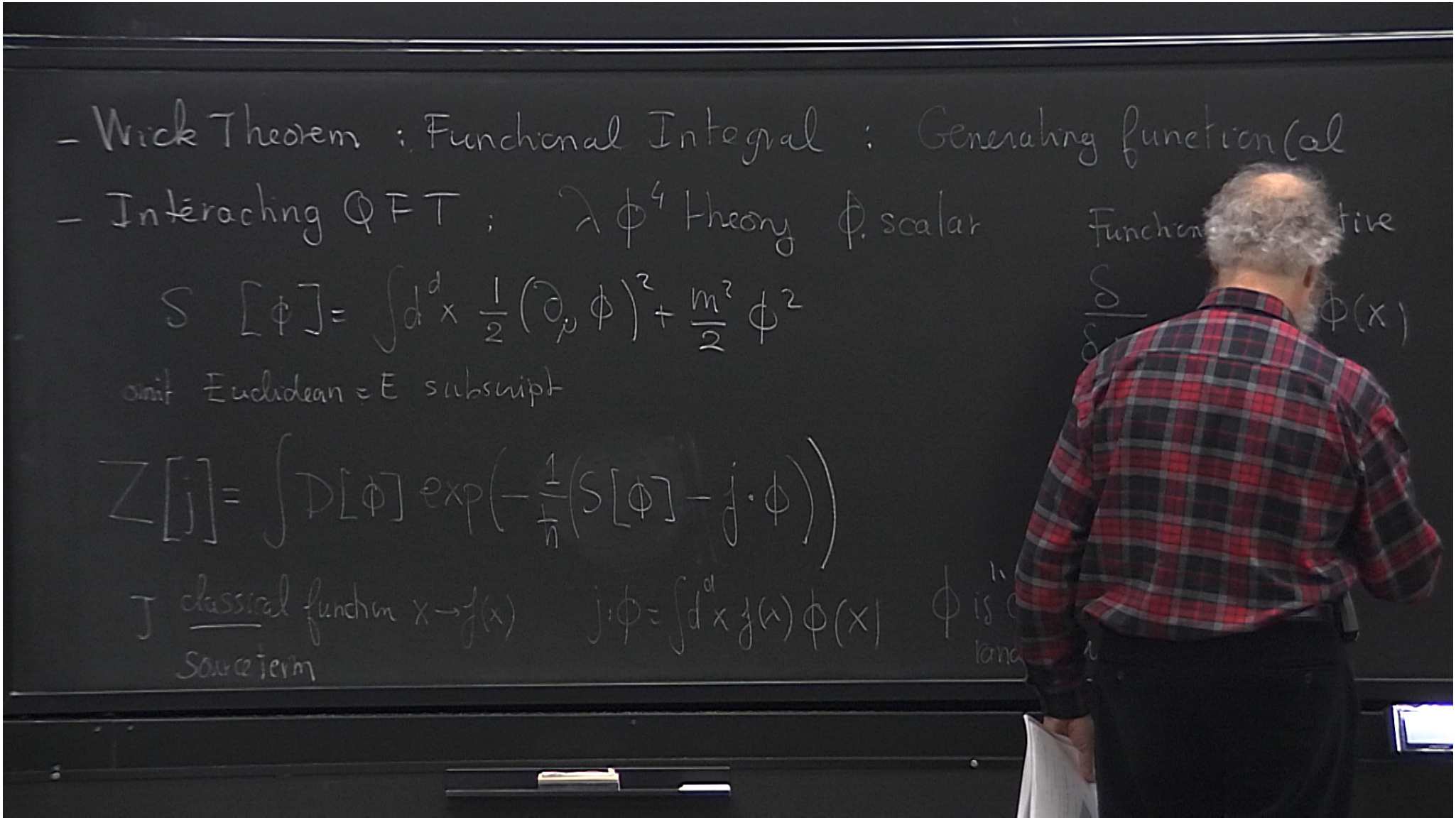
Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 4

Speakers: Francois David

Collection: PSI 2019/2020 - Quantum Field Theory II (David)

Date: November 15, 2019 - 9:00 AM

URL: http://pirsa.org/19110013


- Wick Theorem : Funchional Integral : Generating function(al
- Interacting QFT : $\lambda\phi^4$ theory ϕ scalar
$S[\phi] = \int d^d x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$
omit Euclidean = E subscript
$Z[j] = \int \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar}(S[\phi] - j\cdot\phi)\right)$
J classical function $x \to j(x)$
source term
$j\cdot\phi = \int d^d x\, j(x)\phi(x)$

- Wick Theorem ; Functional Integral ; Generating functional

- Interacting QFT ; $\lambda\phi^4$ theory ϕ scalar

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J classical function $x \to j(x)$ Source term

