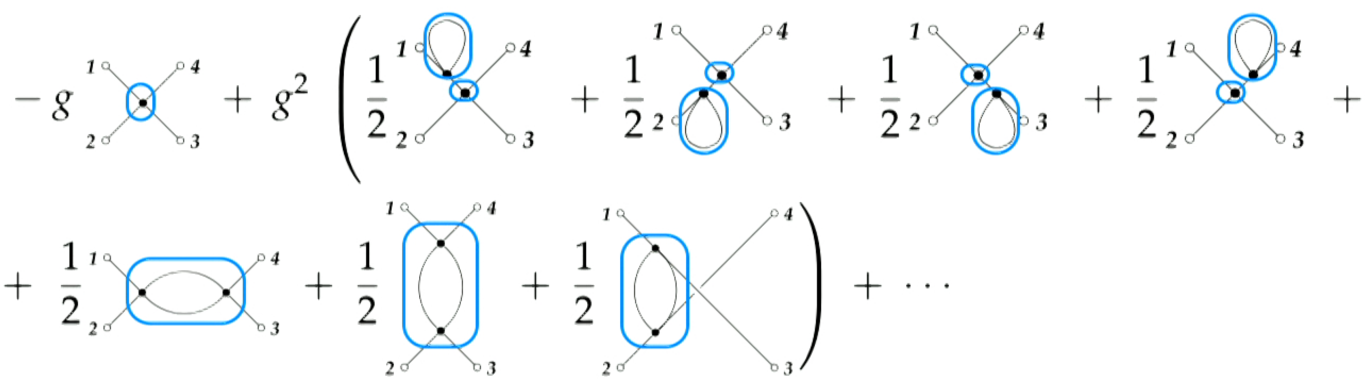
2 points function = connected 2 points function



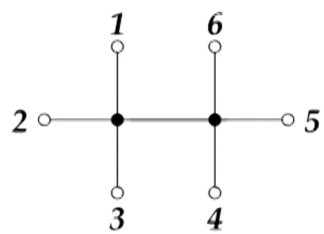
4 points function (up to order 1)



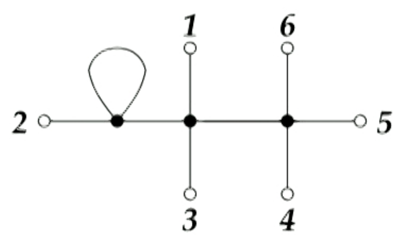
Connected 4 points function (up to order 2)

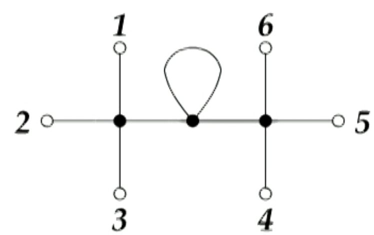


Connected 6 points function (up to order 3)

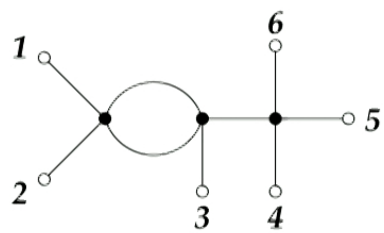
g^2 

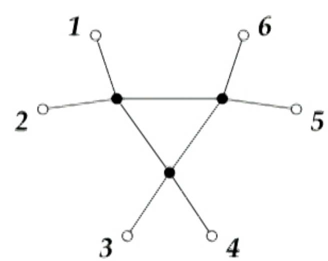
$+\cdots$

$-g^3\frac{1}{2}$ 

$+\cdots$ $-g^3\frac{1}{2}$ 

$+\cdots$

$-g^3\frac{1}{2}$ 

$+\cdots$ $-g^3$ 

$+\cdots$

Generating functionals (Euclidean)

$$Z[j] = \int D[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$$

$\phi = \{\phi(x)\}$ random (quantum) variable

$j = \{j(x)\}$ non-fluctuating (classical) source term

$\approx$ external fixed classical

$j\cdot\phi = \int d^d x \, j(x)\cdot\phi(x)$ scalar product

$$\frac{\delta}{\delta j(z_i)} Z[j]/Z[0]\Big]_{j=0} = \sum_{\substack{\text{diagrams } G \\ \text{with } N \text{ external vertices} \\ \text{without any vaccum} \\ \text{diagram component}}} g^{N} \hbar^{*} \, C_G \, I_G(z_1 \cdots z_N)$$

C_G: combinatorial factor

I_G: Feynman integral

Generating functionals (Euclidean)

$$Z[j] = \int D[\phi] \exp\left(-\frac{1}{\hbar}\left(S[\phi] - j\cdot\phi\right)\right)$$

$\phi = \{\phi(x)\}$ random (quantum) variable

$j = \{j(x)\}$ non-fluctuating (classical) source term
≈ external fixed classical

