Title: PSI 2019/2020 - Quantum Field Theory II (David) - Lecture 3

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Collection: PSI 2019/2020 - Quantum Field Theory II (David)

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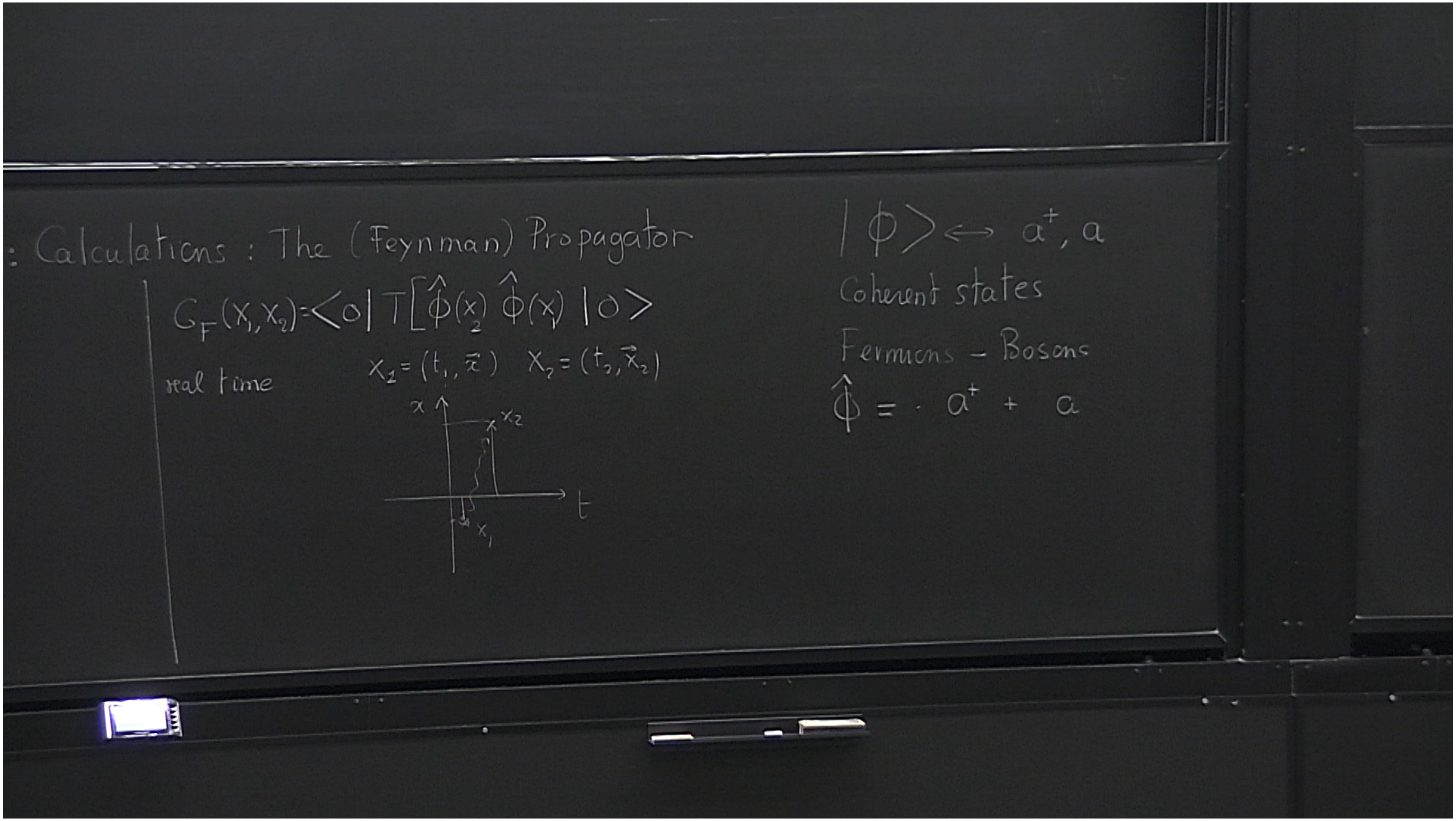
Today: Free Field : Functional Integral Quantization : Calculations : The (Feynman) Propagator



: Calculations : The (Feynman) Propagator
$G_F(x_1,x_2) = \langle 0| T[\hat{\phi}(x_2)\,\hat{\phi}(x_1)\,|0\rangle$
real time
$x_1 = (t_1, \vec{x})$ $x_2 = (t_2, \vec{x}_2)$
x
x_2
t
x_1
$|\phi\rangle \longleftrightarrow a^+, a$
Coherent states
Fermions – Bosons
$\hat{\phi} = \cdot\, a^+ + a$

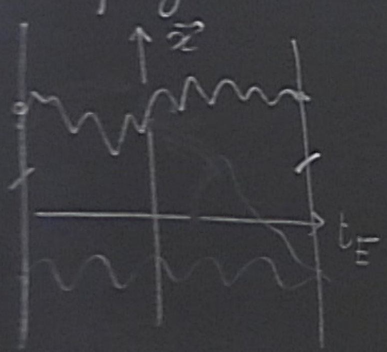
Today: Free Field: Functional Integral Quantization: Calculatio