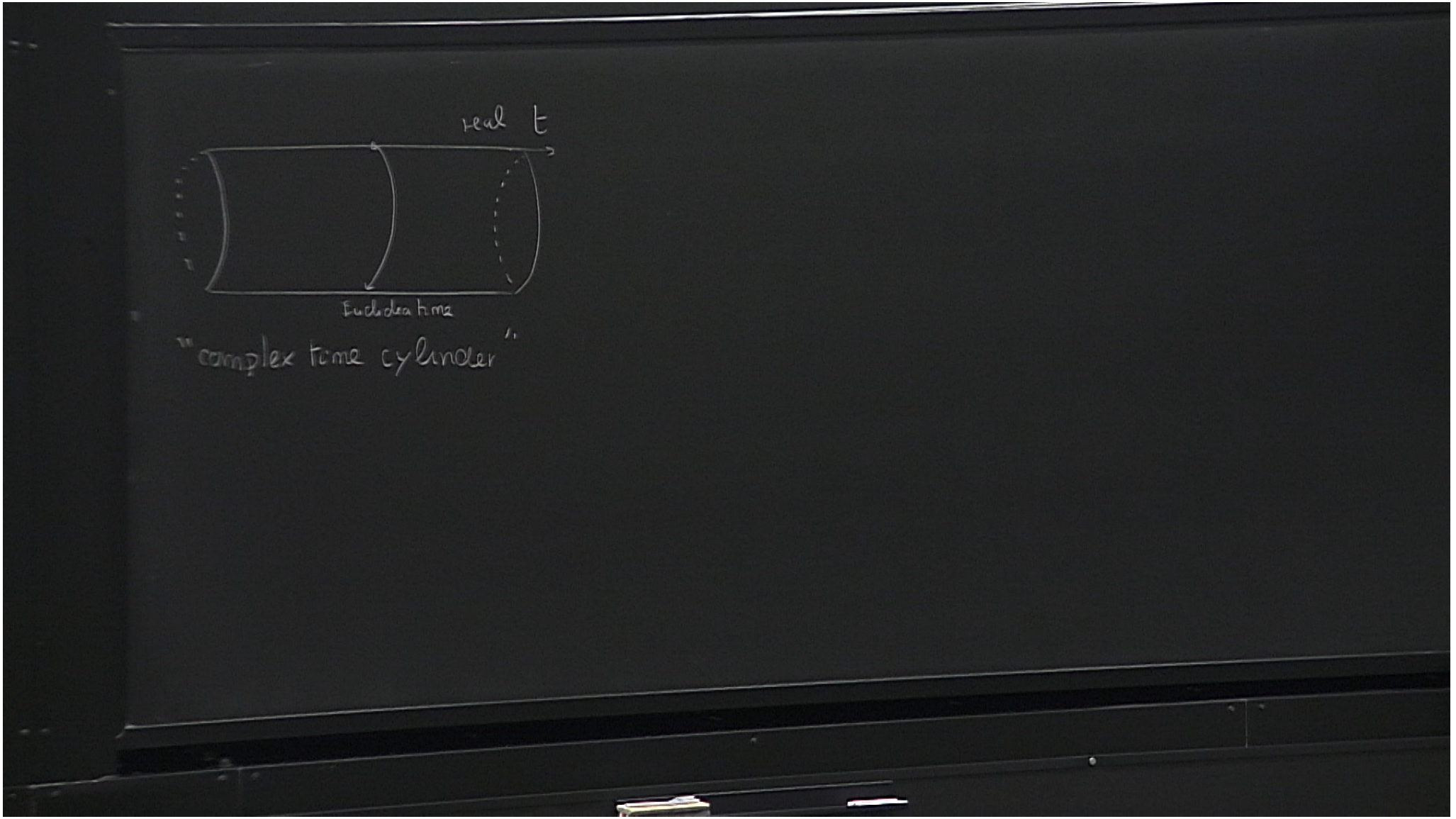
$$j\cdot\phi = \int d^d x \, j(x)\phi(x)$$

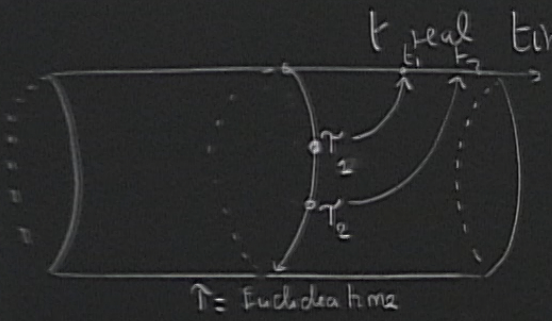
ϕ is "quantum" random variable

Functional derivative

$$\frac{\delta}{\delta j(x)} \longrightarrow \hbar^{-1}\phi(x)$$

$$\frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} Z$$


real t
Euclidean time
"complex time cylinder"



"complex time cylinder"