$j\cdot\phi = \int d^dx\, j(x)\cdot\phi(x)$ scalar product

Green Functions = all diagrams / w.o. vacuum

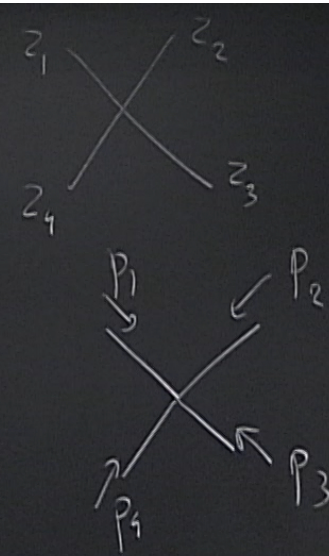
$$G(z_1 \ldots z_n) = \langle \phi(z_1) \cdots \phi(z_n) \rangle_g = \hbar^N \left[\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_n)} Z[j]/Z[0] \right]$$

Connected diagrams

$$W[j] = \hbar \cdot \mathrm{Log}[Z[j]] \iff Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right)$$

$$\int d^dx\, G(z_1-x)\, G(z_2-x)\, G(z_3-x)\, G(z_4-x)$$

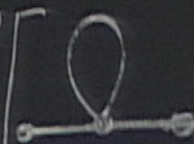
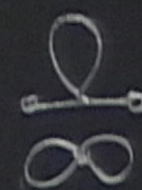
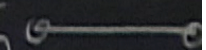
$$\frac{1}{p_1^2+m^2}\,\frac{1}{p_2^2+m^2}\,\frac{1}{p_3^2+m^2}\cdot\frac{1}{p_4^2+m^2}\cdot\delta^d(p_1+p_2+p_3+p_4)$$



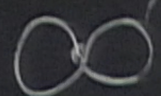
$$\langle\phi(z_1)\,\phi(z_2)\rangle_g = \frac{\left\langle \phi(z_1)\,\phi(z_2)\cdot\exp\left(-\frac{g}{4!\,\hbar}\int d^dx\,\phi^4(x)\right)\right\rangle_0}{\left\langle \exp\left(-\frac{g}{4!}\,\frac{1}{\hbar}\int d^dx\,\phi^4(x)\right)\right\rangle_0}$$

$$= \frac{\hbar \quad - g\hbar^2\left[\ \ \frac{1}{2} + \frac{1}{8}\ \ \right] + \cdots}{1 - g\hbar\,\infty + \cdots}$$

$$\rangle_g = \frac{\left\langle \phi(z_1)\,\phi(z_2)\cdot \exp\left(-\frac{g}{4!\,\hbar}\int d^dx\,\phi^4(x)\right)\right\rangle_0}{\left\langle \exp\left(-\frac{g}{4!}\,\frac{1}{\hbar}\int d^dx\,\phi^4(x)\right)\right\rangle_0}$$

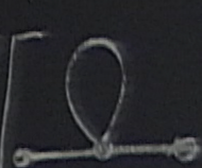
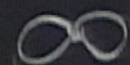
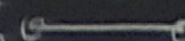
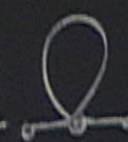
$= \hbar$  $- g\hbar^2 \Big[$  $\cdot \frac{1}{2} + \frac{1}{8}$  $\Big] + \cdots$ $= \hbar$ 

over

$1 - g\frac{\hbar}{8}$  $+ \cdots$

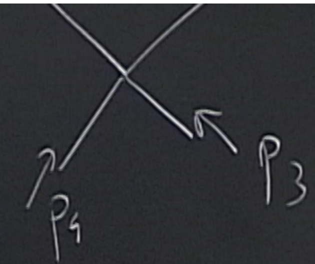
$$z_t) \cdot \exp\left(-\frac{g}{4!\,\hbar}\int d^d x\, \phi^4(x)\right)\Big\rangle_0$$

$$-\frac{g}{4!}\,\frac{1}{\hbar}\int d^d x\, \phi^4(x))\Big\rangle_0$$

$g\hbar^2\Big[$  $\frac{1}{2} + \frac{1}{8}$  /  $\Big] + \cdots = \hbar$  $- g\hbar^2\,\frac{1}{2}$ 

$\frac{\hbar}{8}$  $+ \cdots$

$$\frac{1}{p_3^2+m^2}\cdot\frac{1}{p_4^2+m^2}\cdot\delta^d(p_1+p_2+p_3+p_4)$$



$$\rangle_g = \frac{\left\langle \phi(z_1)\phi(z_2)\cdot\exp\left(-\frac{g}{4!\hbar}\int d^dx\,\phi^4(x)\right)\right\rangle_0}{\left\langle \exp\left(-\frac{g}{4!}\frac{1}{\hbar}\int d^dx\,\phi^4(x)\right)\right\rangle_0}$$

cancells $\Longrightarrow$ all vaccum diagrams

$$= \frac{\hbar - g\hbar^2\left[\frac{1}{2}+\frac{1}{8}\,\infty\right]+\cdots}{1-g\frac{\hbar}{8}\infty+\cdots} = \hbar - g\hbar^2\,\frac{1}{2}$$

$$G(z_1 \ldots z_n) = \langle \phi(z_1) \cdots \phi(z_n) \rangle_g = \hbar^N \left[\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} Z[j]/Z[0] \right]_{j=0} =$$