Euclidean time $\longrightarrow$ real time

$\uparrow$ project on the vaccum

observable $\hat{A}$

$|n\rangle$ energy eigen states



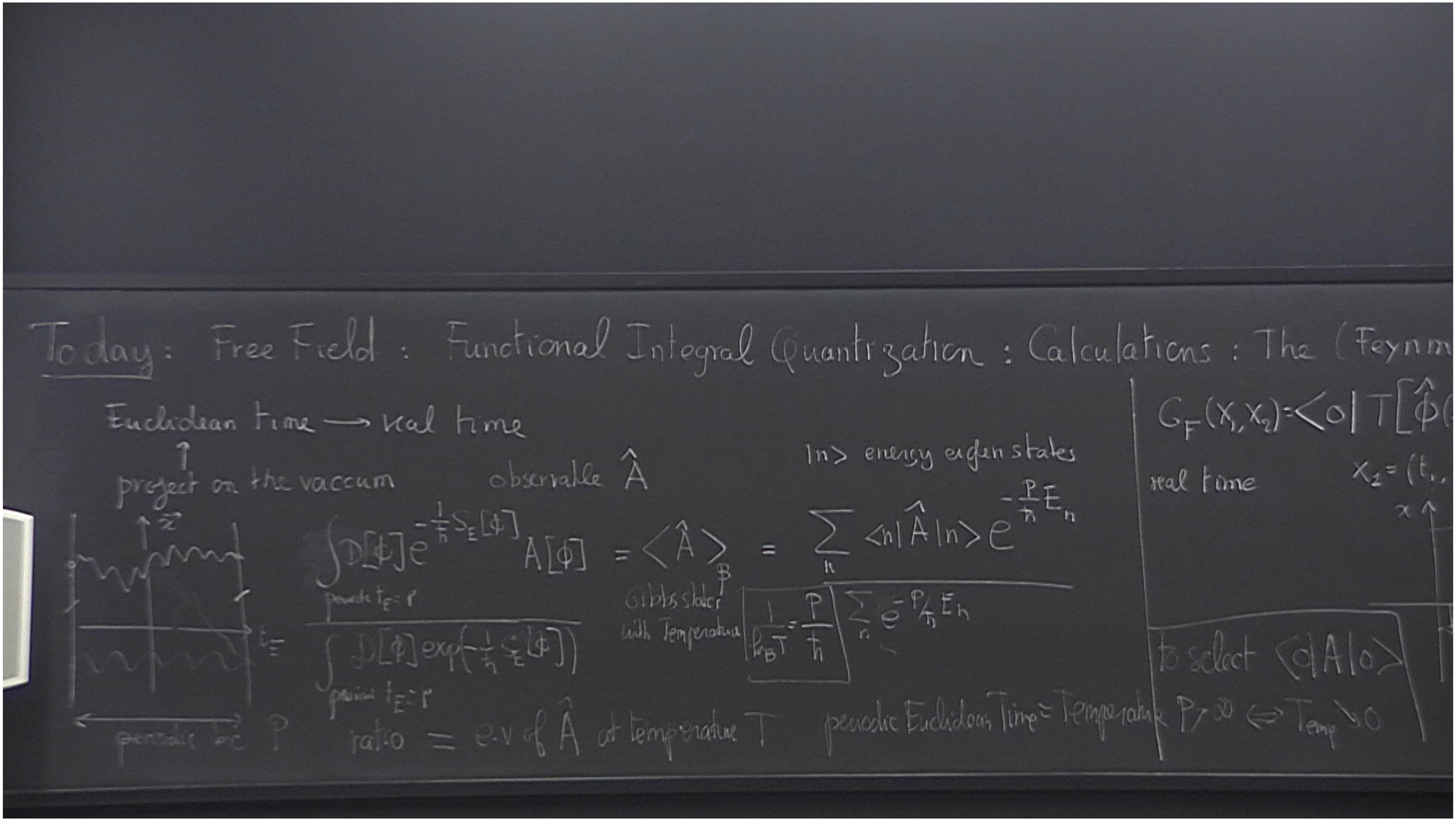
periodic b.c. β

$$\frac{\int_{\text{periodic } t_E=\beta} D[\phi]\, e^{-\frac{1}{\hbar}S_E[\phi]}\, A[\phi]}{\int_{\text{period } t_E=\beta} D[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)} = \langle \hat{A} \rangle_\beta = \frac{\sum_n \langle n|\hat{A}|n\rangle e^{-\frac{\beta}{\hbar}E_n}}{\sum_n e^{-\beta/\hbar\, E_n}}$$

Gibbs state with Temperature $\boxed{\frac{1}{k_B T} = \frac{\beta}{\hbar}}$

ratio = e.v of $\hat{A}$ at temperature T

periodic Euclidean Time = Temperature


Today: Free Field : Functional Integral Quantization : Calculations : The (Feynm
Euclidean time ⟶ real time
project on the vaccum
observable $\hat{A}$
$|n\rangle$ energy eigenstates
$\int \mathcal{D}[\phi] e^{-\frac{1}{\hbar}S_E[\phi]} A[\phi] = \langle \hat{A} \rangle_\beta = \frac{\sum_n \langle n|\hat{A}|n\rangle e^{-\frac{\beta}{\hbar}E_n}}{\sum_n e^{-\beta/\hbar E_n}}$
periodic $t_E = \beta$
$\int \mathcal{D}[\phi] \exp(-\frac{1}{\hbar} S_E[\phi])$
periodic $t_E = \beta$
Gibbs state with Temperature
$\boxed{\frac{1}{k_B T} = \frac{\beta}{\hbar}}$
t_E
periodic b.c. β
ratio = e.v of $\hat{A}$ at temperature T
periodic Euclidean Time ≃ Temperature
$G_F(x_1, x_2) = \langle 0| T[\hat{\phi}($
real time
$x_2 = (t,$
To select $\langle 0|A|0\rangle$
$\beta \to \infty \Leftrightarrow$ Temp $\searrow 0$

periodic b.c β ratio = e.v of $\hat{A}$ at temperature T periodic Euclidean time

Q.M particle $\longrightarrow$ QFT

$$\sum_n \langle n|\hat{A}|n\rangle e^{-\frac{\beta}{\hbar}E_n} = \sum_n \langle n|\hat{A}\, e^{-\frac{\beta}{\hbar}\hat{H}}|n\rangle$$

$$= \int dq \langle q|\hat{A} e^{-\frac{\beta}{\hbar}\hat{H}}|q\rangle = \int dq \int_{q(0)=q(\beta)=q} D[q] e^{-\frac{1}{\hbar}S_E[q]} A(q)$$

bc $q(0)=q(\beta)$

to select $\langle 0|A|0\rangle$

periodic Euclidean Time $\simeq$ Temperature $\beta \to \infty \Leftrightarrow \mathrm{Temp} \searrow 0$

Free Scalar Field $\quad \phi(x) \quad x=(t_E,\vec{x})=(x^0,\dots,x^{d-1}) \quad \partial_\mu=\frac{\partial}{\partial x^\mu} \quad \Delta=\partial_\mu\partial_\mu$ Laplacian operator

$\mu=0,\dots,d-1$

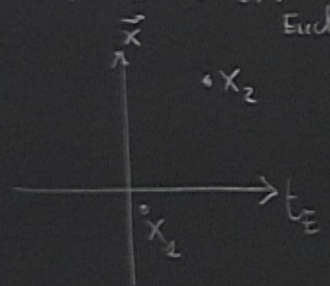
Action $\quad S_E[\phi]=\int_{\mathbb{R}^d} d^dx \; \frac{1}{2}(\partial_\mu\phi)^2+\frac{m^2}{2}\phi^2$

$S_E[\phi]=\frac{1}{2}\int d^dx \; \phi(x)\left(-\Delta+m^2\right)\phi(x)$

is a quadratic form in ϕ

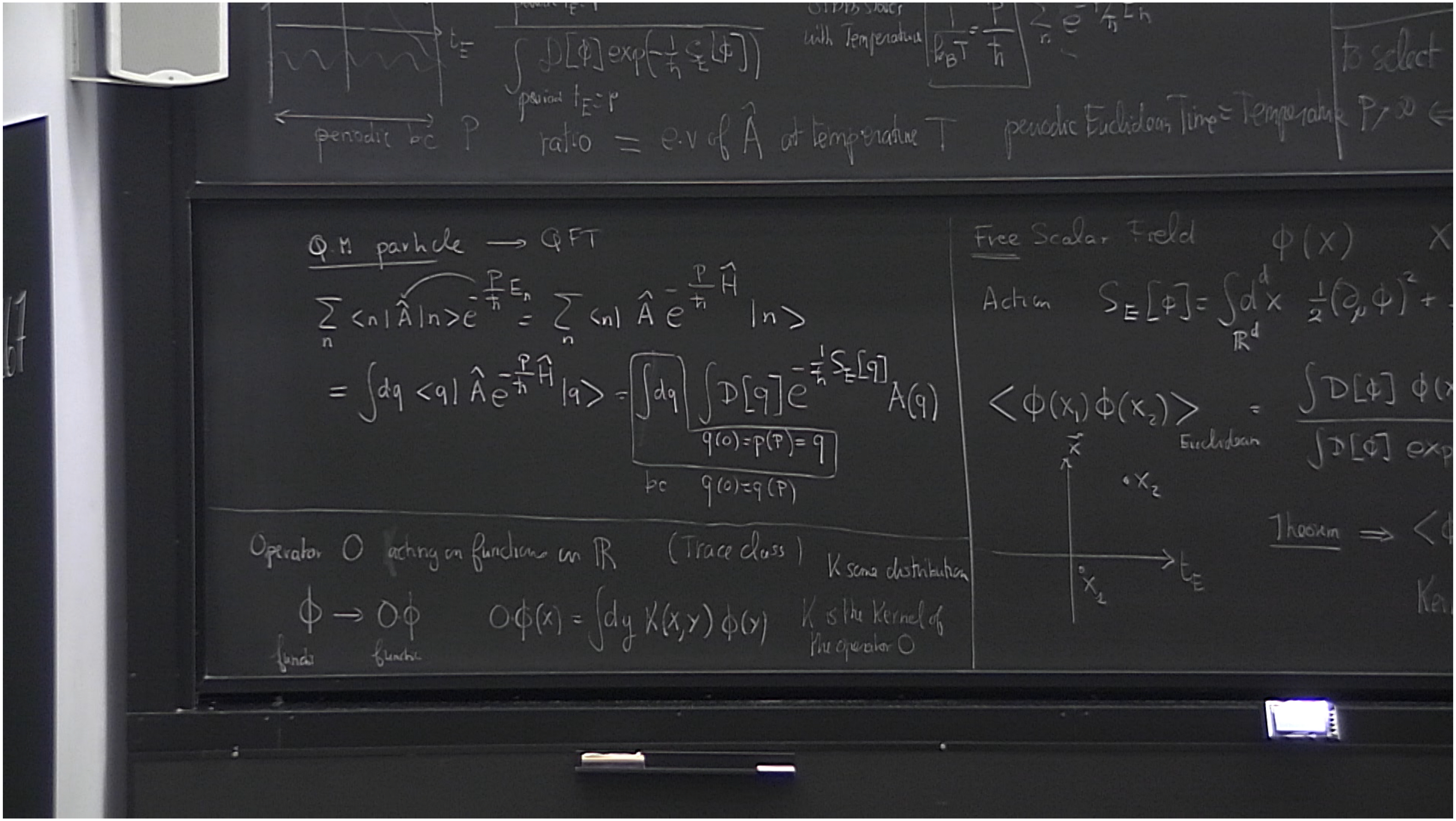
$$\langle\phi(x_1)\phi(x_2)\rangle_{\text{Euclidean}} = \frac{\int D[\phi]\;\phi(x_1)\phi(x_2)\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)}{\int D[\phi]\;\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)}$$