$e^{-\frac{\tau \hat{H}}{\hbar}}$ $\tau > 0$

$0 < \tau_1 < \tau_2 <$ Period

Time ordering in real and euclidean time variable

$$\int d^d x\, \partial_\mu \phi \partial_\mu \phi = \int d^d x\, \partial_\mu (\phi \partial_\mu \phi)$$

total derivative $= 0$

$$- \int d^d x\, \phi \underbrace{\partial_\mu \partial^\mu \phi}_{\Delta}$$

Free theory $\quad S[\phi] = \frac{1}{2} \phi \cdot (-\Delta + m^2) \cdot \phi$

$\cdot$ = scalar product of functions in $\mathbb{R}^d$

$$= \frac{1}{2} \int d^d x\, \phi(x) \left(-\Delta \phi(x) + m^2 \phi(x) \right)$$

$$= \frac{1}{2} \phi \cdot (-\Delta + m^2) \cdot \phi - j \cdot \phi + \frac{1}{2} j \cdot (-\Delta + m^2)^{-1} \cdot j - \frac{1}{2} j \cdot (-\Delta + m^2)^{-1} \cdot j$$

$$= \frac{1}{2} \tilde{\phi} (-\Delta + m^2) \tilde{\phi} - \frac{1}{2} j (-\Delta + m^2)^{-1} \cdot j \qquad \text{with } \tilde{\phi} = \phi - (-\Delta + m^2)^{-1} \cdot j$$

linear shift in ϕ

$$\left. \right]_{j=0} = \frac{1}{i^N} \langle \phi(x_1) \cdots \phi(x_N) \rangle$$

correlation function

$$\langle 0 | T [\hat{\phi}(x_1) \cdots \hat{\phi}(x_N)] | 0 \rangle$$

$$\int d^d x\, \partial_\nu \phi \partial_\nu \phi = \int d^d x\, \partial_\nu (\phi \partial_\nu \phi)$$

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$$- \int d^d x\, \phi \underbrace{\partial_\nu \partial^\nu \phi}_{\Delta}$$

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↑ linear shift in ϕ

$$Z[j] = \underbrace{\int D[\tilde{\phi}] \exp\left(-\frac{1}{\hbar} \tilde{\phi} (-\Delta + m^2) \cdot \tilde{\phi} \right)}_{Z[0]} \times \exp\left(\frac{\hbar}{2} j \cdot (-\Delta + m^2)^{-1} \cdot j \right)$$

$$\left. \frac{\delta^N Z[j]}{\delta j} \right|_{j=0} = \hbar^{-N} \langle \phi(x_1) \cdots \phi(x_N) \rangle$$

correlation function

$$\langle 0 | T[\hat{\phi}(x_1) \cdots \hat{\phi}(x_N)] | 0 \rangle$$

$$\int d^d x\, \partial_\mu \phi \partial_\mu \phi = \int d^d x\, \partial_\mu (\phi \partial_\mu \phi)$$

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$$- \int d^d x\, \phi \underbrace{\partial_\mu \partial^\mu \phi}_{\Delta}$$

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$$= \frac{1}{2} \phi \cdot (-\Delta + m^2) \cdot \phi - j \cdot \phi + \frac{1}{2} j \cdot (-\Delta + m^2)^{-1} \cdot j - \frac{1}{2} j (-\Delta + m^2)^{-1} \cdot j$$

$$= \frac{1}{2} \tilde{\phi} (-\Delta + m^2) \tilde{\phi} - \frac{1}{2} j (-\Delta + m^2)^{-1} \cdot j \qquad \text{with } \tilde{\phi} = \phi - (-\Delta + m^2)^{-1} \cdot j$$

linear shift in ϕ

$$Z[j] = \underbrace{\int D[\tilde{\phi}] \exp\left(-\frac{1}{\hbar} \tilde{\phi} (-\Delta + m^2) \cdot \tilde{\phi} \right)}_{Z[0]} \times \exp\left(\frac{1}{2\hbar} j \cdot (-\Delta + m^2)^{-1} \cdot j \right)$$

change of variable inside the integral

$$\boxed{Z[j] = Z[0] \exp\left(\frac{\hbar}{2} j (-\Delta + m^2)^{-1} j \right)}$$

$$\left. \frac{\delta^N Z[j]}{\delta j^N} \right|_{j=0} = \hbar^{-N} \langle \phi(x_1) \cdots \phi(x_N) \rangle$$

correlation function

$$\langle 0 | T[\hat{\phi}(x_1) \cdots \hat{\phi}(x_N)] | 0 \rangle$$

Source term

random variable

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \hbar^N \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_N)} \exp\left(+ \frac{1}{2\hbar} \int d^d x \int d^d y \; j(x) G(x-y) j(y) \right)$$

↪ co

$N = 2M$ has to be even

$N=2$ $\quad \frac{\delta}{\delta j(x_2)} \left[\frac{1}{\hbar} \int G(x_1 - x) j(x) d^q x \cdot \exp\left(\frac{1}{2\hbar} \int d^q x \; j \cdots j \right) \right]$

$$= \frac{\hbar^2}{\hbar} G(x_1 - x_2)$$

Feynman diagr

$N=4$ 3 possibilities $\hbar^2 \left[G(x_1-x_2) G(x_3-x_4) + G(x_1-x_3) G(x_2-x_4) + G(x_1-x_4) G(x_2-x_3) \right] = \langle \phi(x_1)$

random variable

$$\frac{\delta}{\delta j(x_n)} \exp\left(\pm \frac{1}{2\hbar} \int d^d x \int d^d y \; j(x) G(x-y) j(y)\right)$$