Connected diagrams

$$W[j] = \hbar \cdot \mathrm{Log}[Z[j]] \iff Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right)$$

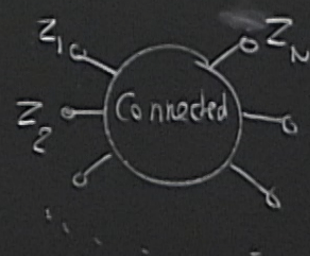
$$\left.\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} W[j] \right|_{j=0} = \sum_{\substack{\text{connected} \\ \text{diagrams } G \\ \text{with } N \text{ external} \\ \text{vertices}}} g^{K(G)} \; C_G$$

$$\ldots\phi(z_N)\rangle_g = \hbar^N \left[\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} Z[j]/Z[0]\right]_{j=0} = \sum_{\substack{\text{diagrams } G \\ \text{with } N \text{ external vertices} \\ \text{without any vacuum} \\ \text{diagram component}}} g^{K(G)} \hbar$$

$K(G) = \#$ internal vertices

$$\Leftrightarrow Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right)$$

$$= \sum_{\substack{\text{connected} \\ \text{diagrams } G \\ \text{with } N \text{ external} \\ \text{vertices}}} g^{K(G)} \hbar^{[illegible](G)} c_G I_G(z_1 \ldots z_N) = \sum_G$$



$$G(z_1 \ldots z_n) = \langle \phi(z_1) \cdots \phi(z_n) \rangle_g = \hbar^N \left[\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} Z[j]/Z[0] \right]_{j=0} = \sum$$

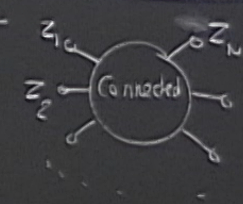
diagrams G with N external
without any vac
diagram compone

$K(G) = \#$ internal ve

Connected diagrams

$$W[j] = \hbar \cdot \mathrm{Log}[Z[j]] \iff Z[j] = \exp\left(\frac{1}{\hbar} W[j]\right)$$

$$\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} W[j] \Big|_{j=0} = \sum_{\substack{\text{connected} \\ \text{diagrams } G \\ \text{with } N \text{ external} \\ \text{vertices}}} g^{K(G)} \hbar^{[illegible](G)} c_G I_G(z_1 \ldots z_N) = \sum_G$$



$$\hbar \left[\frac{\delta}{\delta j(z_1)} \cdots \frac{\delta}{\delta j(z_N)} Z[j]/Z[0] \right]_{j=0} = \sum g^{K(G)} \hbar \; c_G \, I_G(z_1 \cdots z_N)$$

diagrams G
with N external vertices
without any vaccum
diagram component

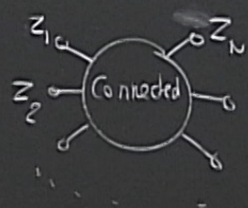
combinatorial factor

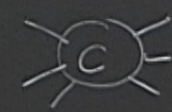
Feynman integral

$K(G) = \#$ internal vertices of G

$$\exp\left(\frac{1}{\hbar} W[j]\right)$$

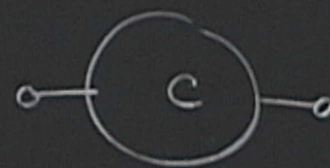
$$g^{K(G)} \frac{1}{\hbar}^{(G)} c_G \, I_G(z_1 \ldots z_N) = \sum_G$$



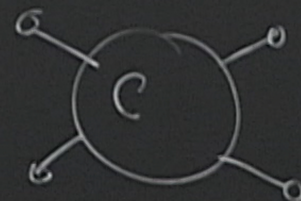


$$N=0 \qquad \sum_{\substack{\text{connected Vaccum} \\ \text{diagrams}}}$$

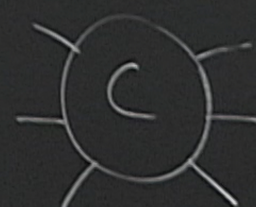
$$N=2 \qquad \sum_{\text{Connected 2pts}}$$



$$N=4 \qquad \sum_{\text{Connected 4pts}}$$



$$N=6 \qquad \sum_{\text{connected 6pts}}$$



$$Z = \exp(W) = 1 + W + \frac{W^2}{2} + \frac{W^3}{6} + \cdots$$

$$= \text{nothing} + \text{connected} + \frac{1}{2}\,\text{conn.}\;\text{conn.} + \frac{1}{6}\,\text{Conn}\;\text{Conn}\;\text{Conn} + \cdots$$

↑ combinatoric factor

$$= \text{all diagrams with any \# of connected comp}$$

c_1 c_2

symmetries between connected components

check the combinatoric factors

$$I_{c_1} \times I_{c_2} = I_{c_1 c_2}$$

$$C_{c_1} \cdot C_{c_2} = C_{c_1 c_2} + C_{c_2 c_1}$$