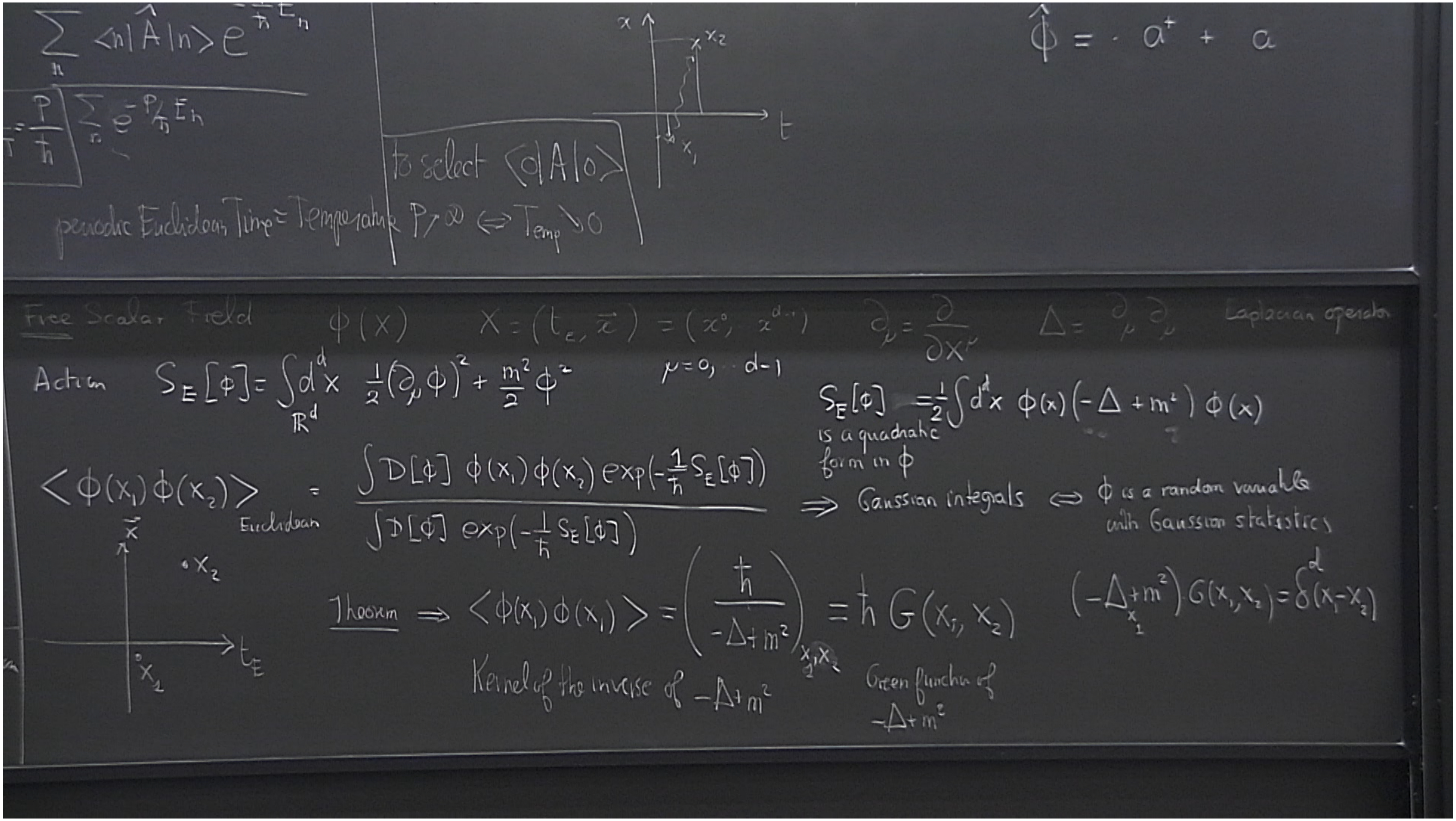
$\Rightarrow$ Gaussian integrals $\Leftrightarrow$ ϕ is a random variable with Gaussian statistics



Theorem $\Rightarrow$ $\langle\phi(x_1)\phi(x_1)\rangle = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1,x_2}$

Kernel of the inverse of $-\Delta+m^2$


periodic bc P
ratio = e.v of Â at temperature T
periodic Euclidean Time ≃ Temperature
with Temperature
to select
Q.M particle → QFT
$\sum_n \langle n|\hat{A}|n\rangle e^{-\frac{P}{\hbar}E_n} = \sum_n \langle n|\hat{A}e^{-\frac{P}{\hbar}\hat{H}}|n\rangle$
$= \int dq \langle q|\hat{A}e^{-\frac{P}{\hbar}\hat{H}}|q\rangle = \int dq \int D[q] e^{-\frac{1}{\hbar}S_E[q]} A(q)$
$q(0)=p(P)=q$
bc $q(0)=q(P)$
Operator O acting on functions on ℝ (Trace class)
K some distribution
$\phi \to O\phi$
function function
$O\phi(x) = \int dy\, K(x,y)\, \phi(y)$
K is the Kernel of the operator O
Free Scalar Field $\phi(x)$
Action $S_E[\phi] = \int_{\mathbb{R}^d} d^dx\, \frac{1}{2}(\partial_\mu\phi)^2 +$
$\langle \phi(x_1)\phi(x_2)\rangle_{\text{Euclidean}} = \frac{\int D[\phi]\,\phi(x}{\int D[\phi]\, \exp}$
x_2
x_1
t_E
Theorem ⇒



$\sum_n \langle n|\hat{A}|n\rangle e^{-\frac{1}{\hbar}E_n}$
$\sum_n e^{-\frac{P}{\hbar}E_n}$
to select $\langle 0|A|0\rangle$
periodic Euclidean Time = Temperature $P \nearrow \infty \Leftrightarrow \text{Temp} \searrow 0$
$\hat{\phi} = \cdot\, a^+ + a$
Free Scalar Field $\phi(x)$ $x = (t_E, \vec{x}) = (x^0, x^{d-1})$ $\partial_\mu = \frac{\partial}{\partial x^\mu}$ $\Delta = \partial_\mu \partial_\mu$ Laplacian operator
Action $S_E[\phi] = \int_{\mathbb{R}^d} d^dx \ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2$ $\mu = 0, \dots d-1$
$S_E[\phi] = \frac{1}{2}\int d^dx\ \phi(x)(-\Delta + m^2)\phi(x)$
is a quadratic form in ϕ
$\langle \phi(x_1)\phi(x_2)\rangle_{\text{Euclidean}} = \frac{\int D[\phi]\ \phi(x_1)\phi(x_2)\exp(-\frac{1}{\hbar}S_E[\phi])}{\int D[\phi]\ \exp(-\frac{1}{\hbar}S_E[\phi])}$
$\Rightarrow$ Gaussian integrals $\Leftrightarrow$ ϕ is a random variable with Gaussian statistics
Theorem $\Rightarrow \langle \phi(x_1)\phi(x_1)\rangle = \left(\frac{\hbar}{-\Delta + m^2}\right)_{x_1, x_2} = \hbar\, G(x_1, x_2)$
$(-\Delta_{x_1} + m^2)\, G(x_1, x_2) = \delta^d(x_1 - x_2)$
Kernel of the inverse of $-\Delta + m^2$
Green function of $-\Delta + m^2$