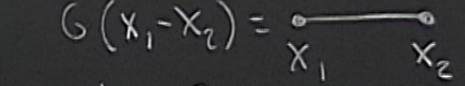
↖ computed yesterday

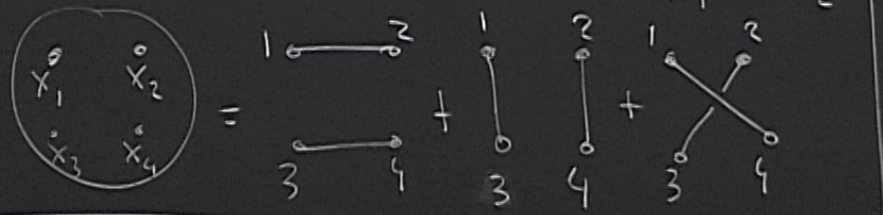
$j = 0$

and so on

$$= \exp\left(\frac{1}{2\hbar} \int d^d x \; j \cdots j\right)\Big]$$

Feynman diagrams

$G(x_1 - x_2) =$ 



$$x_4) + G(x_1 - x_3) G(x_2 - x_4)\Big] = \langle \phi(x_1) \cdots \phi(x_4) \rangle$$

$$(x_2 - x_3)$$

and so on $N=6 = \sum_{\text{pairing}} G \cdot G \cdot G$

in 3 propagator

15 pairings

Wick Theorem

$\Uparrow$

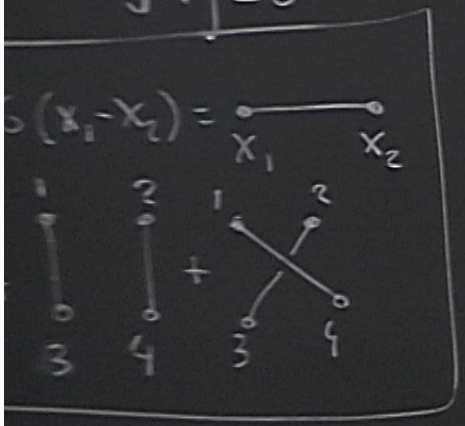
Gaussian integrals or

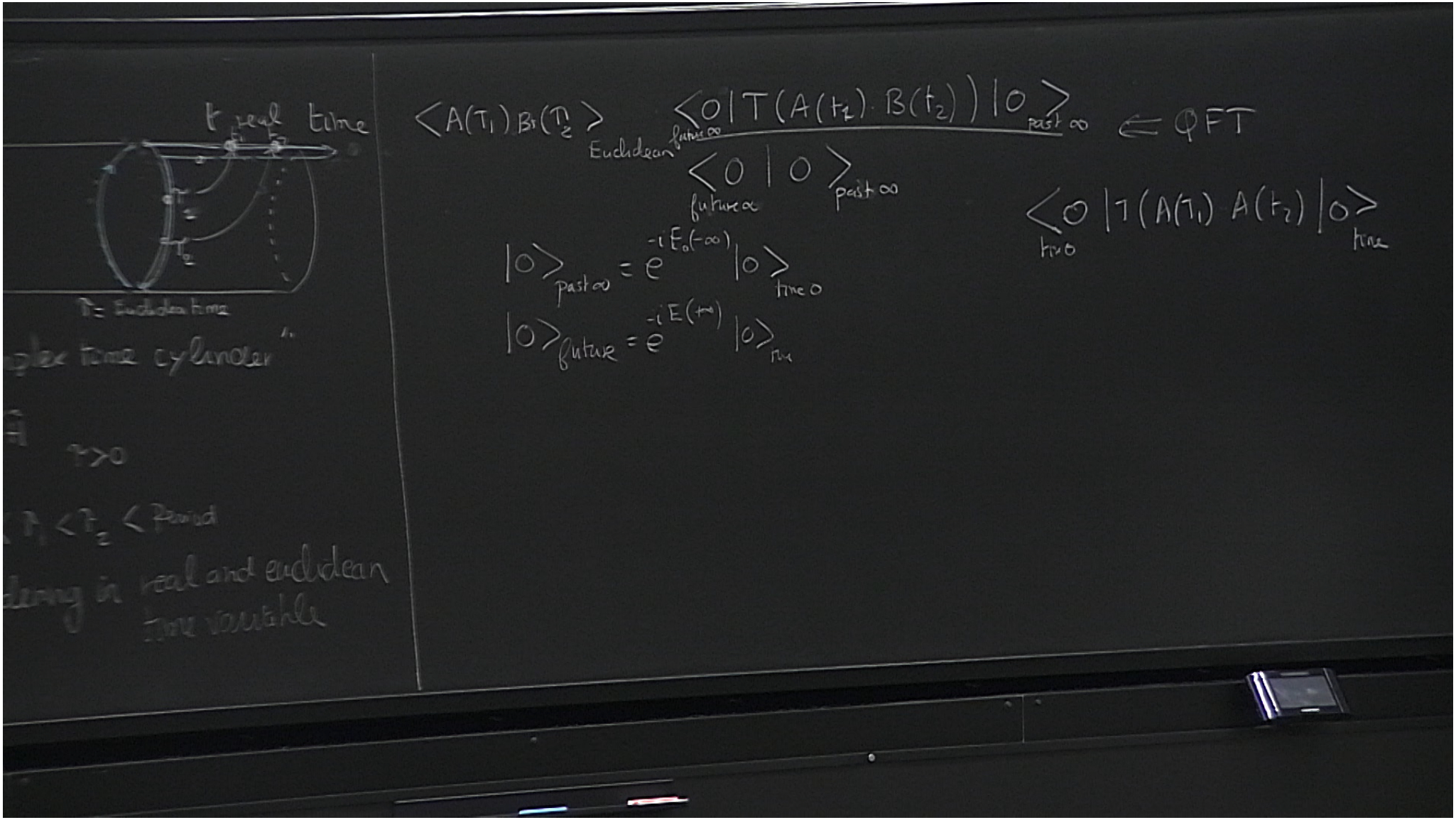
expectations of Gaussian random variables

Exercise

pairing $N=2M$

$$\frac{2M!}{2^N \cdot M!}$$




t real time
T= Euclidean time
$\langle A(T_1) B(T_2) \rangle_{\text{Euclidean}}$
$\frac{\langle 0 | T(A(t_2) \cdot B(t_2)) | 0 \rangle_{\text{past}\infty}}{\langle 0 | 0 \rangle_{\text{future}\infty}}$
$\in$ QFT
$\langle 0 | T(A(T_1) A(T_2) | 0 \rangle_{\text{time}}$
$|0\rangle_{\text{past}\infty} = e^{-iE_0(-\infty)} |0\rangle_{\text{time}}$
$|0\rangle_{\text{future}} = e^{-iE(+\infty)} |0\rangle_{\text{time}}$
plex time cylinder
$T_1 < T_2 <$ Period
ering in real and euclidean
time variable

$$+G(x_1-x_4)\,G(x_2-x_3)\Big]$$

Interacting ϕ^4 theory

simple, condensed matter, Higgs Sector

$$S[\phi] = \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right]$$

g coupling constant $\quad g \geq 0$

$4!$ for convenience

why ϕ^4

ϕ describes bosons with mass m

$$+ G(x_1 - x_4)\, G(x_2 - x_3) \Big]$$

Interacting ϕ^4 theory

simple, condensed matter, Higgs Sector

$$S[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4 \right]$$

g coupling constant $\quad g \geq 0$

$4!$ for convenience

why ϕ^4 . 2 particle interaction local



ϕ describes bosons with mass m

$\phi^2(x) \longrightarrow$ # of bosons (density) at x

e, condensed matter, Higgs Sector

$\phi^2 + \frac{g}{4!}\phi^4 \Big]$

$0 \Rightarrow$ repulsive interaction



n local

classical equation of motion

$$(-\Delta + m^2)\phi + \frac{g}{6}\phi^3 = 0$$

$$\Delta = -\frac{\partial^2}{\partial t^2} + \left(\frac{\partial}{\partial \vec{x}}\right)^2$$

Minkowski

non-linear KG Equation

correlation functions $\langle \phi(x_1) \cdots \phi(x_N) \rangle_g$

$$Z[j] = \int \mathcal{D}[\phi] \cdot \exp\left[-\frac{1}{\hbar}\left(S[\phi] - j \cdot \phi\right)\right]$$

$$S[\phi] = S_0[\phi] + S_I[\phi] \qquad S_I[\phi] = \frac{g}{4!}\int d^D x\, \phi^4(x)$$

S_0: free field; S_I: interaction

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = \frac{\int \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right) \phi(x_1) \cdots \phi(x_N)}{\int \mathcal{D}[\phi] \exp\left(-\frac{1}{\hbar} S[\phi]\right)}$$

← depend on g

numerator : $Z_0 \cdot \left[\frac{\int D[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right) \cdot \exp\left(-\frac{1}{\hbar}\frac{g}{4!}\int \phi^4(x)\, d^d x\right) \phi(z_1)\cdots\phi(z_N)}{\int D[\phi] \exp\left(-\frac{1}{\hbar} S_0[\phi]\right)} \right]$