$\phi \to O\phi$ $\quad O\phi(x) = \int dy\, K(x,y)\,\phi(y)$ $\quad$ K is the Kernel of the operator O

function $\quad$ function

Sol of homogeneous eq $=$ $\exp(\vec{x}\cdot\vec{u}\cdot m)$ $\quad |\vec{u}| = 1$

grow to ∞ in $\vec{u}_+$ direction

$\hat{G}(k) =$

$k^2 = k_\mu$

<u>But</u>: periodic space-time ; then period to ∞

$\Rightarrow$ $G(x_1, x_2)$ is <u>bounded</u> at ∞ (and in fact $\searrow 0$)

Which solutions ? $\quad$ b.c in Euclidean space time (only one sensible b.c)

<u>periodic</u>

- $G(x_1, x_2) = G(x_1 - x_2)$ translation invariance ; Fourier transform

$$\hat{G}(k) = \int_{\mathbb{R}^d} dx\, e^{-ik\cdot x} G(x) \quad ; \quad (k^2 + m^2)\hat{G}(k) = 1$$

$$k = (k_0 \cdots k_{d-1}) = (k_\mu)$$

Kernel of the inverse of $-\Delta + m^2$

Green function of $-\Delta + m^2$

$= 1$

$$\hat{G}(k) = \frac{1}{k^2 + m^2} \Rightarrow \hat{G}(x) = \int \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \frac{1}{k^2+m^2}$$

$$k^2 = k_\mu k_\mu$$

$0)$

ne sensible b.c)

transform

$x_1 \longrightarrow t_E$

Theorem $\Rightarrow \langle \phi(x_1)\phi(x_1)\rangle = \left(\frac{\hbar}{-\Delta+m^2}\right)_{x_1,x_2} = \hbar\, G(x_1,x_2)$

Kernel of the inverse of $-\Delta+m^2$

Green function of $-\Delta+m^2$

$(-\Delta_{x_1}+m^2)\,G(x_1,x_2) = \delta(x_1-x_2)$

$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}$

$\hat{G}(k) = \frac{1}{k^2+m^2} \Rightarrow \boxed{\hat{G}(x) = \int \frac{d^d k}{(2\pi)^d} e^{ikx}\frac{1}{k^2+m^2}}$

Bessel Function

$k^2 = k_\mu k_\mu$

$= \frac{1}{2\pi}\left(\frac{2\pi|x|}{m}\right)^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(|x|\cdot m)$

Green Function in Euclidean space-time

rotational invariance

Properties: * $x = x_1 - x_2 \nearrow \infty$

large distance

$G(x) \simeq \exp(-|x|\cdot m)$ exponential decay

$m = \frac{1}{\xi}$ ξ is the correlation length (of the quantum field)

* small distance $x \searrow 0$

mass term m is irrelevant

$\boxed{G(x) \approx |x|^{2-d}}$

divergent if $d>2$

A bit more $d=2$

$\boxed{G(x) \simeq \frac{-1}{2\pi}\log(|x|\cdot m)}$

$\boxed{\text{IMPORTANT}}$

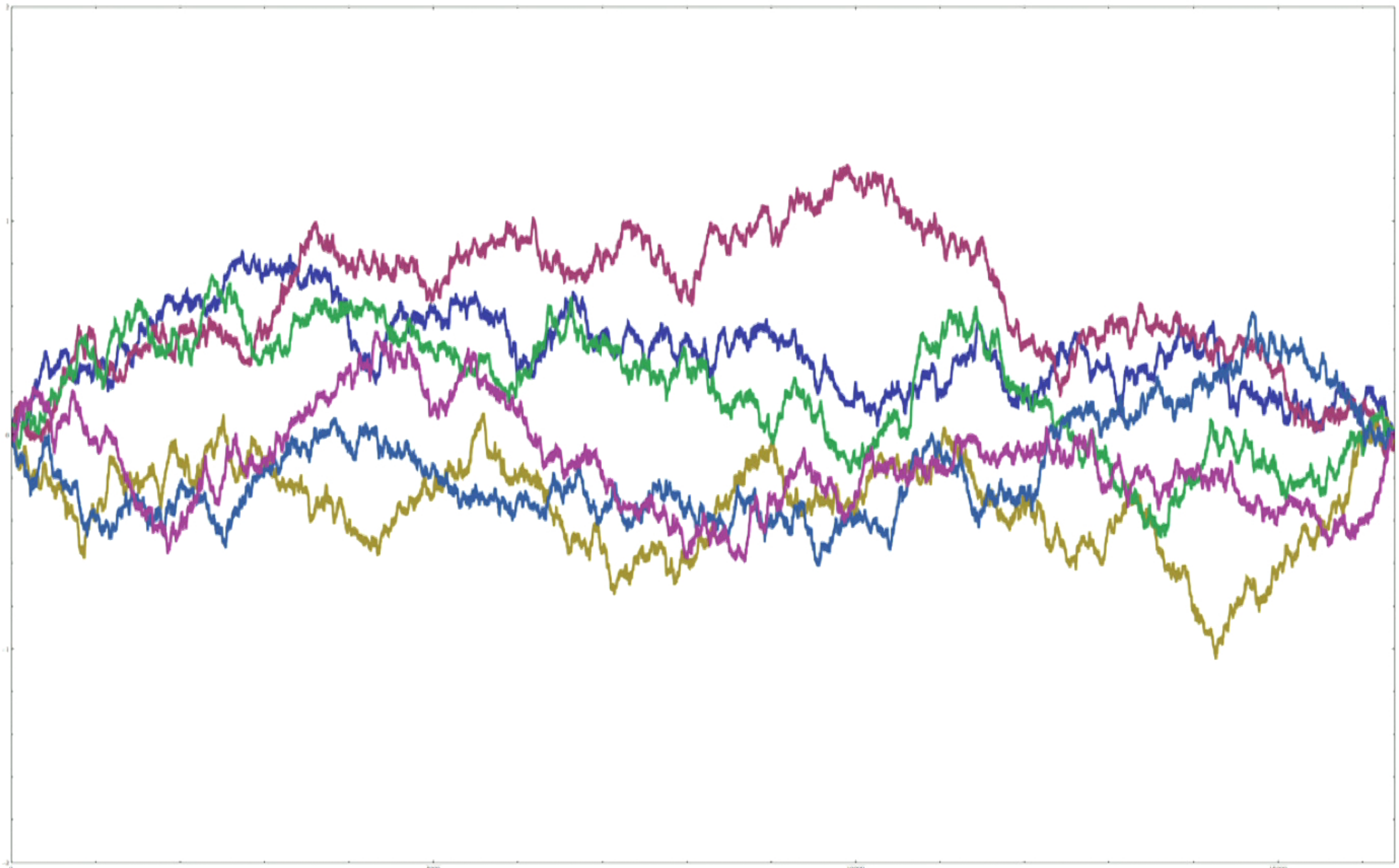
U.V. singularities

Singular at small distance as long as $d \geqslant 2$

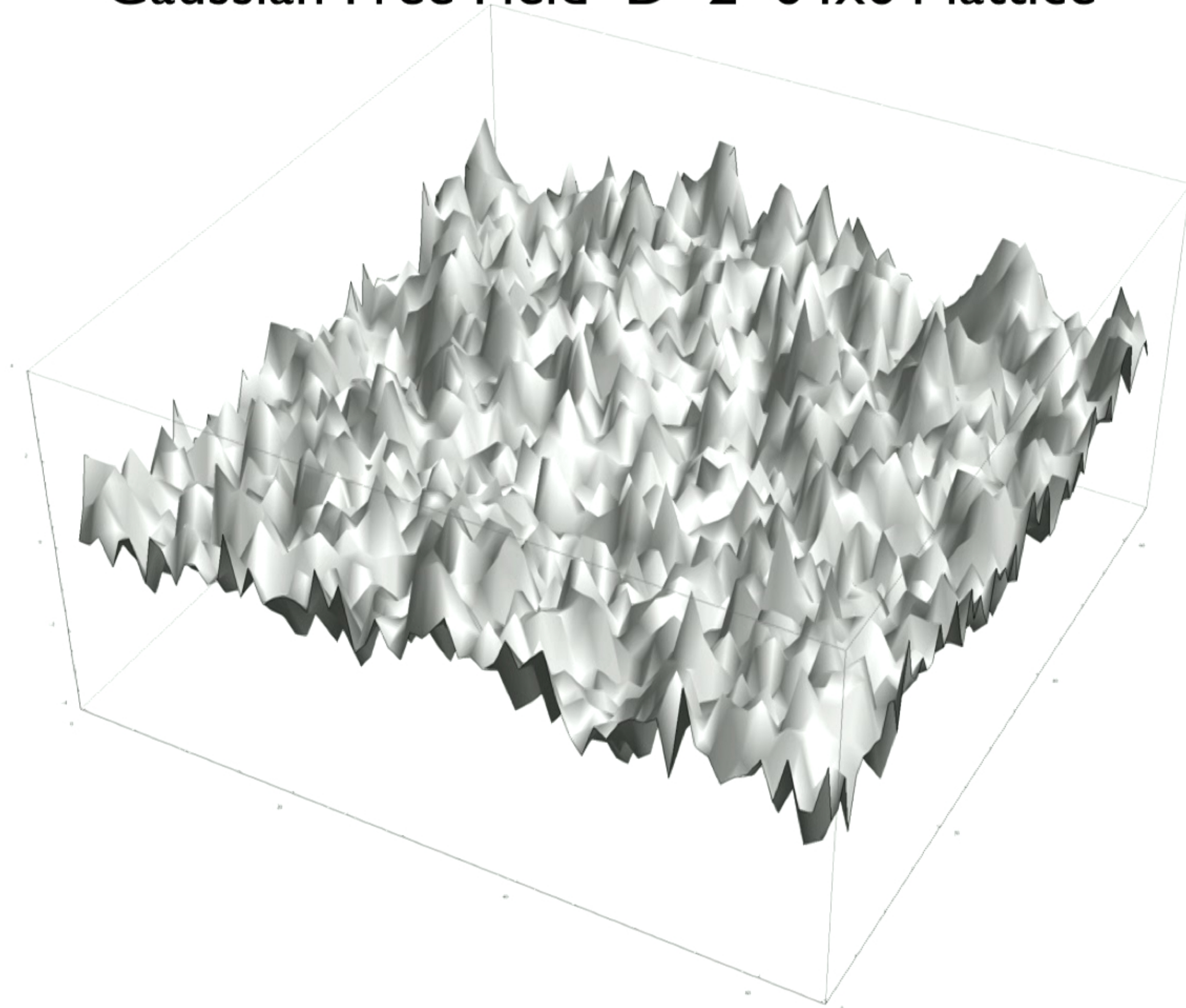
$QFT \neq Q.M$

Samples of Gaussian free field configurations (Euclidean, periodic b.c.)

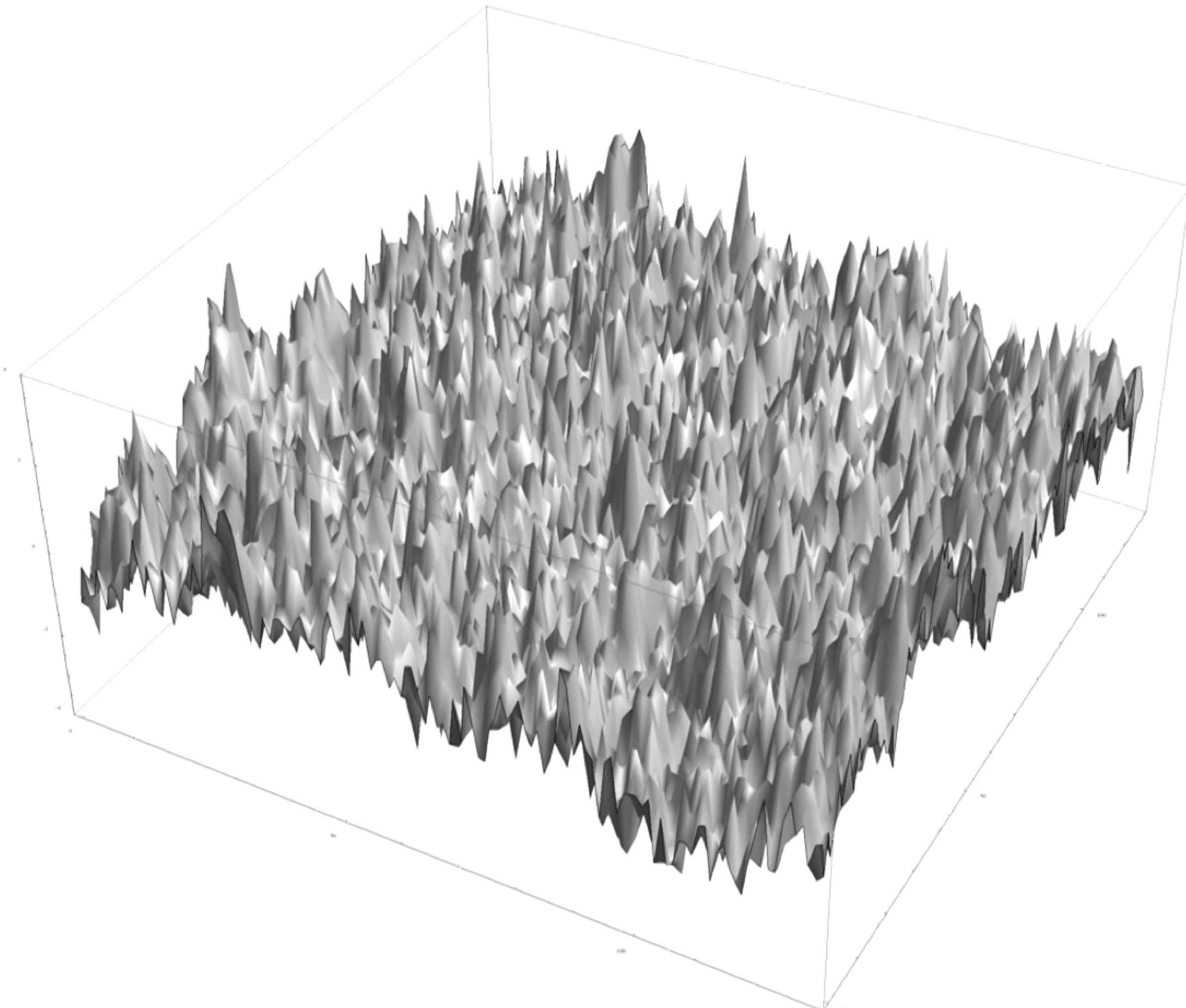
D=1 : Gaussian Free Fields = Random Walk
(i.e. Brownian or Wiener Process)