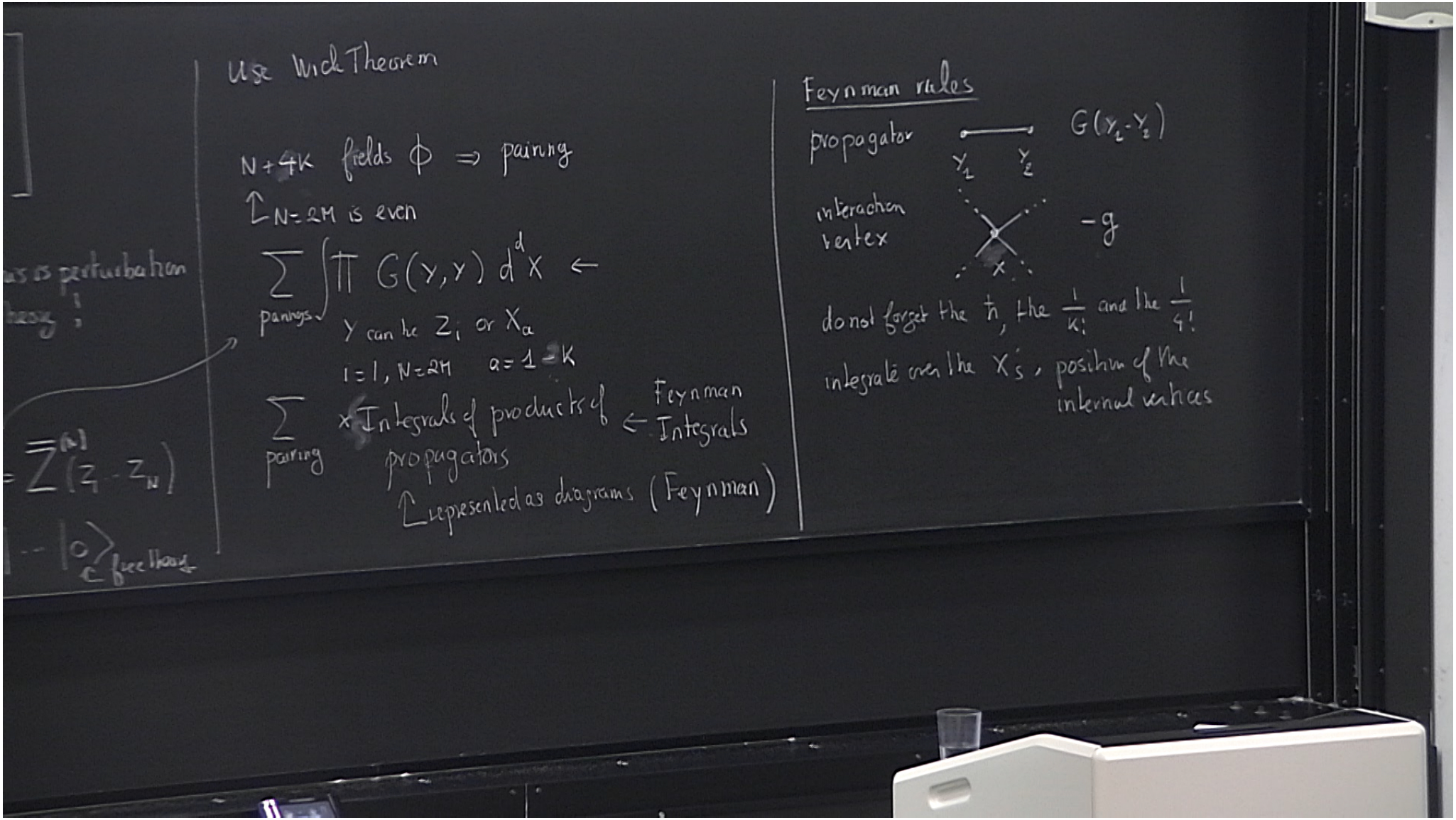
Z_0 : generating function of the free theory at $J=0$