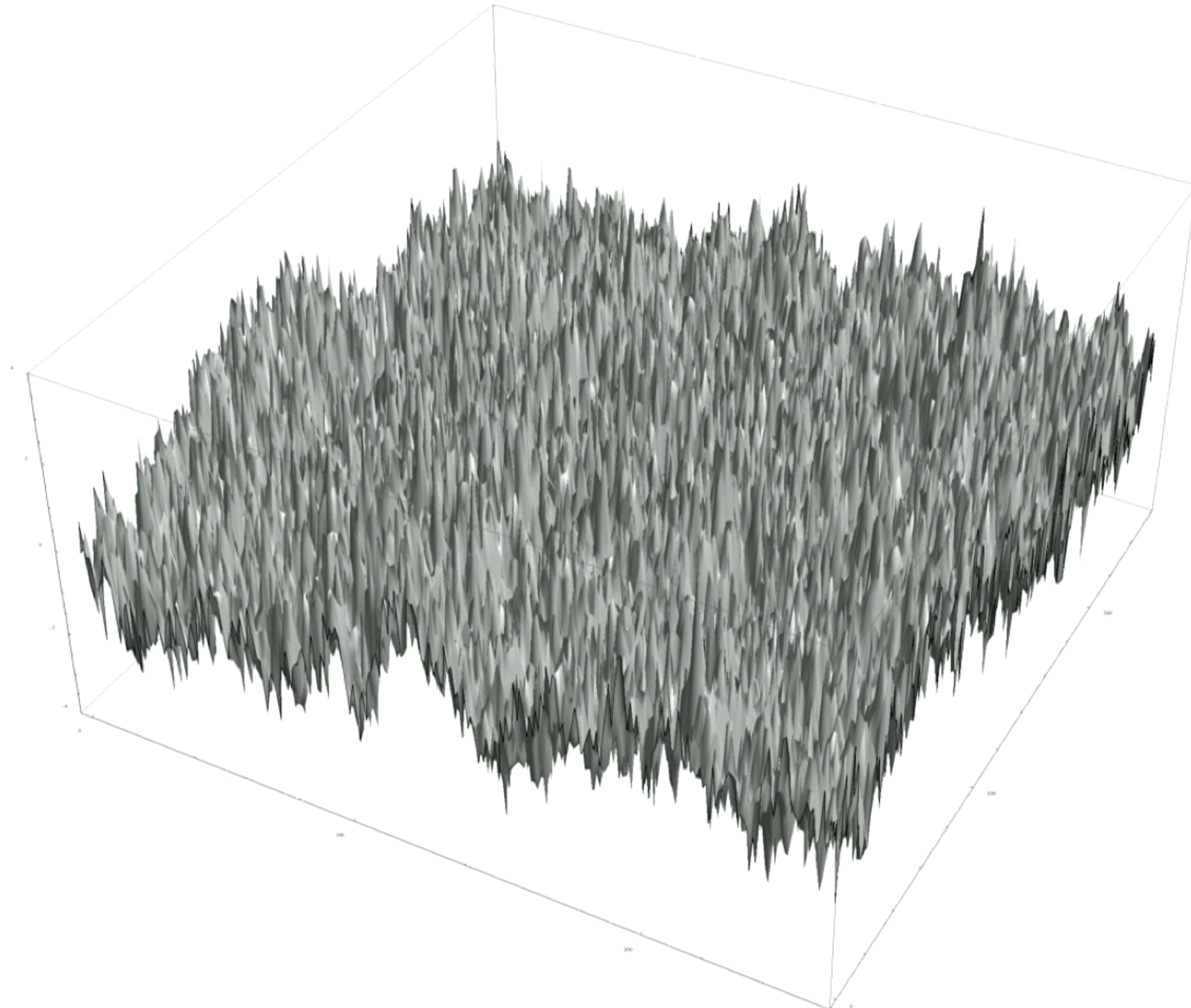
Gaussian Free Field D=2 64x64 lattice



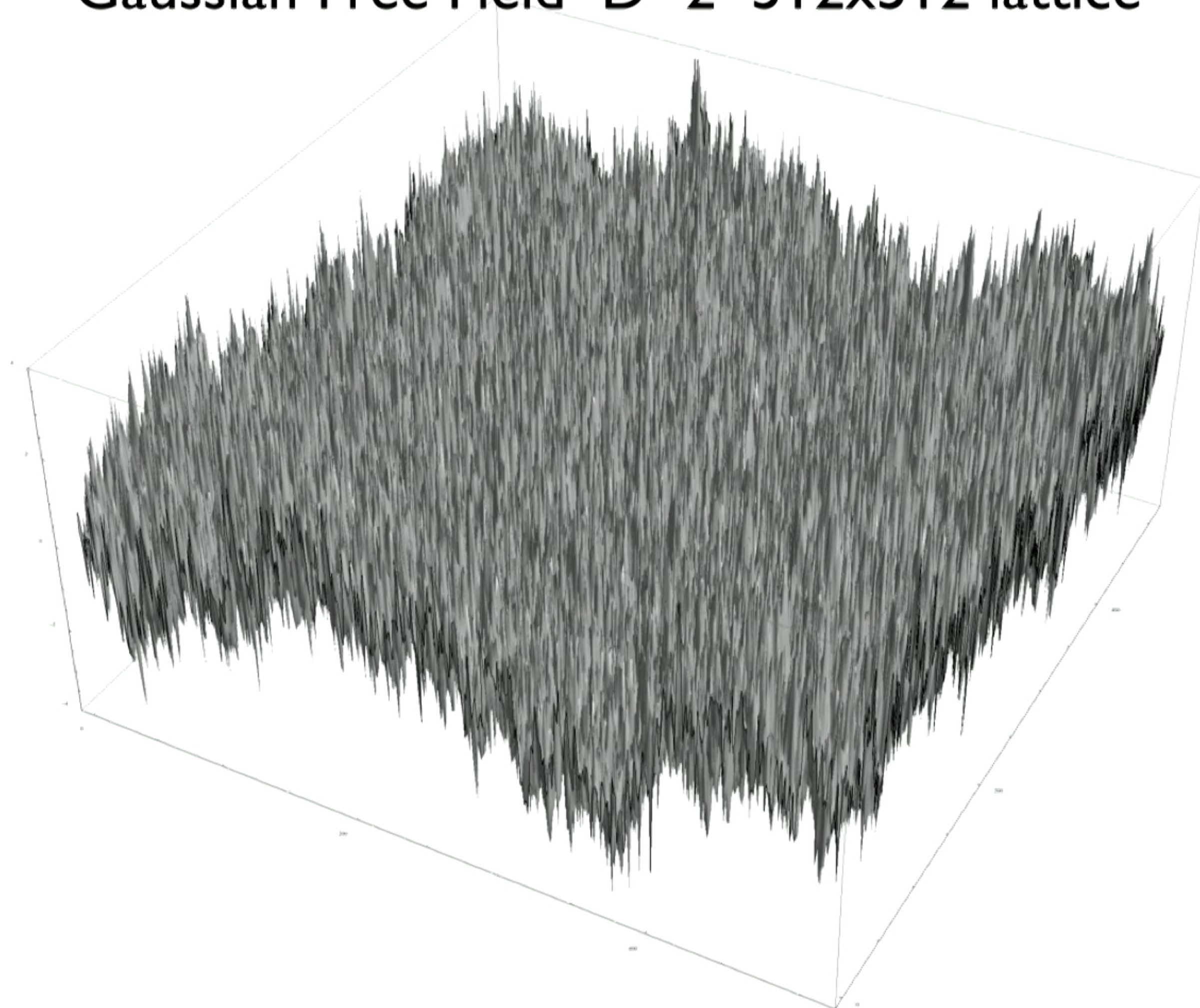
Gaussian Free Field D=2 128x128 lattice



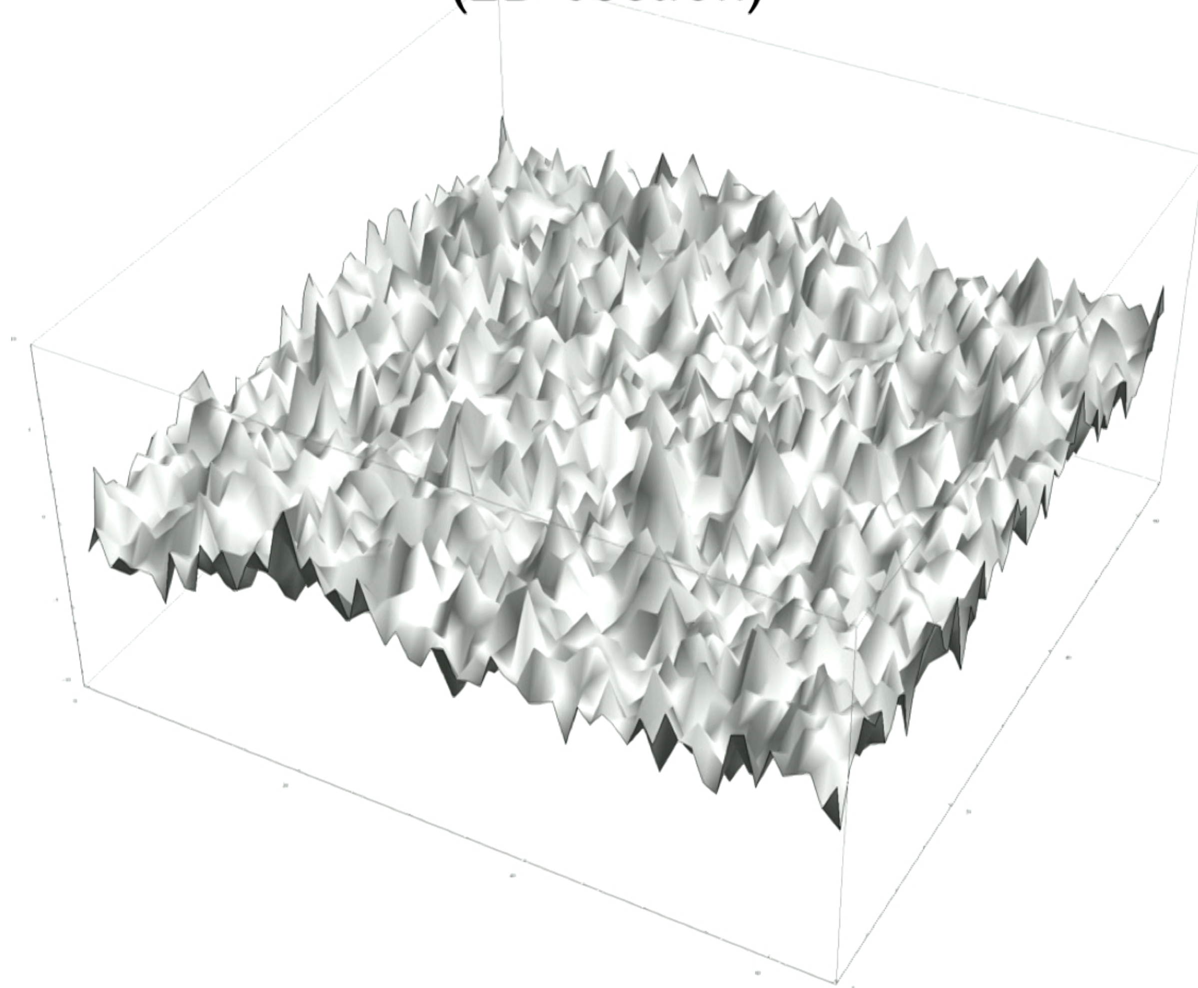
Gaussian Free Field D=2 256x256 lattice



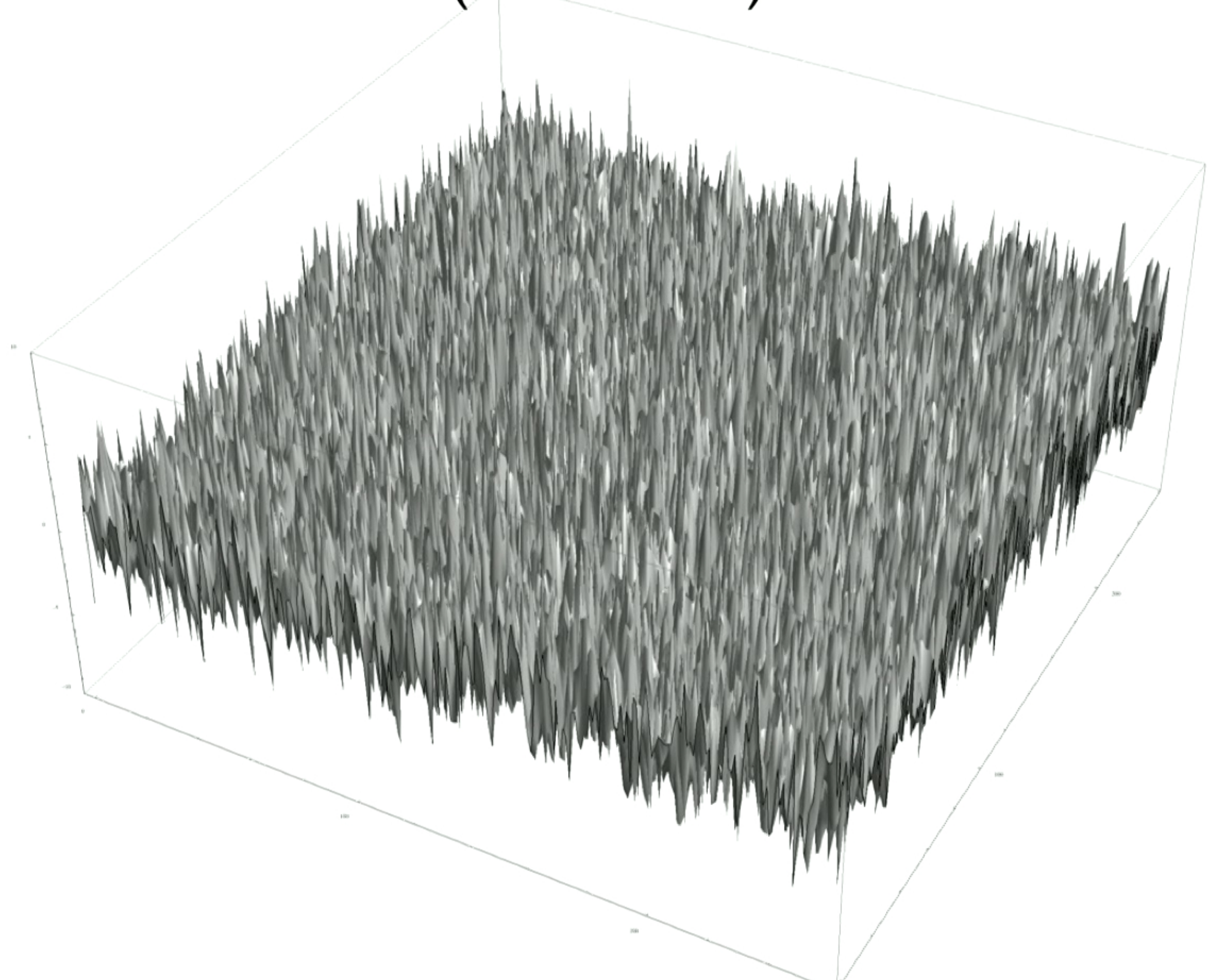
Gaussian Free Field D=2 512x512 lattice



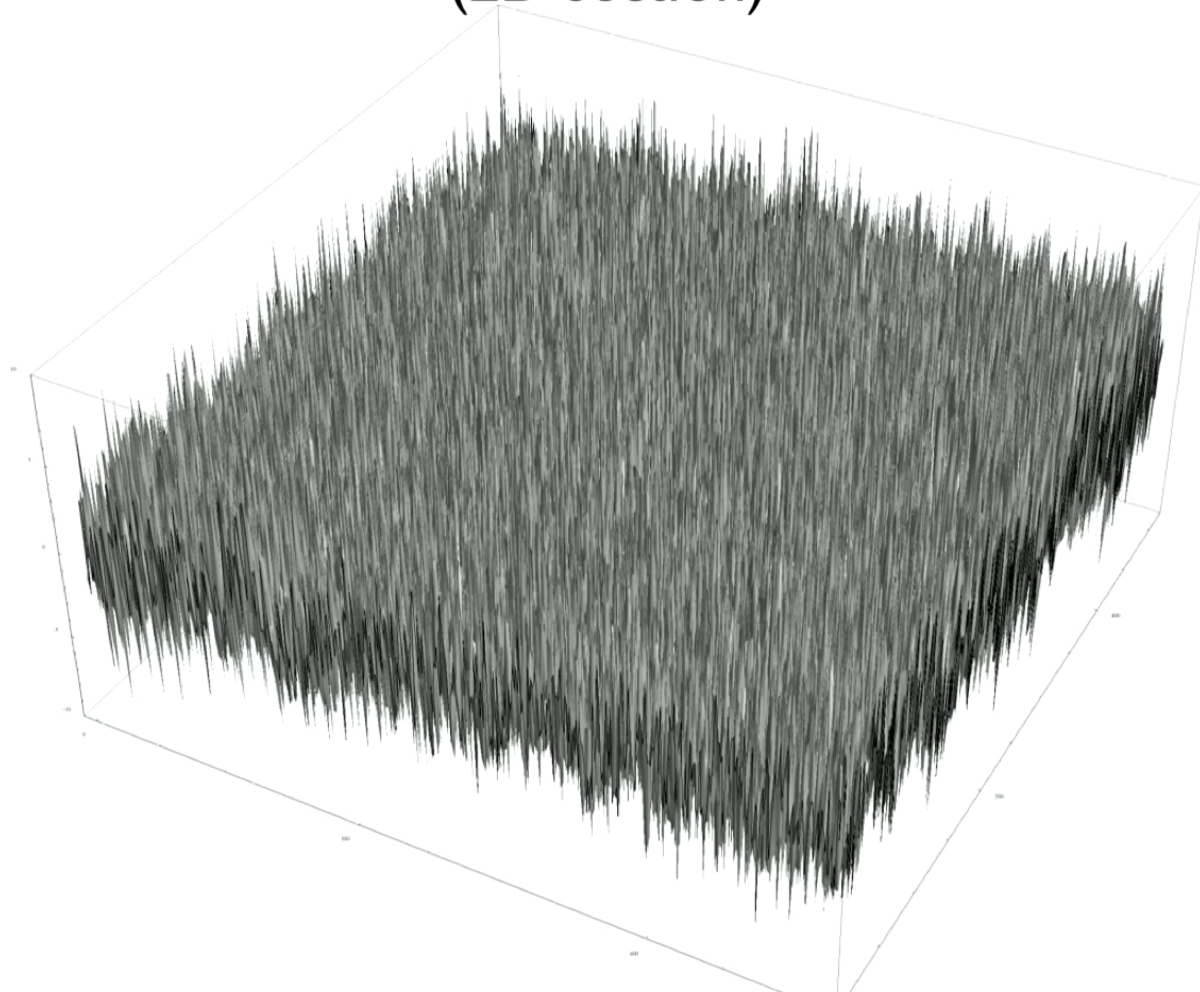
Gaussian Free Field D=3 64x64x64 lattice
(2D section)



Gaussian Free Field D=3 256x256x256 lattice
(2D section)



Gaussian Free Field D=3 512x512x512 lattice
(2D section)



Euclidean → Minkowski (Real time)

$$G_E(x,0) \xrightarrow[\text{Expect}]{} G_F(x,0) = \langle 0 | T[\hat{\Phi}(x)\hat{\Phi}(0)] | 0 \rangle$$

Feynman

position: $X_E = (x^0, \vec{x})$

momentum: $K_E = (k_0, \vec{k})$

$$G(X_E) = \int \frac{d^d K_E}{(2\pi)^d} \boxed{\frac{e^{iK_E \cdot X_E}}{K_E^2 + m^2}}$$

analytic

$$\Longrightarrow \boxed{x^0 \to it} \quad \boxed{k_0 \to -i\omega}$$

keep this real

$$e^{iK_E \cdot X_E} = e^{i(x^0 k_0 + \vec{x}\cdot\vec{k})}$$

$$\int d^d K_E = \int dk_0 \int d\vec{k}$$

deform the contour (analyticity)

$\Uparrow$

locality

only possible choice

t, ω are real

$\vec{x}$, x^0

Euclidean

$\vec{x}$, t

Mink.

M, t

Wick rotation

$E \downarrow x^0 = t_E = \uparrow$

$+\frac{\pi}{2}$ rotation

$\alpha - i\theta$

$t_E = x_0$

$-\infty$, $+\infty$, k_0, $-\theta$, $-\frac{\pi}{2}$, ω

keep this real

$$e^{i K_E \cdot X_E} = e^{i(x^0 k_0 + \vec{x}\cdot\vec{k})} \longrightarrow e^{i(t\omega + \vec{x}\cdot\vec{k})}$$

$$\boxed{x^0 \longrightarrow it}$$

$$\boxed{k_0 \longrightarrow -i\omega}$$

$$\int d^d K_E = \int dk_0 \int d\vec{k} \longrightarrow -i\int d\omega \int d\vec{k}$$

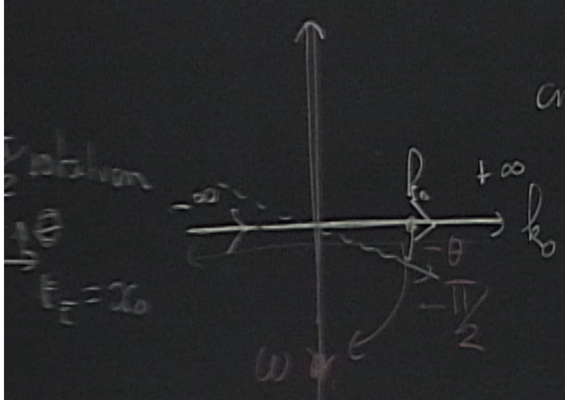
deform the contour (analyticity)

$\Uparrow$

locality