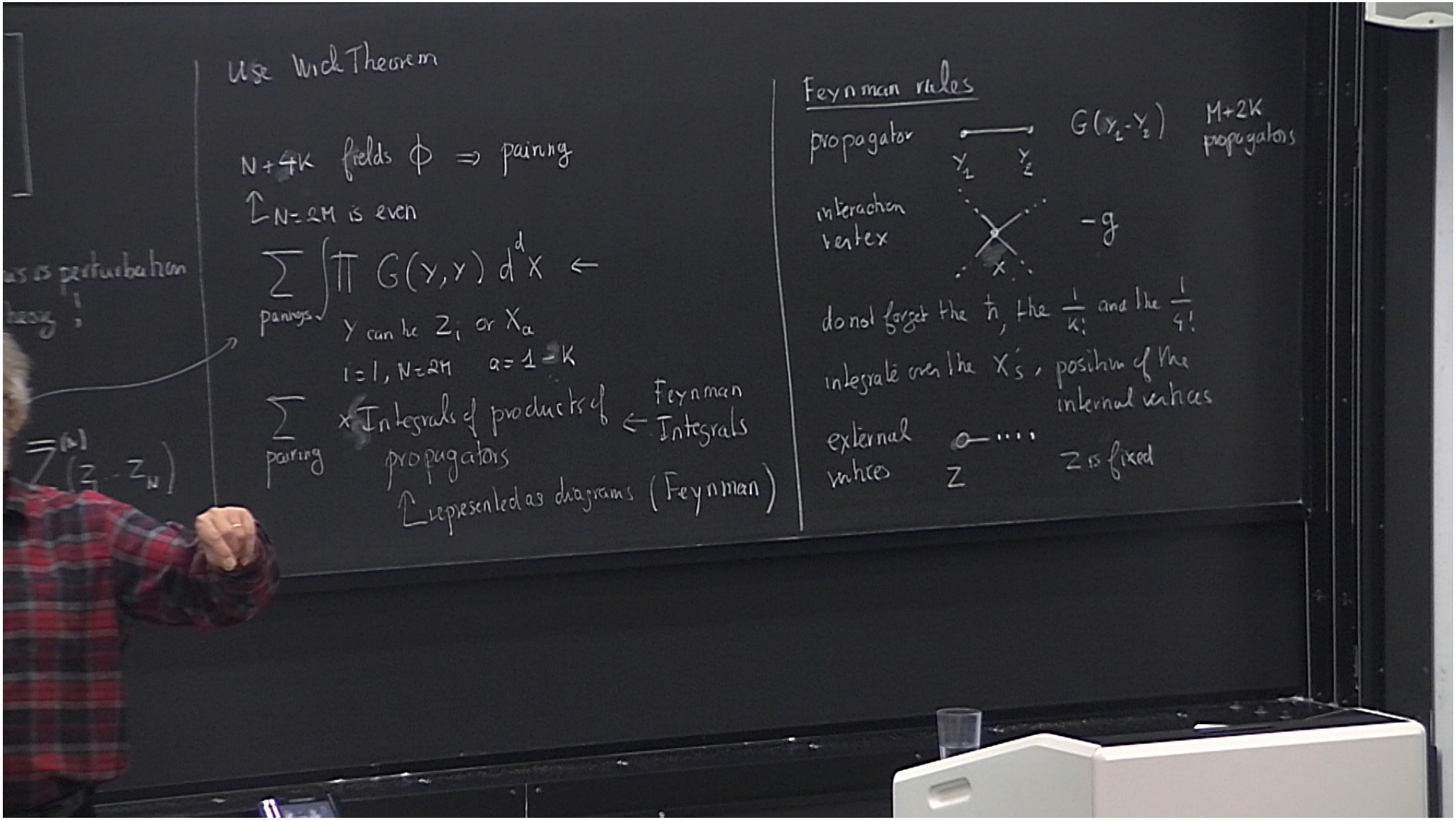
$$\exp\left(-\frac{1}{\hbar}\frac{g}{4!}\int d^d x\, \phi^4(x)\right) = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{-g}{\hbar \cdot 4!}\right)^K \int\cdots\int d^d x_1 \cdots d^d x_K\, \phi^4(x_1)\cdots\phi^4(x_K)$$

Invert $\int D[\phi]$ and $\sum_K$

$$[\ ] = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{-g}{\hbar \cdot 4!}\right)^K \int d^d x_1 \cdots d^d x_K \cdot \langle \phi(z_1)\cdots\phi(z_N)\, \phi^4(x_1)\cdots\phi^4(x_K) \rangle_0$$

expectation value for the free theory

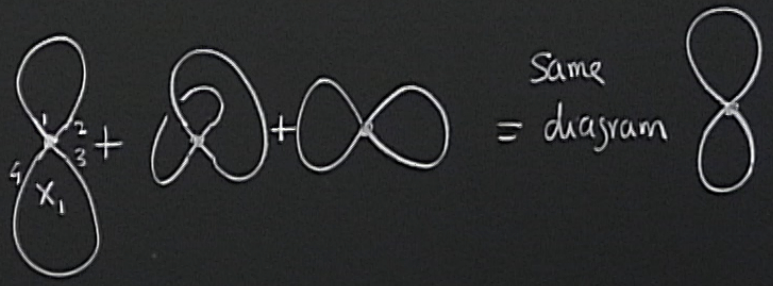

Use Wick Theorem
N+4K fields φ ⇒ pairing
↑ N=2M is even
∑ pairings ∫ ∏ G(Y,Y) d^d X ←
Y can be Z_i or X_a
i=1, N=2M a=1-K
∑ pairing × Integrals of products of propagators ← Feynman Integrals
↑ represented as diagrams (Feynman)
Feynman rules
propagator
G(Y_1 - Y_2)
Y_1 Y_2
interaction vertex
-g
X
do not forget the ħ, the 1/k! and the 1/4!
integrate over the X's, position of the internal vertices
as is perturbation
theory !
Z^(N)(J_1 ... J_N)
|0⟩ free theory


Use Wick Theorem
N+4K fields φ ⇒ pairing
↑ N=2M is even
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G(Y_1 - Y_2)
M+2K propagators
interaction vertex
-g
do not forget the ħ, the 1/k! and the 1/4!
integrate over the X's, position of the internal vertices
external vertices Z
Z is fixed