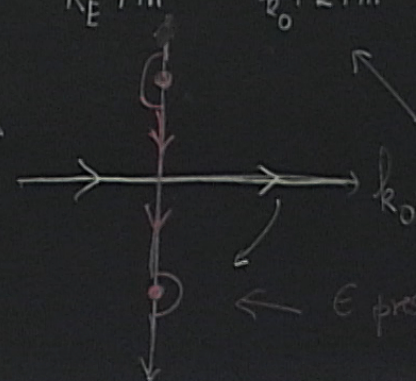
$$\frac{1}{K_E^2+m^2} = \frac{1}{k_0^2+\vec{k}^2+m^2} \longrightarrow \frac{1}{-\omega^2+\vec{k}^2+m^2-i\epsilon_+}$$

poles at $k_0 = \pm i\sqrt{\vec{k}^2+m^2}$



only possible choice

t, ω are real



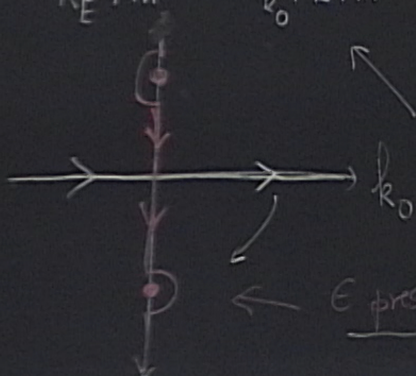
$\leftarrow$ ϵ prescription of Feynman

$\longrightarrow e^{i(t\omega+\vec{x}\cdot\vec{k})}$

$\longrightarrow -i\int d\omega\int d\vec{k}$

$$\frac{1}{K_E^2+m^2} = \frac{1}{k_0^2+\vec{k}^2+m^2} \longrightarrow \frac{1}{-\omega^2+\vec{k}^2+m^2-i\epsilon_+}$$

poles at $k_0 = \pm i\sqrt{\vec{k}^2+m^2}$



ϵ prescription of Feynman

$$\boxed{G(t,\vec{x}) = \int d\omega\int d\vec{k}\, e^{i(\omega t+\vec{k}\vec{x})}\frac{i}{\omega^2-\vec{k}^2-m^2+i\epsilon}}$$

$\omega = k_0$ $\quad K=(\omega,\vec{k})$ $\quad K^\mu = (E,\vec{k})$

$\vec{k} = (k_\mu)$

$\boxed{E = K^0 = -\omega \quad (-,+++)}$

$G(t,\vec{x}) = G_F(t,\vec{x})$