Examples $N=0$

$K=0 \Rightarrow 1$

$K=1$ 1 vertex

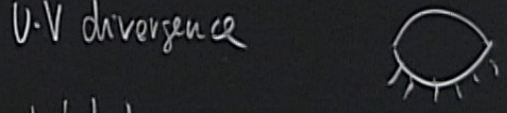


$\langle \phi(x)\phi(y) \rangle = \hbar \cdot G(x-y)$

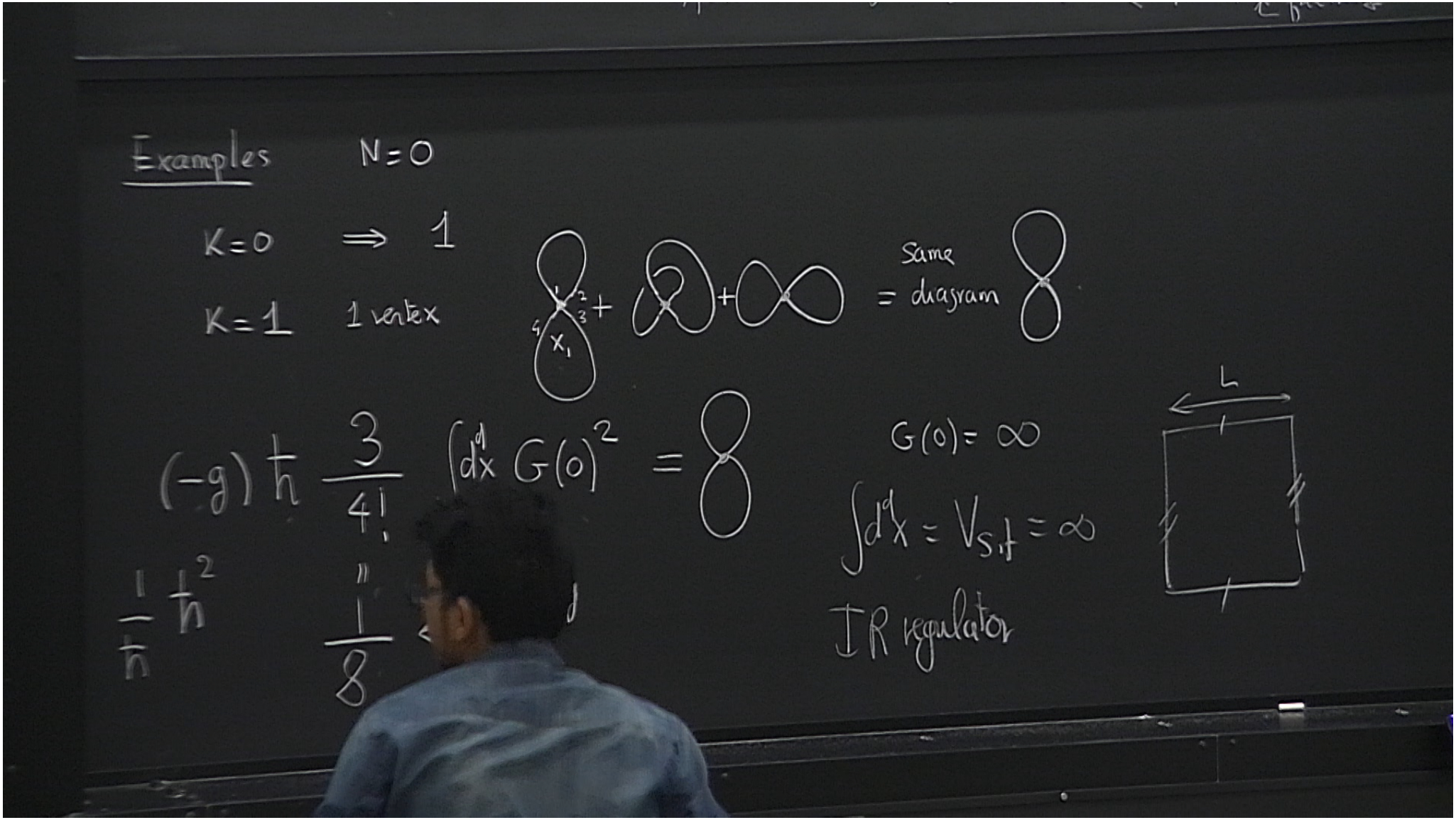
$G(x-y) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik(x-y)}}{k^2+m^2}$

$G(0) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} = \infty$ if $d \geq 2$

U.V divergence






Examples N=0
K=0 ⇒ 1
K=1 1 vertex
Same
= diagram
$(-g)\hbar\frac{3}{4!}\int d^dx\, G(0)^2 =$
$G(0)=\infty$
$\int d^dx = V_{S,t} = \infty$
IR regulator
L
$\frac{1}{\hbar}\hbar^2$
$\frac{1}{8}$